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TIME EVOLUTION OF ALMOST PERIODIC SOLUTIONS OF THE KdV EQUATION

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ABSTRACT. In [8], Lax has developed a theory describing a large class of space periodic solutions of the KdV equation which are almost periodic in time. In this note we numerically construct and study an example in this class.

We consider the KdV equation,

$$u_t + uu_x + u_{xxx} = 0 \tag{1}$$

with periodic initial data on the real axis. The solution u is a real-valued function of time t and space x. The solution exists for all time and has the remarkable property of possessing an infinite number of conservation laws. The first four conserved quantities are

$$\begin{aligned} F_{-1}(u) &= \int u\,dx \\ F_0(u) &= \int \frac{1}{2} u^2\,dx \\ F_1(u) &= \int \left(\frac{1}{3} u^3 - u_x{}^2 \right) dx \\ F_2(u) &= \int \left(\frac{1}{4} u^4 - 3uu_x{}^2 + \frac{9}{5} u_{xx}^2 \right) dx. \end{aligned} \tag{2}$$

In [8] Lax investigated the following variational problem suggested by Kruskal and Zabusky: Given the values $F_i(u) = A_i$, $i = -1, 0, 1, 2, \cdots, N-1$ find the function $\tilde{u}(x)$ which minimizes $(-1)^N F_N(u)$. Lax showed that if the constraints are chosen so that A_j is not a stationary value of $F_j(u)$ when the other constraints are imposed, then this minimization problem has a solution. The solution is not unique; in fact, the solutions form an N-dimensional manifold. This manifold is an N-dimensional torus of solutions of the minimization problem.

Since the F_i are conserved quantities it follows that if $u(x, t)$ is a solution of the KdV equation such that for a particular value of t, $u(x, t_0)$ is a solution of the minimization problem, then for any other value of t,

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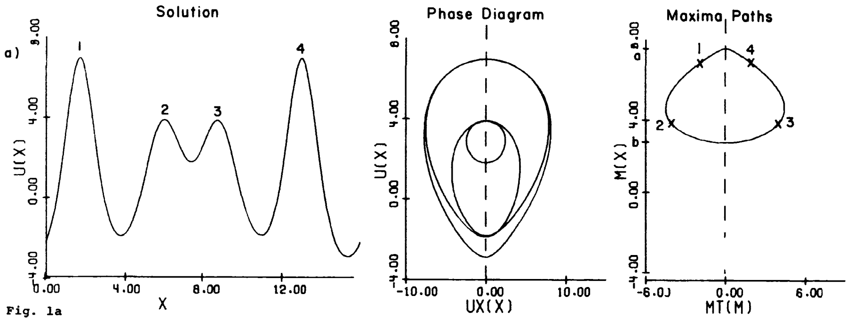

Fig. 1a

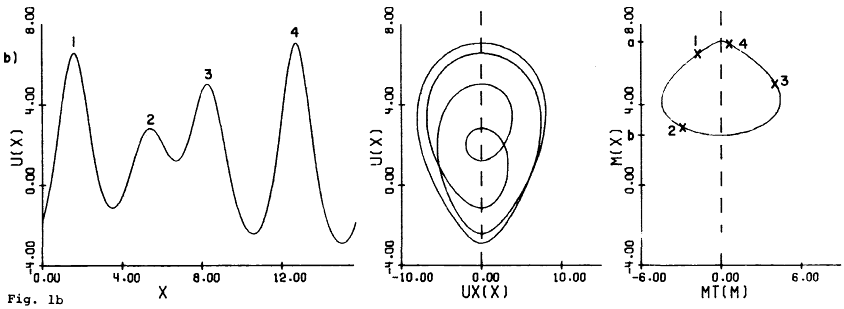

Fig. 1b

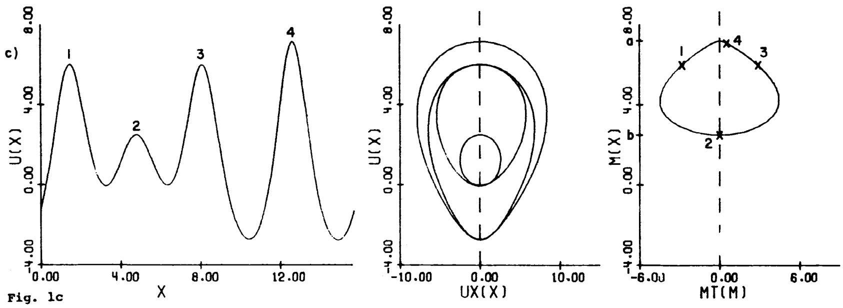

Fig. 1c

$u(x, t)$ is a solution of the minimization problem. In other words, a solution of the KdV equation which starts on the torus remains on the torus. It is shown in [8] that the motion of such a solution is quasi-periodic in time.

The minimized functions satisfy the classical Euler equation,

$$G = G_N + \sum a_j G_j = 0, \tag{3}$$

with Lagrange multipliers corresponding to the constraints $A_j. G_j$, the gradient of F_j, is a nonlinear differential operator of order $2j$. It is easy to show, see [8], that all $\tilde{u}(x)$ on the same torus satisfy the *same* Euler equation.

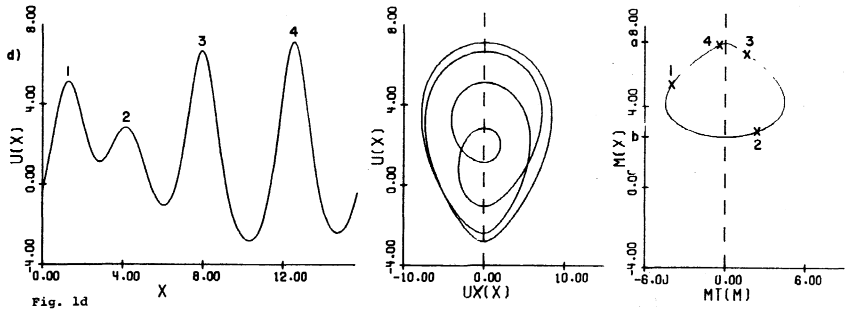

Fig. 1d

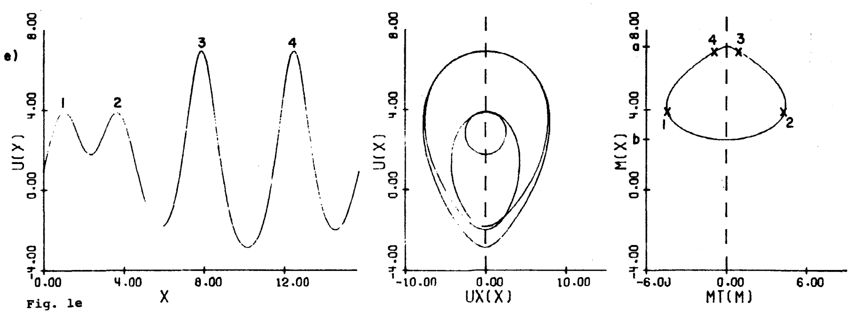

Fig. 1e

Figure 1. Time history of a particular $\tilde{u}(x)$, its corresponding phase diagram and the position of the maxima as $\tilde{u}(x)$ evolves under *KdV* flow. This particular $\tilde{u}(x)$ has spatial period $P = 15.7$ and temporal period $T = (1/4)T_0 = 1.11$. The constants in (3) and (4) are $a_{-1} = 0$, $a_0 = -8$, $a_1 = 2$, $c_1 = 56$, $c_2 = -54$. The solution is shown at times: (a) $t = 0$, (b) $t = (1/16)T_0$; (c) $t = (1/8)T_0$, (d) $t = (3/16)T_0$, (e) $t = (1/4)T_0$.

For the case $N = 1$ the minimizing function is uniquely defined up to translation and the *KdV* flow is translation at constant speed. In other words the special solutions constructed in this way are the classical traveling waves discovered by Korteweg-deVries [5]. In this paper we describe some further results for the case $N = 2$. In particular we investigate the stability of these solutions by numerical experimentation.

In the rest of this paper we shall deal exclusively with solutions of the *KdV* equation which minimize $F_2(u)$. For the sake of brevity we shall call these *special solutions.*

To construct a special solution we minimized $F_2(u)$ numerically, using a combination of the Fletcher-Powell-Davidon minimization algorithm [3] and numerical shooting of the Euler equation. A complete description of the numerical technique appears in [8].

Next we solve the *KdV* equation numerically using F. Tappert's split-step FFT method [10] with the minimizing function $\tilde{u}(x)$ as initial data. Figure 1 displays the time evolution of $\tilde{u}(x)$ under the *KdV* flow and its corresponding phase diagram.

As the figures show there is a time T, called the *recurrence time,* such that $u(x, T)$ is identical with $u(x, 0)$ shifted by the amount L; i.e.,

$$u(x, t) = u(x - L, 0).$$

This is rigorously proved in [8]. We now give a method of calculating the recurrence time. The method is based on this observation: At any time t the special solution $u(x, t)$, as a function of x, has a number of local maxima, for example four in Figure 1. Denote by $m(t)$ the value of u at such a local maximum; then as shown in [6] each $m(t)$ satisfies the same first order ordinary differential equation, of the form

$$m_t = \pm R^{1/2}(m), \tag{4}$$

where

$$\begin{aligned}
R(m) &= \alpha(m)\, P^{1/2}(m) - \beta(m),\\
P(m) &= \frac{5}{9}\left(\frac{1}{4}m^4 + \frac{1}{3}a_1 m^3 \right.\\
&\qquad \left. + \frac{1}{2}a_0 m^2 + a_{-1}m + c_1\right),\\
\alpha(m) &= \frac{5}{9}(m^3 + a_1 m^2 + a_0 m + a_{-1}),\\
\beta(m) &= \frac{2}{3}\, mP(m) + \frac{1}{9}m^5 + \frac{5}{9}a_1 P(m)\\
&\qquad + \frac{5}{36}a_1 m^4 + \frac{5}{27}a_0 m^3 + c_2.
\end{aligned}$$

The constants a_j also appear in (3) and the c_i are constants dependent upon the a_j. Furthermore as shown in [6] the value of u_{xx} at a local maximum of $u(x, t)$ is equal to $-P^{1/2}(m)$, thus completely determined as a function of m. Also, at a maximum, $u_x = 0$ and $u_t = m_t$ and the *KdV* equation reduces to the simple form

$$-u_{xxx} = u_t = m_t = \pm R^{1/2}(m).$$

Thus the first three derivatives of u at a maximum are completely determined by m and the sign of m_t.

We saw earlier that for each t the special solution $u(x, t)$ of the *KdV* equation under investigation satisfies the Euler equation (3). This is a fourth order ordinary differential equation and therefore knowledge of the value of u and its first three derivatives at any point x completely determine u. Since (3) does not contain x explicitly it follows that if u_0 and u_1 are two solutions of (3) such that the value of u_0 and its first three derivatives at a point x_0 equal to the value of u_1 and its first three derivatives at a point x_1 then

$$u_1(x) = u_0(x + x_0 - x_1).$$

Suppose $u(x, t)$ is a special solution of the KdV equation and T is a time such that a maximum m_0 of $u_0(x) = u(x, 0)$ at $x = x_0$ equals a maximum m_1 of $u_1(x) = u(x, T)$ at $x = x_1$. Also suppose that the third derivative of u_0 at x_0 and u_1 at x_1 have the same sign, which would imply that m_t at $t = 0$ and at $t = T$ have the same sign. Then, as we saw before, this implies that the value of u_0 and its first three derivatives at x_0 equal u_1 and its first three derivatives at x_1. This leads to

LEMMA 1. *Under the above condition*

$$u_1(x) = u_0(x + x_0 - x_1)$$

or

$$u(x, T) = u(x + x_0 - x_1, 0),$$

i.e., T is a recurrence time for the solution of the KdV.

Equation (4) can be integrated by quadrature. Every solution $m(t)$ is a *periodic function* in time with period

$$T_0 = 2 \int_a^b \frac{dm}{R^{1/2}(m)}, \tag{5}$$

where a and b are the zeros of $R(m)$ between which all the maxima of $\tilde{u}(x)$ lie. It follows from Lemma 1 that

LEMMA 2. *T_0 is a recurrence time.*

A periodic solution of (3) has at least one maximum, but may have many maxima. The special solutions we have calculated here have four, and in some cases five, local maxima. According to (4), the value of a local maximum, m, is restricted to an interval (a, b) in which $R(m)$ is nonnegative. Since $u_{xx} = -P^{1/2}(u)$ at a maximum if $P(m)$ is strictly positive over this interval, no local maximum can disappear, nor can a new one appear from nowhere. Thus in this case the number of local maxima remains constant.

THEOREM. *Let u be a special solution, and suppose that the total number M of local maxima is independent of t. Then there is a recurrence time $T \leqq T_0/M$, where T_0 is defined by* (5).

PROOF. During the time T_0 each of the M local maxima traverse the interval (a, b) completely, both up and down. Denote by t_0 the time when one of the local maxima reaches the point $m = b$, and denote by t_j the next M times when a local maximum reaches b, $j = 1, 2, \cdots, M$. It follows from Lemma 1 that all the times $t_j - t_{j-1}$, $j = 1, 2, \cdots, M$ are recurrence times. Since

$$\sum_1^M t_j - t_{j-1} = t_M - t_0 = T_0,$$

the smallest of these recurrence times is $\leqq T_0/M$. Also the quantities $t_{j+1} - t_j$ must be integer multiples of the smallest period. Our numerical calculations indicate that in all the cases we have considered the quantities $t_j - t_{j-1}$ are equal, and the smallest recurrence time is T_0/M. Thus at recurrence each peak assumes the role of another peak. Numerical evidence indicates that this peak is an adjacent one. This explains the billiard ball effect seen when viewing a movie of the solution or in the space-time plot of the solution's peaks in Figure 2.

Figure 1 also includes a graph of the values of m and m_t as the minimizing solution $\tilde{u}(x)$ evolves under KdV flow. The maxima move counterclockwise around the closed curve (m, m_t), $t \in [0, T]$, and at recurrence each maximum has assumed the position of the maximum previously to its left. This particular $\tilde{u}(x)$ has four peaks; therefore at $t = (1/4)T_0$ the solution has resumed its original shape in a shifted position. After four of these shape recurrences each peak has returned to its original position with respect to the other peaks and the cycle repeats.

Figure 2 traces the paths of the maxima as the solution shown in Figure 1 evolves under KdV. Figure 2 shows that the peaks move with two distinct speeds. In any spatial period three of the peaks are traveling with one speed while the fourth is traveling faster. The faster speed is indicated by a dashed line and is visible as a phase shift traversing the solution in a billiard ball type motion. This phase shift seems similar to the shift observed when two solitary waves with different speeds interact on the infinite line, see [4].

Figure 3 shows the time history of a solution where the initial value was obtained by superimposing a random disturbance on the function shown in Figure 1a. Note that the disturbance is not magnified during the evolution, and that the average of this solution is very close at all times to the undisturbed solution pictured in Figure 1. This calculation demonstrates convincingly the great stability of the KdV flow in the $N = 2$ case. The stability of the travelling wave solutions (the case $N = 1$) has been proved by T. B. Benjamin [1] and J. Bona [2].

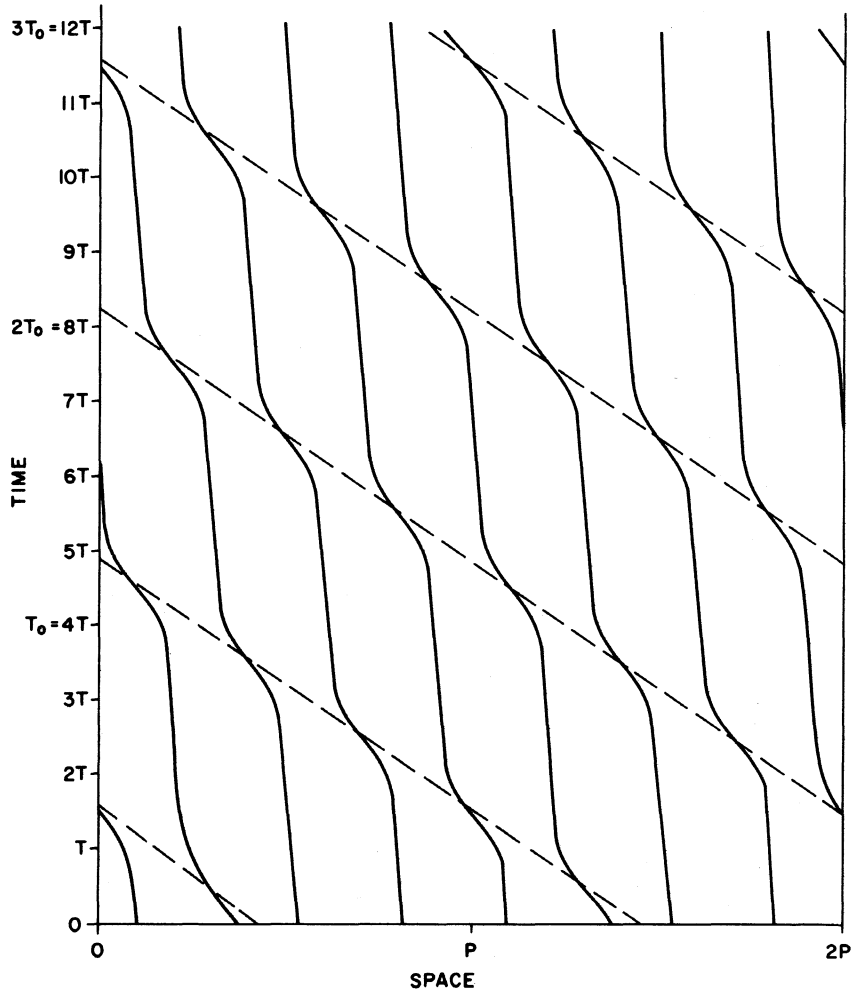

Figure 2. Trajectory of the maxima as the solution in Figure 1 evolves under KdV flow.

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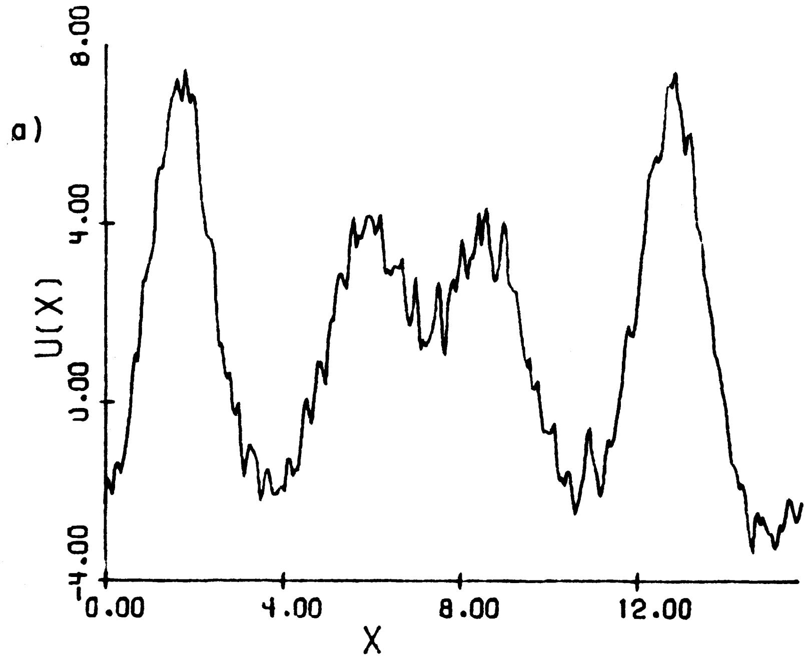
a)
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Fig. 3a

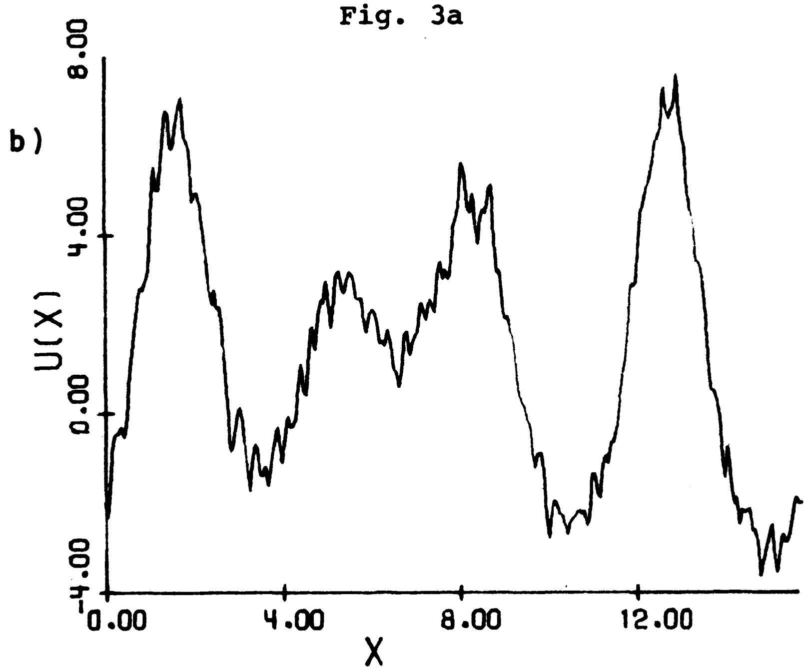
b)
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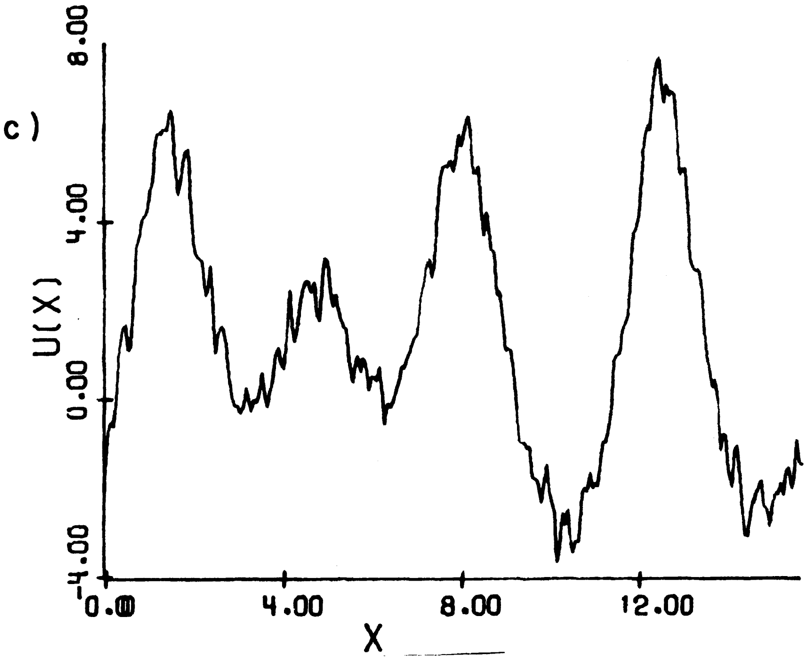
Fig. 3b


c)
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U(X)
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Fig. 3c

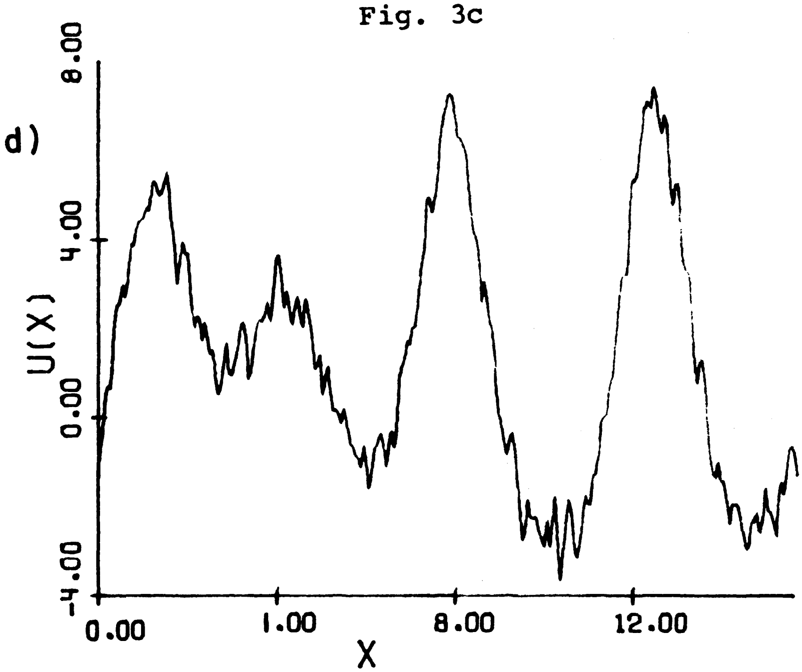
d)
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Fig. 3d

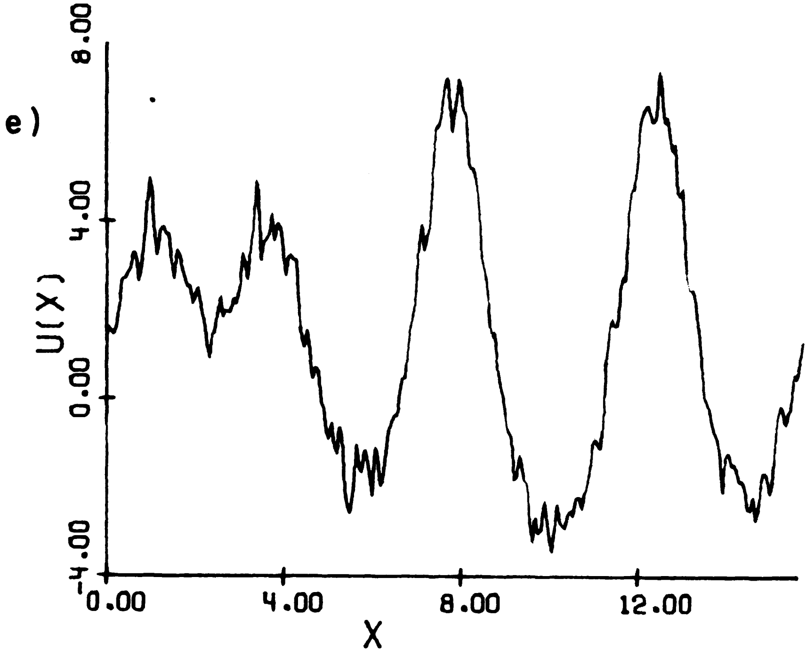

Figure 3. A random disturbance is superimposed on the solution shown in Figure 1. The mean of the perturbed solution is very close to the solution in Figure 1 as it evolves under KdV flow. The solution is shown at times: (a) $t = 0$, (b) $t = (1/16)T_0$, (c) $t = (1/8)T_0$, (d) $t = (3/16)T_0$, (e) $t = (1/4)T_0$.

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