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A NOTE ON SPECIAL CLASSES OF p-VALENT FUNCTIONS

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ABSTRACT. Let $V_k^\lambda(p)$ $(k \geqq 2,\ |\lambda| < \pi/2,\ p \geqq 1)$ denote the class of functions f analytic in $\mathscr{U}: \{z/|z| < 1\}$ having $(p-1)$ critical points there and satisfying

$$\limsup_{r\to 1^-} \int_0^{2\pi} \left| \operatorname{Re}\left\{ e^{i\lambda} \left(1 + \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right)\right\}\right| d\theta \leqq kp\pi\cos\lambda.$$

From $V_k^\lambda(p)$, we can obtain many interesting known subclasses including the class of functions of bounded boundary rotation and the class of p-valent functions $f(z)$ for which $zf'(z)$ is λ-spiral-like. In the present paper, the results obtained for $f \in V_k^\lambda(p)$ include a domain of values for $(1 + (zf''(z)/f'(z)))$, a distortion theorem for $\operatorname{Re} e^{i\lambda}\log[f'(z)/z^{p-1}]$, and the Hardy classes to which f' and f belong.

1. **Introduction.** Let A_q $(g \geqq 1)$ denote the class of functions $f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n$ which are analytic in $\mathscr{U}: \{z/|z| < 1\}$. For $f \in A_q$, we say f belongs to the class $V_k^\lambda(p, q)$ $(k \geqq 2,\ |\lambda| < \pi/2,\ p \geqq q,\ p$ an integer) if there exists $\delta > 0$ such that

$$(1) \qquad \int_0^{2\pi} \operatorname{Re}\left\{1 + \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right\} d\theta = 2p\pi \quad (1 - \delta < r < 1)$$

and

$$(2) \quad \limsup_{r\to 1^-} \int_0^{2\pi} \left| \operatorname{Re}\left\{ e^{i\lambda} \left(1 + \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right)\right\}\right| d\theta \leqq k\,p\,\pi\cos\lambda.$$

Condition (1) implies that f has $(p-1)$ critical points in $\mathscr{U}$. Further, $V_2^\lambda(p, q)$ is the class of p-valent functions f for which zf' is λ-spiral-like in $\mathscr{U}$.

The class $V_k^\lambda(p, q)$ was recently introduced by the author [11]. For special parametrizations, $V_k^\lambda(p, q)$ coincides with several interesting classes. For instance, from condition (2), $V_k^0(1, 1)$ is the class of functions of bounded boundary rotation introduced by Löwner [5] and Paatero [7], [8]. The class $V_k^\lambda(1, 1)$ was investigated by Moulis [6] and Silvia [10], while $V_k^0(p, q)$ was recently studied by Leach [3].

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In the following, we restrict ourselves to the case where $p = q$. For the class $V_k^\lambda(p, p) = V_k^\lambda(p)$, the transformation satisfying

$$F_\alpha'(z) = \frac{p\alpha^{p-1}z^{p-1}f'((z+\alpha)/(1+\bar{\alpha}z))}{f'(\alpha)(z+\alpha)^{p-1}(1+\bar{\alpha}z)^{pe^{-2i\lambda}+1}}$$

for $|\alpha| < 1$, $p \geqq 1$ is shown to be $V_k^\lambda(p)$-preserving. This result enables us to obtain a domain of values for $1 + (zf''(z)/f'(z))$ whenever $f \in V_k^\lambda(p)$ ($|z| \leqq r$) and a disc where f is convex. Additional results are obtained concerning the Hardy classes for $V_k^\lambda(1)$.

2. **A $V_k^\lambda(p)$-preserving Transformation.** In order to obtain the desired transformation we need the following lemmas which are proved in [6] and [11], respectively.

LEMMA A. *If* $h \in V_k^\lambda(1)$ *then* H *defined by* $H'(z) = h'((z+\alpha)/(1+\bar{\alpha}z))/h'(\alpha)\,(1+\bar{\alpha}z)^{e^{-2i\lambda}+1}$, ($|\alpha| < 1$, $|z| < 1$ *and* $H(0) = 0$) *is in* $V_k^\lambda(1)$.

LEMMA B. *The function* $f \in V_k^\lambda(p)$, $p \geqq 1$, *if and only if* $f'(z) = pz^{p-1}[h'(z)]^p$ *for some* $h \in V_k^\lambda(1)$.

From Lemmas A and B we easily obtain

THEOREM 1. *If* $f \in V_k^\lambda(p)$ *then the transformation* F_α *satisfying*

$$(3) \qquad F_\alpha'(z) = \frac{p\alpha^{p-1}z^{p-1}f'((z+\alpha)/(1+\bar{\alpha}z))}{f'(\alpha)(z+\alpha)^{p-1}(1+\bar{\alpha}z)^{pe^{-2i\lambda}+1}} \quad (z \in \mathscr{V},\ F_\alpha(0) = 0)$$

is in $V_k^\lambda(p)$ *for all* α, $|\alpha| < 1$.

PROOF. By Lemma B, there exists $h \in V_k^\lambda(1)$ such that

$$(4) \qquad f'(z) = pz^{p-1}[h'(z)]^p.$$

For such an $h \in V_k^\lambda(1)$, we define $H \in V_k^\lambda(1)$ by

$$(5) \qquad H'(z) = h'((z+\alpha)/(1+\bar{\alpha}z))/h'(\alpha)(1+\bar{\alpha}z)^{e^{-2i\lambda}+1},$$

where $H(0) = 0$. Using Lemma B, and (5) we see that an F_α such that

$$(6) \qquad F_\alpha'(z) = pz^{p-1}[H'(z)]^p$$

is in $V_k^\lambda(p)$. Finally, from (4) we obtain

$$(7) \qquad \begin{cases} p\alpha^{p-1}[h'(\alpha)]^p = f'(\alpha) \\ p\left(\dfrac{z+\alpha}{1+\bar{\alpha}z}\right)^{p+1}\left[h'\left(\dfrac{z+\alpha}{1+\bar{\alpha}z}\right)\right]^p = f'\left(\dfrac{z+\alpha}{1+\bar{\alpha}z}\right) \end{cases}$$

and (3) follows from (6) and (7).

REMARK. For $p = 1$, Theorem 1 reduces to Lemma A. For $p = 1$, and $k = 2$, we have the result obtained by Libera and Ziegler [4]. If $p > 0$, $k = 2$, Theorem 1 gives us a transformation that preserves the class of p-valent functions f for which zf' is a λ-spiral-like function.

It is known [11] that for $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in V_k^{\lambda}(p)$,

$$(p + 1)|a_{p+1}| \leqq p^2 k \cos\lambda \tag{8}$$

with equality for f satisfying $f'(z) = pz^{p-1}[F'(z)]^p$ where

$$F'(z) = \left\{ \frac{(1 + \epsilon z)^{k/2-1}}{(1 - \epsilon z)^{k/2+1}} \right\}^{e^{-i\lambda}\cos\lambda}, \ |\epsilon| = 1.$$

We now use this coefficient bound and Theorem 1 to obtain

THEOREM 2. *For $|z| \leqq r$ and f ranging over $V_k^{\lambda}(p)$ the domain of values of $1 + (zf''(z)/f'(z))$ is the disc with center $(p(1 + r^2\cos 2\lambda)/(1 - r^2)$, $-pr^2\sin 2\lambda/(1 - r^2)$ and radius $pkr\cos\lambda/(1 - r^2)$.*

PROOF. Whenever $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in V_k^{\lambda}(p)$, $\lim_{z\to 0}(f''(z) - p(p - 1)z^{p-2})/z^{p-1} = p(p + 1)a_{p+1}$. For $f \in V_k^{\lambda}(p)$, let $F_\alpha(z) = z^p + \sum_{n=p+1}^{\infty} A_n z^n \in V_k^{\lambda}(p)$ be given by (3) for $0 < |\alpha| 3\ 1$. By direct calculation we have

$$p(p + 1)A_{p+1} = p(1 - |\alpha|^2)\frac{f''(\alpha)}{f'(\alpha)} - \frac{p(pe^{-2i\lambda} + 1)|\alpha|^2 + p(p - 1)}{\alpha}. \tag{9}$$

Combining (8) and (9), we obtain

$$\left| \frac{f''(\alpha)}{f'(\alpha)} - \frac{(pe^{-2i\lambda} + 1)|\alpha|^2 + (p - 1)}{\alpha(1 - |\alpha|^2)} \right| \leqq \frac{pk\cos\lambda}{1 - |\alpha|^2}. \tag{10}$$

From (10), it follows that, for $|z| = r < 1$,

$$\left|\left(1 + \frac{zf''(z)}{f'(z)} \right) - \frac{p(1 + e^{-2i\lambda}r^2}{1 - r^2} \right| \leqq \frac{pkr\cos\lambda}{1 - r^2}, \tag{11}$$

the desired result.

COROLLARY 1. *If $f \in V_k^{\lambda}(p)$ then*

$$\log\left\{ \frac{(1 - |z|)^{k-2}}{(1 + |z|)^{k+2}} \right\}^{p/2\cos\lambda} \leqq \operatorname{Re}\{e^{i\lambda}\log[f'(z)/pz^{p-1}]\} \tag{12}$$

$$\leqq \log\left\{ \frac{(1 + |z|)^{k-2}}{(1 - |z|)^{k+2}} \right\}^{p/2\cos\lambda},$$

and these bounds are sharp.

PROOF. From (11), for $|z| = r < 1$, we have

$$\left| \frac{zf''(z)}{f'(z)} - \frac{(pe^{-2i\lambda} + 1)r^2 + (p-1)}{1-r^2} \right| \leqq \frac{pk\, r \cos\lambda}{1-r^2}.$$

It follows that

$$\left| e^{i\lambda} \left\{ \frac{zf''(z)}{f'(z)} - (p-1) \right\} - \frac{2p\, r^2 \cos\lambda}{1-r^2} \right| \leqq \frac{pk\, r \cos\lambda}{1-r^2}$$

and

$$\frac{2p\, r\cos\lambda - pk\cos\lambda}{1-r^2} \leqq \operatorname{Re}\left[e^{i\lambda} \left\{ \frac{e^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})} - \frac{p-1}{r} \right\} \right]$$
$$\leqq \frac{2p\, r\cos\lambda + pk\cos\lambda}{1-r^2}.$$

We obtain (12) by integrating with respect to r. The upper and lower bounds in (12) are obtained for f satisfying $f'(z) = pz^{p-1}[F'(z)]^p$, where

$$F'(z) = \left\{ \frac{(1-z)^{k-2}}{(1+z)^{k+2}} \right\}^{e/2^{-i\lambda}\cos\lambda}$$

with $z = r$ and $z = -r$, respectively.

COROLLARY 2. *If $f \in V_k^\lambda(p)$ then f is convex for $|z| < 2/(k\cos\lambda + (k^2\cos^2\lambda - 4\cos 2\lambda)^{1/2})$.*

PROOF. From (11), we have

$$\operatorname{Re}\left\{ 1 + \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})} \right\} \geqq \frac{p(1 + r^2\cos 2\lambda - kr\cos\lambda)}{1-r^2}.$$

Thus, f will be convex if

$$(1 - kr\cos\lambda + r^2\cos 2\lambda) > 0$$

and the result follows.

3. **Hardy classes for $V_k^\lambda(1)$.** For real μ, $\mu > 0$, we say that a function h analytic in U belongs to the class H^μ if

$$\int_{-\pi}^{\pi} |h(re^{i\theta})|^\mu \, d\theta < M$$

for $0 \leqq r < 1$, M a constant determined by h and μ.

In order to obtain the H^μ classes for $V_k^\lambda(1)$, we will use the following well known lemmas.

LEMMA C. *A necessary and sufficient condition for $f \in V_k^{\lambda}(1)$ is that there exist an $h \in V_k^0(1)$ such that*

$$[f'(z)] = [h'(z)]^{e^{-i\lambda}\cos\lambda}.$$

LEMMA D. *Let $f \in V_k^{\lambda}(1)$. Then, for $|z| = r$,*

$$|\arg f'(re^{i\theta})| \leqq k\cos\lambda \text{ arc sin } r.$$

LEMMA E. *If $f' \in H^{\mu}$, $o < \mu \leqq 1$ then $f \in H^{\mu/1-\mu}$ where, for $\mu = 1$, H^{∞} is the class of bounded functions.*

LEMMA F. *Let $h \in V_k^0(1)$. Then $h' \in H^{\mu}$ for all $\mu < 2/(k+2)$ and $h \in H^{\eta}$ for $\eta < 2/k$. Furthermore, if h' is not of the form*

$$(13) \quad h'(z) = (1 - ze^{-it_0})^{-(k/2+1)} \exp\left\{ \int_{-\pi}^{\pi} - \log(1 - ze^{-it})\, dm(t) \right\}$$

($m(t)$ a probability measure on $[-\pi, \pi]$), then $f' \in H^{\mu}$ for some $\mu > 2/(k+2)$ and $f \in H^{\eta}$ for some $\eta > 2/k$.

Lemmas C and D were proved in [**10**]. Lemma E can be found in [**2**, p. 88] and Lemma F is due to Pinchuk [**9**].

THEOREM 3. *If $f \in V_k^{\lambda}(1)$ then $f' \in H^{\mu}$ for all $\mu < 2\sec^2\lambda/(k+2)$ and $f \in H^{\eta}$ for $\eta < 2/((k+2)\cos^2\lambda - 2)$, $2/(k+2) < \cos^2\lambda$. Furthermore, if f' is not of the form $f'(z) = [h'(z)]^{e^{-i\lambda}\cos\lambda}$ where h is given by (13) then there exists $\delta = \delta(f) > 0$ and $\epsilon = \epsilon(f) > 0$ such that $f' \in H^{(2+\delta)\sec^2\lambda/(k+2)}$ and $f \in H^{(2+\epsilon)/((k+2)\cos^2\lambda-2)}$ for $2/(k+2) < \cos^2\lambda$.*

PROOF. For $f \in V_k^{\lambda}(1)$, let h be given by Lemma C. Thus $[f'(z)] = [h'(z)]^{\cos^2\lambda - i\sin\lambda\cos\lambda}$ and $|f'(z)|^{\mu} = |h'(z)|^{\mu\cos^2\lambda}\exp\{\mu\sin\lambda\cos\lambda\arg h'(z)\}$. By Lemma D, the exponential factor is bounded. Thus the result follows from Lemmas E and F.

Note that for $\lambda = 0$, Theorem 3 reduces to Lemma F. When $k = 0$, we have the result obtained by Başgöze and Keogh [1] for the class of λ-spiral-like functions.

For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in V_k^{\lambda}(1)$ the sharp upper bounds for $|a_2|$ and $|a_3|$ are known [10] and [11]. From Theorem 3 and the well known result [**2**, p 98] that $f(z) = \sum a_n z^n \in H^{\mu}$ $(0 < \mu < 1)$ implies $a_n = o(n^{1/\lambda - 1})$, we obtain a growth estimate for the Taylor coefficients of $f \in V_k^{\lambda}(1)$.

COROLLARY. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^{\nu} \in V_k^{\lambda}(1)$ and $(k+2)\cos^2\lambda > 2$ then*

$$a_n = o(n^{[(k+2)\cos^2\lambda - 4]/2}).$$

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