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STEPANOFF FLOWS ON ORIENTABLE SURFACES

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ABSTRACT. We consruct flows on compact orientable surfaces of positive genus which are analogous to Stepanoff flows on the torus; i.e., these flows admit a finite number of nontrivial ergodic measures, their orbits are dense except for a finite number of fixed points, and they are topologically mixing. Our method is to lift Stepanoff flows along with their ergodic measures from the torus to a surface of higher genus through a branched covering map.

0. **Introduction.** In this paper we construct flows on compact orientable surfaces of higher positive genus analogous to the Stepanoff flows on the torus constructed by J. C. Oxtoby [11]. Our method is to lift Oxtoby's Stepanoff flows through a branched covering map described by E. Hemmingsen and W. Reddy [5] (cf. also T. O'Brien [9]). To do this we introduce additional fixed points (by splitting orbits) into Oxtoby's flows but basically these flows are the same. We show that the topological dynamical properties of these flows, and also most of the ergodic properties, lift through this branched covering map. We also show that these lifted flows admit only a finite number of ergodic measures. To do this we develop a Kryloff-Bogoliouboff theory for a transformation group called a *deck flow* which is a composite of the action of the flow and the branched covering translation.

The research in this paper is also reported in the author's thesis written at Wesleyan University under the direction of Professor W. L. Reddy to whom the author is indebted.

1. **Stepanoff Flows on the Torus and Their Ergodic Measures.** We shall follow the notation and definitions found in Gottschalk and Hedlund [4] unless otherwise stated. However, we shall denote a transformation group (X, G, γ) as (X, γ^g) where $\gamma^g(x) = \gamma(x, g)$ for $x \in X$ and $g \in G$. We shall say that (X, γ^g) is a *lift* through $f: X \rightarrow Y$ of (Y, γ^g) if f is a homomorphism of (X, γ^g) onto (Y, γ^g), and also say that (Y, γ^g) is a *projection*. Throughout let T be the torus defined as R^2/Z^2, where Z is the set of integers.

DEFINITION. By a *Stepanoff flow on the torus* we shall mean a 1-parameter continuous flow isomorphic to a flow (T, γ^t) for which (i) there are $n \geqq 2$ fixed points and (ii) the points of each orbit satisfy an equa-

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tion $y - \alpha x = c$ for $c \in R$ (reals), where α is an irrational number corresponding to γ^t.

It is well-known [8] that the semi-orbit of each point which is not asymptotic to a fixed point is dense since α is irrational. Note by (i) and (ii) there may exist a non-fixed point p which is positively and negatively asymptotic to two distinct fixed points. Then the orbit-closure of p will be an arc. Because there are dense orbits positively and negatively asymptotic to fixed points, it follows that these flows are topologically mixing (for any two nonempty open sets U and V there is an $N > 0$ for which $\gamma(U, t) \cap V \neq \Phi$ for all $|t| > N$).

The results of J. C. Oxtoby [11] for Stepanoff flows having exactly one fixed point extend easily when we consider additional fixed points. Thus we state the extension of his results in the following propositions 1.1-1.3, and sketch the modifications when necessary.

PROPOSITION 1.1. *If $n \geqq 2$, there is an analytic area preserving Stepanoff flow on the torus with exactly n fixed points. Moreover the orbit of each non-fixed point is dense.*

PROOF. Consider the functions on R^2 defined by

$$X(x, y) = \alpha(1 - \cos 2\pi(x - y)) + (1 - \alpha)(1 - \cos 2\pi n y)$$

$$Y(x, y) = \alpha(1 - \cos 2\pi(x - y))$$

where $0 < \alpha < 1$ is irrational. The system $dx/dt = X$, $dy/dt = Y$ defines a planar flow (R^2, γ^t) whose projection through $R^2 \rightarrow T$ is a flow on the torus (T, γ^t) [8]. Both flows are area preserving since

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0, \tag{1}$$

and both are analytic since X and Y are. The set F of all points in R^2 whose coordinates are rational numbers $(k_1/n, k_2/n)$ such that $k_1 - k_2$ is congruent to $0 \bmod(1)$ is the set of fixed points under (R^2, γ^t), so under (T, γ^t) there are n such points. By (1) the expression $-Ydx + Xdy$ is an exact differential which generates the integral function

$$H = (1 - \alpha)\left(y - \frac{\sin 2n\pi}{2n\pi}\right) - \alpha\left(x - y - \frac{\sin 2\pi(x - y)}{2\pi}\right).$$

The contour $H(x, y) = c$ intersects F if and only if $c = k_2/n - \alpha k_1/n$ for $k_1 - k_2 \equiv 0 \bmod(1)$, so $H(x, y) = c$ contains at most one point $(k_1/n, k_2/n)$ in F. When $c \neq k_2/n - \alpha k_1/n$, the argument given on [1, p. 985] shows that the flow is Stepanoff, and the line $y' - \alpha x' = c$ containing a fixed point is composed of three orbits, hence each non-fixed point has a dense orbit.

PROPOSITION 1.2. *If μ is a normalized Borel (probability) measure invariant under a Stepanoff flow on the torus for which the set F of fixed points is μ-null, then μ is unique, and the flow is ergodic with respect to μ.*

PROOF. Since the fixed points are isolated and since the set consisting of points in F and the points asymptotic to F is μ-null, the proof of Oxtoby applies here (see [11, pp. 984–985]).

We assume the reader is familiar with *Kryloff-Bogoliouboff* (*K-B*) *Theory* [6]; see also [8] and [10]. For a summable function f on a compact space X, we denote the *time mean* under a continuous flow (X, γ^t) at point $p \in X$ by

$$f^*(p) = \lim_{t \to \infty} f_t(p)$$

where

$$f_t(p) = \frac{1}{t} \int_0^t f \circ \gamma(p, \tau)\, d\tau.$$

Recall that a point $q \in X$ is *quasi-regular* under (X, γ^t) if $f^*(q)$ exists for all continuous f. Also recall that corresponding to the point q is a functional called the *individual measure* μ_q defined by $\mu_q(f) = f^*(q)$. We shall say an ergodic measure is *trivial* if it is a unit point mass (Dirac) measure.

PROPOSITION 1.3. *A Stepanoff flow on the torus with fixed point set F either* (1) *admits only trivial ergodic measures or* (2) *admits, besides the trivial ones, a unique ergodic measure for which* $\mu(F) = 0$. *Moreover if the orbit of each non-fixed point is dense then the set Q of quasi-regular points is a set of first category dense in the torus.*

PROOF. Each individual measure μ_p for $p \in F$ is a trivial ergodic measure, so it follows from 1.2 that either (1) or (2) holds. The set Q is dense because it contains both F and the orbits positively asymptotic to F. Thus we need only show that Q is a set of first category for case (1) since the arguments found in [11, p. 986] apply for case (2). Assume that the Stepanoff flow (T, γ^t) admits only trivial ergodic measures. There is a continuous function f which separates the fixed points p_1 and p_2. Let a and b be rationals such that $f(p_1) < a < b < f(p_2)$. Let

$$E_1 = \bigcap_{n=1}^{\infty} \bigcup_{t>n} \{p \mid f_t(p) < a\}$$

and

$$E_2 = \bigcap_{n=1}^{\infty} \bigcup_{t>n} \{p \mid f_t(p) > b\}.$$

Observe that E_1 is a dense G_δ-set since E_1 contains both p_1 and the dense orbit positively asymptotic to p_1. *Similarly the set* E_2 is a dense G_δ-set. Then Q is first category since $Q \subset T - E_1 \cap E_2$.

The flows given in 1.1 are examples of case (2). To obtain examples of Stepanoff flows with set F of n fixed points admitting only trivial ergodic measures, consider the system $dx/dt = X$, $dy/dt = \alpha X$ with $\alpha > 0$ irrational and $X = \sin^2\pi(x - y) + \sin^2 2\pi ny$. Recall that [10, 7.2, p. 128] the sub-flow $(T - F, \gamma^t)$ defined by this system admits no invariant probability measure if and only if $(\chi_K)^*(p) = 0$ for all $p \in T - F$ and for all characteristic functions on compact sets $K \subset T - F$. W. Stepanoff [12] (cf. [8, pp. 368–373]) has shown this for such a system when $n = 1$. His arguments apply also for $n > 1$, and thus we obtain examples of case (1).

2. **Lifting Flows Through Branched Coverings.** Represent points in R^3 as (z, w) where $z = r\exp(i\theta)$ is a complex number. As described in [5, § 5, pp. 13–14], let M_h be a compact orientable surface represented as a sphere centered at the origin with $h \geqq 1$, handles attached. Consider the handles attached so that M_h is invariant under the rotation ρ through $2\pi/h$ radians defined by $\rho(r\exp(i\theta), w) = (r\exp(i(\theta + 2\pi/h), w)$. Also let $\psi : M_h \to M_1$ be the usual branched covering map [4] from M_h onto M_1 defined by $\psi(z, w) = (z^h, w)$ for which the branch set $B_\psi = \{(0, \pm 1)\}$. Henceforth we shall consider a Stepanoff flow on the torus with M_1 as its phase space, and assume that its set of fixed points contains ψB_ψ.

THEOREM 2.1. *There exists a unique flow* $\bar{\lambda} : M_h \times R \to M_h$ *which is a lift through the branched covering* $\psi : M_h \to M_1$ *of a Stepanoff flow* $\lambda : M_1 \times R \to M_1$ *whose set F of fixed points contains* ψB_ψ *such that the following diagram commutes*:

$$\begin{array}{ccc} M_h \times R & \xrightarrow{\ \bar{\lambda}\ } & M_h \\ \downarrow{\scriptstyle \psi \times \mathrm{id}} & & \downarrow{\scriptstyle \psi} \\ M_1 \times R & \xrightarrow{\ \lambda\ } & M_1 \end{array}$$

PROOF. Define $\bar{\lambda}(p, t) = p$ if $p \in B_\psi$. Off B_ψ, ψ is an (unbranched) covering map, hence we may lift the flow λ to unique flow ξ on $M_h - B_\psi$ by [7, Lemma 1]. It is easy to check that the extension of ξ to $\bar{\lambda}$ is a continuous flow on M_h.

DEFINITION. By a *Stepanoff flow* on M_h ($h \geqq 2$) we shall mean a flow isomorphic to a lift through ψ of a Stepanoff flow on M_1 in which the orbit of each point except for the fixed points is dense.

Henceforth let $(M_h, \bar{\lambda}^t)$ denote the Stepanoff flow which is a lift through ψ of a Stepanoff flow on the torus (M_1, λ^t).

COROLLARY 2.2. *The group $\langle \rho \rangle$ of self-homeomorphisms of M_h generated by ρ is a group of automorphisms of $(M_h, \bar{\lambda}^t)$.*

PROOF. The flow $(M_h, \rho \circ \bar{\lambda}^t \circ \rho^{-1})$ is also a lift through ψ of (M_1, λ^t) since $\psi \circ \rho = \psi$; hence $\rho \circ \bar{\lambda}^t \circ \rho^{-1} = \bar{\lambda}^t$ by 2.1.

We now consider the dynamical properties of these flows on M_h.

THEOREM 2.3. *Each non-fixed point p of a Stepanoff flow on M_h has a dense semi-orbit. If p is also not asymptotic to any fixed point then both its semi-orbits are dense.*

PROOF. The branched covering $\psi : M_h \longrightarrow M_1$ induces a natural decomposition of M_h as follows. We cut M_1 open along a segment I (avoiding the handle) connecting the two points in ψB_ψ to obtain V with two segments I^+ and I^- as its boundary. Corresponding to V there are h copies of V in M_h under ψ, $V_1, \cdots V_i, \cdots, V_h$ with boundary I_i^+ and I_i^- respectively. We glue V_i and V_j together along I_i^- and I_j^+ for $i = j - 1$ or $i = h$ and $j = 1$ to obtain M_h so that $\psi(I_i^\pm) = I$ and $\psi(V_i - (I_i^+ \cup I_i^-)) = V - I$. We obtain the flow $\bar{\lambda}$ on M_h by matching the flow lines (orbits) of λ_i on each V_i induced from λ on M_1 along I_i^- and I_i^+. Then the theorem follows from the properties of λ on M_1.

THEOREM 2.4. *A Stepanoff flow $(M_h, \bar{\lambda}^t)$ is topologically mixing.*

PROOF. The argument in the proof of Theorem 2.3 and the fact (M_1, λ^t) is mixing gives us Theorem 2.4.

3. **Lifting Invariant Measures Through Branched Coverings.** Throughout let $C(M_h)$ be the space of all continuous functions on M_h ($h \geqq 1$) equipped with usual norm (sup norm). Let $I(M_h, \bar{\lambda}^t)$ denote the space of normalized Borel measures invariant under $(M_h, \bar{\lambda}^t)$ with the weak-star topology on $C(M_h)^*$, and similarly let $I(M_1, \lambda^t)$ denote the space of normalized Borel measures invariant under (M_1, λ^t) with weak-star topology on $C(M_1)^*$. The epimorphism $\psi : (M_h, \bar{\lambda}^t) \longrightarrow (M_1, \lambda^t)$ induces a con-

tinuous linear operator $\hat{\psi}$ from $I(M_h, \bar{\lambda}^t)$ *onto* $I(M_1, \lambda^t)$ defined by $\hat{\psi}\,\mu(f) = \mu(f \circ \psi)$ for $\mu \in I(M_h, \bar{\lambda}^t)$ and $f \in C(M_1)$; this is well known [3, § 3, pp. 370–373] for discrete flows and easily extends for continuous flows. Also the operator $\hat{\psi}$ is equivalently defined by $\hat{\psi}\,\mu(B) = \mu(\psi^{-1}B)$ for $B \subset M_1$ Borel. Since ergodic measures under continuous flows are extreme points, it follows that ψ maps the set of all ergodic measures under $(M_h, \bar{\lambda}^t)$ onto the set of all ergodic measures under (M_1, λ^t). Similarly the automorphism ρ of $(M_h, \bar{\lambda}^t)$ described in § 2 induces a self-isomorphism of $I(M_h, \bar{\lambda}^t)$ defined by $\hat{\rho}\,\mu(f) = \mu(f \circ \rho)$ for $f \in C(M_h)$ or equivalently by $\hat{\rho}\,\mu(B) = \mu(\rho^{-1}B)$ for $B \subset M_h$ Borel.

Proposition 3.1. *Let μ_p be an individual measure under $(M_h, \bar{\lambda}^t)$. Then* (1) $\hat{\psi}\,\mu_p = \mu_{\psi(p)}$ *is an individual measure under (M_1, λ^t),* (2) $\hat{\rho}\,\mu_p = \mu_{\rho(p)}$ *is one under $(M_h, \bar{\lambda}^t)$, and* (3) $\hat{\psi} \circ \hat{\rho}\,\mu_p = \mu_{\psi(p)}$.

Proof. Since $\lambda \circ (\psi \times \mathrm{id}) = \psi \circ \bar{\lambda}$ (2.1) the equation $\mu_{\psi(p)}(f) = \lim_{t\to\infty}(1/t) \int_0^t f \circ \lambda(\psi(p), \tau)\, d\tau = \mu_p(f \circ \psi)$ for all $f \in C(M_1)$, so (1) follows. Similarly (2) follows from the equation $\bar{\lambda}^t \circ \rho = \rho \circ \bar{\lambda}^t$ (2.2). Finally (3) follows from (1) and (2) since $\rho \circ \psi = \psi$.

Theorem 3.2. *The set U of quasi-regular points under a Stepanoff flow on M_h $(h \geqq 2)$ is a ρ-invariant set of first category dense in M_h.*

Proof. By 3.1 ψU is contained in the set of quasi-regular points under (M_1, λ^t) and $U = \psi^{-1}\psi U$ is ρ-invariant. Since ψ is open it follows from 1.3 that U is a set of first category dense in M_h.

It would be interesting to know whether the lift of a quasi-regular point through ψ is quasi-regular.

We now assume that the reader is familiar with the results of invariant measures under general transformation groups on compact spaces; see [1]. The compact subspace of measures in $I(M_h, \bar{\lambda}^t)$ invariant under the group $\langle\rho\rangle$ corresponds to a transformation which we shall call a *deck flow* associated with $(M_h, \bar{\lambda}^t)$.

Definition. By a *deck flow* on M_h associated with Stepanoff flow $(M_h, \bar{\lambda}^t)$ we shall mean a group $R \times Z/h$ which acts as a transformation group with respect to $\rho^j \circ \bar{\lambda}^t (j \in Z/h,\ t \in R)$. We shall denote the deck flow by $(M_h, \rho^j \circ \bar{\lambda}^t)$ and its corresponding subspace of invariant deck measures in $I(M_h, \lambda^t)$ by $I(M_h, \rho^j \circ \lambda^t)$.

Proposition 3.3. *The linear operator $\hat{\psi}$ maps $I(M_h, \rho^j \circ \bar{\lambda}^t)$ onto $I(M_1, \lambda^t)$.*

Proof. Consider the group of operators on $C(M_h)$ defined by $g \to g \circ \rho^j \circ \bar{\lambda}^t (j \in Z/h,\ t \in R)$, and apply Agnew-Morse Theorem.

4. **Ergodic Measures Under Stepanoff Flows on M_h.** To count the ergodic measures under $(M_h, \bar{\lambda}^t)$, we first count the ergodic measures σ under $(M_h, \rho^j \circ \bar{\lambda}^t)$ and show that σ is a finite linear combination of ergodic measures under $(M_h, \bar{\lambda}^t)$. Recall ([8] or [10] that a quasi-regular point p under $(M_h, \bar{\lambda}^t)$ is called *regular* if μ_p is ergodic and if $\mu_p(W) > 0$ for every open set W containing p. Throughout let Q_R denote the set of all regular points under $(M_h, \bar{\lambda}^t)$. For each $p \in Q_R$ we associate an *individual deck* measure σ_p defined by $\sigma_p = (1/h) \sum_{j=0}^{h-1} \mu_{\rho^j(p)}$ where each $\mu_{\rho^j(p)}$ is an individual measure under $(M_h, \bar{\lambda}^t)$. An invariant deck measure σ is called deck ergodic if whenever $B \subset M_h$ is a $\rho^j \circ \bar{\lambda}^t$-invariant Borel set then $\sigma(B)\sigma(M_h - B) = 0$. Observe from 3.1 that σ_p is deck ergodic for $p \in Q_R$. To each σ we associate the (*deck ergodic*) set $\mathscr{E}(\sigma)$ of all $p \in Q_R$ for which $\sigma_p = \sigma$. Finally for each $f \in C(M_h)$ let $f^*(p) = (1/h) \sum_{j=0}^{h-1} (f \circ \rho^j)^*(p)$ where $(f \circ \rho^j)^*$ denotes the time mean under $(M_h, \bar{\lambda}^t)$; see § 1.

PROPOSITION 4.1. *Let σ be a deck ergodic measure. Then*

(1) $f^*(p) = \int f\,d\sigma$ σ-a.e. *for* $f \in C(M_h)$.

(2) *The set $\mathscr{E}(\sigma)$ is $\rho^j \circ \bar{\lambda}^t$-invariant and $\sigma(\mathscr{E}(\sigma)) = 1$. Moreover for each deck ergodic measure $\nu \neq \mu$, $\nu(\mathscr{E}(\sigma)) = 0$.*

(3) *σ is an extreme point of $I(M_h, \rho^j \cdot \bar{\lambda}^t)$.*

(4) *σ is a linear combination of ergodic measures under $(M_h, \bar{\lambda}^t)$; that is $\sigma = \sigma_p$ for some $p \in Q_R$.*

PROOF. Since each $(f \circ \rho^j)^*$ exists σ-a.e. and is $\bar{\lambda}^t$-invariant (Ergodic Theorem), it follows that f^* exists σ-a.e. and also that f^* is $\rho^j \circ \bar{\lambda}^t$-invariant. We then obtain (1) by applying the same argument given for continuous ergodic flows [8, pp. 468–469]. Since $\sigma_p(f) = f^*(p)$ by 3.1, (1) implies (2). As in the case for continuous flows, (2) implies (3). Recall that $\sigma(Q_R) = 1$, so (4) follows from (2) also.

In general note that for arbitrary transformation groups 4.1.(2) is not valid; see [1].

PROPOSITION 4.2. *Let* $\operatorname{erg}(M_h, \rho^j \circ \bar{\lambda}^t)$ *be the set of all ergodic measures under the deck flow and* $\operatorname{erg}(M_1, \lambda^t)$ *the ergodic measures under* (M_1, λ^t). *Then* $\hat{\psi} : \operatorname{erg}(M_h, \rho^j \circ \bar{\lambda}^t) \to \operatorname{erg}(M_1, \lambda^t)$ *is a bijection.*

PROOF. Since each deck ergodic measure has the form σ_p for some $p \in Q_R$ (4.1.(4)), it follows from 3.1 that $\hat{\psi}$ is onto. To show $\hat{\psi}$ is 1-to-1, we need to show that [*] $\hat{\psi}^{-1}(\nu) \cap \operatorname{erg}(M_h, \rho^j \circ \bar{\lambda}^t) = \hat{\psi}^{-1}(\nu) \cap I(M_h, \rho^j \circ \bar{\lambda}^t)$ for $\nu \in \operatorname{erg}(M_1, \lambda^t)$. For if [*] holds, we obtain a nonempty convex set of extreme points by 4.1.(3); hence there can be *only one* such deck ergodic measure (extreme point). So let σ be an invariant deck measure for which $\hat{\psi}\sigma = \nu$, and $B \subset M_h$ be a $\rho^j \circ \overline{\lambda}^t$-invariant σ-measurable set for which $\sigma(B) > 0$. By measurability, B is the union of an F_σ-set and a set P such that $\sigma(P) = 0$. Since σ is ρ-invariant it follows that $\nu(\psi P) = \sigma(\psi^{-1}\psi P) = 0$. Then $\sigma(B) = \nu(\psi B) = 1$ since ν is ergodic. Hence σ is ergodic and [*] holds.

COROLLARY 4.3. *If all the nontrivial ergodic measures under* $(M_h, \overline{\lambda}^t)$ *are* ρ*-invariant then there exists at most one nontrivial measure* μ *for which the set of fixed point is* μ*-null.*

PROOF. This follows from 1.3, 4.2, and 3.1.

We remark that it is still an open question whether all the nontrivial ergodic measures under $(M_h, \overline{\lambda}^t)$ are ρ-invariant. However, the following theorem shows there are a finite number of such ergodic measures.

THEOREM 4.4. *A Stepanoff flow on* M_h *with set* $\tilde{F}$ *of fixed points either* (i) *admits only trivial ergodic measures or* (ii) *admits, in addition to these trivial ones, at least* 1 *but at most* h *ergodic measures* μ *for which* $\mu(\tilde{F}) = 0$.

PROOF. By 1.3 (M_1, λ^t), the projection of $(M_h, \overline{\lambda}^t)$, either (1) admits trivial ergodic measures or (2) admits a unique ergodic measure ν which assign zero to the fixed points under (M_1, λ^t). If (1) holds, then $(M_h, \overline{\lambda}^t)$ admits only trivial ergodic measures. If (2) holds, consider a non-trivial ergodic measure μ_p under $(M_h, \overline{\lambda}^t)$ for which $\hat{\psi}\mu_p = \nu$, where $p \in Q_R$. Then by 3.1 there are at *least* h such ergodic measures $\mu_{\rho^i(p)}$. Suppose that $(M_h, \overline{\lambda}^t)$ admits a μ_q for which $\mu_q \neq \mu_p$ and $\hat{\psi}\mu_q = \nu$ where $q \in Q_R$. Consider the deck ergodic measures σ_p and σ_q corresponding to p and q respectively. Since $\hat{\psi}\sigma_p = \hat{\psi}\sigma_q$, $\sigma_p = \sigma_q$ by 4.2. Note that $\sigma_q(E(\mu_p)) \geqq 1/h$ for the ergodic set corresponding to μ_p. Hence it follows from 3.1 that each $\mu_{\rho^j(q)} = \mu_{\rho^i(p)}$ for some $i \in Z/h$. Thus in (2) there are at most h nontrivial ergodic measures $\mu_{\rho^i(p)}$.

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