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GOING-BETWEEN RINGS AND CONTRACTIONS OF SATURATED CHAINS OF PRIME IDEALS

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ABSTRACT. The concept of a going-between ring A (that is, if $A \subseteq B$ are rings such that B is integral over A, if $P \subset Q$ are prime ideals in B, and if there exists a prime ideal p' in A such that $P \cap A \subset p' \subset Q \cap A$, then there exists a prime ideal P' in B such that $P \subset P' \subset Q$) is introduced and a number of characterizations of such rings in terms of factor rings, quotient rings, and contractions of saturated chains of prime ideals are given. The relationship between such rings and catenary-like conditions on a ring is considered, and two additional characterizations of Noetherian going-between rings are given.

1. **Introduction.** All rings in this article are assumed to be commutative with non-zero identity element. The teminology is, in general, the same as that in [6].

This paper is concerned with two concepts which have recently been deeply investigated. First, somewhat analogous to GU-rings and GD-rings which have been studied in many papers, including [1, 4], we consider GB-rings (going-between rings (see (2.1))), characterize them in a number of ways, consider the special case of Noetherian GB-rings, and, finally, polynomial extensions of such rings are also considered. Second, the results are related to catenary-like conditions on a ring. This is due to the following result (3.8): A ring A satisfies the c.c. (3.7.4) if and only if A is catenary and a GB-ring. Now, rings which satisfy the c.c. have been investigated in many papers, including [5, 7, 8], and catenary rings were investigated in [9]. The results in this paper are thus in this line of reasearch, since they have to do with the other property (being a GB-ring) of rings which satisfy the c.c.

In § 2 we first define GB-rings (2.1), and then list four useful facts about when $A \subseteq B$ satisfy GB.

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§ 3 contains characterizations of GB-rings in terms of certain factor rings (3.3) and certain quotient rings (3.5). Next, it is shown that the Henselization of a quasi-local GB-ring is a GB-ring (3.6), and then a characterization of rings which satisfy the c.c. and the s.c.c. (see (3.7)) in terms of GB-rings is given in (3.8) and (3.10), respectively.

In § 4, it is shown that a Noetherian ring A is a GB-ring if and only if $A \subseteq B$ satisfy GB, for all principal integral extension rings B of A (4.1). Another characterization as a subclass of a class C of rings which has appeared in the literature is given in (4.3).

In (5.1), it is shown that if $A(X)$ is a Noetherian GB-ring, then A is, and, in (5.3), a related result concerning $A[X]$ is given.

§ 6 contains six questions on the subject of this paper which the author has been unable to answer.

I am indebted to the referee for several suggestions which improved the readability of this paper.

2. **GB-rings and SC-rings.** In this section, we define the concept of a GB-ring and list a few of their basic properties which will be needed in what follows.

Recall that for rings $A \subseteq B$ it is said that $A \subseteq B$ satisfy GD in case, whenever $p \subset q$ are prime ideals in A and Q is a prime ideal in B such that $Q \cap A = q$, then there exists a prime ideal $P \subset Q$ in B such that $P \cap A = p$. A is said to be a GD-ring in case $A \subseteq B$ satisfy GD, for all rings B which contain A. $A \subseteq B$ satisfy GU, and A is a GU-ring are defined analogously (using the Going Up Theorem). The following definition introduces related concepts.

DEFINITION 2.1. Let $A \subseteq B$ be rings. Then it will be said that $A \subseteq B$ *satisfy GB* (going between) in case, whenever $P \subset Q$ are prime ideals in B such that there exists a prime ideal p' in A such that $P \cap A \subset p' \subset Q \cap A$, then there exists a prime ideal P' in B such that $P \subset P' \subset Q$. A is said to be a *GB-ring* in case, for every integral extension ring B of A, $A \subseteq B$ satisfy GB.

Two comments are in order at this point. First, to be more in analogy to GU and GD, the condition in (2.1) should be: there exists a prime ideal P' in B such that $P \subset P' \subset Q$ and $P' \cap A = p'$. However, this is *very* strong (even complete regular local rings fail to satisfy it), so this weaker version was decided on. Second, again to be more in analogy to the literature on GU and GD, a ring A should be called a GB-ring only when, for every extension ring (instead of every *integral* extension ring) B of A, $A \subseteq B$ satisfy GB. However, our main interest in this paper is in the analogy between GB-rings and certain chain conditions for prime ideals (which concern integral extension rings), so the above definition was settled on.

If A is a ring which satisfies the c.c. (see (3.7.4) for the definition), then A is a GB-ring (as is readily seen from the definitions). On the other hand, it is an open problem whether the integral closure of a Noetherian domain is a GB-ring. However, in answer to a question of W. Krull [3, p. 755], I. Kaplansky showed in [2] that not every integrally closed domain is a GB-ring.

(2.2) gives four readily verified useful facts concerning when $A \subseteq B$ satisfy GB. (For (2.2.2), recall that a saturated chain of prime ideals in a ring B is a chain $P_1 \subset \cdots \subset P_n$ of prime ideals in B such that, for $i = 1, \cdots, n-1$, there does not exist a prime ideal Q in B such that $P_i \subset Q \subset P_{i+1}$; that is, height $P_{i+1}/P_i = 1$.)

Remark 2.2. Let $A \subseteq B$ be rings such that B is integral over A. Then the following statements hold:

(2.2.1) $A \subseteq B$ satisfy GB if and only if, for all prime ideals $P \subset Q$ in B such that height $Q/P = 1$, it holds that height $(Q \cap A)/(P \cap A) = 1$.

(2.2.2) $A \subseteq B$ satisfy GB if and only if every saturated chain of prime ideals in B contracts in A to a saturated chain of prime ideals, by (2.2.1).

(2.2.3) If $A \subseteq B$ satisfy GB and B is a domain, then, for all height one prime ideals Q in B, height $Q \cap A = 1$, by (2.2.1).

(2.2.4) Assume further that C is a ring such that $B \subseteq C$ and C is integral over A. Then $A \subseteq C$ satisfy GB if and only if $A \subseteq B$ and $B \subseteq C$ satisfy GB.

3. **GB-rings and catenary considerations.** The goal of this section is to show that GB-rings have many properties analogous to properties of rings which are either catenary or satisfy the c.c. (see (3.7) for the definitions). Namely, such rings are closed under passing to arbitrary integral extension rings, factor rings, and quotient rings. Also, in (3.8) and (3.10) we characterize rings which satisfy the c.c. and s.c.c., respectively, in terms of GB-rings.

We begin with the following result.

Proposition 3.1. *Let $A \subseteq B$ be rings such that B is integral over A. If A is a GB-ring, then B is.*

Proof. This follows immediately from (2.2.4).

Several times in what follows we need to use a result which is certainly known, but for which the author has no reference. For this reason, we state and sketch a proof of the following lemma.

LEMMA 3.2. *If J is an ideal in a ring A and B^* is an integral extension ring of $A^* = A/J$, then there exists an integral extension ring B of A which has an ideal J' such that $J' \cap A = J$ and $B/J' = B^*$.*

PROOF. Let B^* be generated over A^* by $\{b_i; i \in I\}$, and let K^* be the kernel of the natural homomorphism on $A^*[\{X_i\}_{i \in I}]$ onto B^*. Let K be the pre-image of K^* in $P = A[\{X_i\}_{i \in I}]$. Then $K \cap A = J$, K is the kernel of the natural homomorphism on P onto B^*, and, for each $i \in I$, there exists a monic polynomial $f_i(X_i) \in K$. Let K_1 be the ideal generated in P by $\{f_i(X_i); i \in I\}$. Then it readily follows that $B = P/K_1$ and $J' = K/K_1$ satisfy the required conditions.

The next result characterizes a GB-ring in terms of certain factor rings. Because of (2.2.1), to verify that a ring A is a GB-ring, it will normally be shown that, for every adjacent pair of prime ideals $P \subset Q$ in an arbitrary integral extension ring of A, it holds that height $(Q \cap A)/P \cap A) = 1$.

PROPOSITION 3.3. *The following statements are equivalent for a ring A:*

(3.3.1) *A is a GB-ring.*

(3.3.2) *A/I is a GB-ring, for all ideals I in A.*

(3.3.3) *A/p is a GB-ring, for all prime ideals p in A.*

(3.3.4) *A/z is a GB-ring, for all minimal prime ideals z in A.*

PROOF. That (3.3.1) $\Rightarrow$ (3.3.2) follows readily from (3.2) and (2.2.1), and it is clear that (3.3.2) $\Rightarrow$ (3.3.3) $\Rightarrow$ (3.3.4).

Assume that (3.3.4) holds, let p be a prime ideal in A, and let B^* be an integral extension domain of A/p. Let z be a minimal prime ideal in A such that $z \subseteq p$. Then there exists an integral extension domain B of A/z and a prime ideal p' in B such that $p' \cap (A/z) = p/z$ and $B/p = B^*$ (3.2). From this it easily follows that (3.3.4) $\Rightarrow$ (3.3.3).

Finally, assume that (3.3.3) holds and let B be an integral extension ring of A. Let $P \subset Q$ be prime ideals in B such that height $Q/P = 1$. Then B/P is an integral extension domain of $A/(P \cap A)$, so it readily follows (using (3.3.3)) that height $(Q \cap A)/(P \cap A) = 1$. Therefore A is a GB-ring (2.2.1).

REMARK 3.4. The following statements hold for a ring A:

(3.4.1) A is a GB-ring if and only if $A/(\mathrm{Rad}\, A)$ is a GB-ring.

(3.4.2) If $A = \oplus_1{}^n A_i$, then A is a GB-ring if and only if each A_i is a GB-ring.

(3.4.3) If $A[X]$ is a GB-ring, where X is an indeterminate, then A and every ring $A[x]$ is a GB-ring.

(3.4.4) If A is a complete semi-local ring, then A is a GB-ring.

PROOF. (3.4.1)–(3.4.3) follow immediately from (3.3), and (3.4.4) also follows from (3.3), since a complete local domain is a GB-ring (since it satisfies the s.c.c. [**6**, (34.4)]).

The next result characterizes a GB-ring in terms of certain quotient rings. Since its proof is quite similar to the proof given for (3.3), it will be omitted.

PROPOSITION 3.5. *The following statements are equivalent for a ring* A:

(3.5.1) *A is a GB-ring.*

(3.5.2) *A_S is a GB-ring, for all multiplicatively closed subsets* $S(0 \notin S)$ *in* A.

(3.5.3) *A_p is a GB-ring, for all prime ideals p in A.*

(3.5.4) *A_M is a GB-ring, for all maximal ideals M in A.*

If R is a local domain which is a GB-ring, then the completion of R is a GB-ring, since every complete local ring is (3.4.4). (3.6) shows, in particular, that a similar result holds concerning the Henselization of R.

PROPOSITION 3.6. *Let S be a quasi-local ring, and assume S is a GB-ring. Then the Henselization S^H of S is a GB-ring.*

PROOF. It follows from the definition of S^H [**6**, p. 180] that S^H is a quotient ring of a ring B which contains and is integrally dependent on S. Therefore B is a GB-ring (3.1), hence its quotient ring S^H is (3.5).

It will next be shown that rings which satisfy the c.c. or the s.c.c. can be characterized in terms of GB-rings. For this, we need the following definitions.

DEFINITION 3.7. Let A be a ring.

(3.7.1) A satisfies the *first chain condition for prime ideals* (*f.c.c.*) in case each maximal chain of prime ideals in A has length = altitude A.

(3.7.2) A is *catenary* in case, for each pair of prime ideals $p \subset q$ in A, $(A/p)_{q/p}$ satisfies the f.c.c.

(3.7.3) A satisfies the *second chain condition for prime ideals* (*s.c.c.*) in case, for each minimal prime ideal z in A, every integral extension domain B of A/z satisfies the f.c.c. and depth z = altitude A.

(3.7.4) A satisfies the *chain condition for prime ideals* (c.c.) in case, for each pair of prime ideals $p \subset q$ in A, $(A/p)_{q/p}$ satisfies the s.c.c.

Many properties of rings which satisfy one of the above conditions are known. A fairly useful summary of basic properties of such rings is given in [9, Remarks 2.22–2.25].

(3.8) characterizes a ring which satisfies the c.c. in terms of a GB-ring.

PROPOSITION 3.8. *A ring A satisfies the c.c. if and only if A is catenary and a GB-ring.*

PROOF. If A satisfies the c.c., then by definition A is catenary. To see that A is also a GB-ring, let B be an integral extension ring of A, let $P \subset Q$ be adjacent prime ideals in B, and let $p = P \cap A$ and $q = Q \cap A$. Then, by definition, $R = (A/p)_{q/p}$ satisfies the s.c.c. Also, $S = (B/P)_{((A/p)-(q/p))}$ is integral over R. Further, QS is a height one maximal ideal, so, by the s.c.c., height q/p = altitude $R = 1$. Therefore A is a GB-ring (2.2.1).

Conversely, it suffices to prove that if $p \subset q$ are prime ideals in A, then $R = (A/p)_{q/p}$ satisfies the s.c.c. For this, let S be an integral extension domain of R, and let $(0) \subset P_1 \subset \cdots \subset P_n$ be a maximal chain of prime ideals in S. Now R is a GB-ring, by (3.3) and (3.5), so $(0) \subset P_1 \cap R \subset \cdots \subset P_n \cap R$ is a saturated chain of prime ideals in R (2.2.2), hence it is a maximal chain of prime ideals in R (since $P_n \cap R$ is the maximal ideal in R). Therefore, since R is catenary (since A is), n = altitude R = altitude S. Therefore S satisfies the f.c.c., so R satisfies the s.c.c.

The following remark follows readily from (3.8).

REMARK 3.9. The following statements hold for a ring A:

(3.9.1) If A is a domain and altitude $A = 2$, then A is a GB-ring if and only if A satisfies the c.c.

(3.9.2) If altitude $A \leqq 1$, then A satisfies the c.c.

This section will be closed with the following characterization of a ring which satisfies the s.c.c.

PROPOSITION 3.10. *Let A be a ring such that altitude $A < \infty$. Then A satisfies the s.c.c. if and only if A satisfies the f.c.c. and is a GB-ring.*

PROOF. If A satisfies the s.c.c., then by definition A satisfies the f.c.c. and the c.c., so A is a GB-ring (3.8) and satisfies the f.c.c.

Conversely, let z be a minimal prime ideal in A, and let B be an integral extension domain of A/z. Then depth $z =$ altitude A (by the f.c.c.). Also, a maximal chain of prime ideals in B contracts in A/z to a maximal chain of prime ideals (as in the proof of (3.8)), so both chains have length = altitude A (by the f.c.c.). It follows that B satisfies the f.c.c., hence A satisfies the s.c.c.

4. **Noetherian GB-rings.** In this section we give two additional characterizations of a GB-ring A, when A is Noetherian.

It is known that a Noetherian domain A satisfies the s.c.c. if and only if every finite integral extension domain of A satisfies the f.c.c. if and only if every principal integral extension domain of A satisfies the f.c.c. [6, (34.3)] and [7, Theorem 3.11]. (4.1) and (4.2) show that the analogous results hold for a Noetherian GB-ring.

PROPOSITION 4.1. *Let A be a Noetherian ring. Then A is a GB-ring if and only if, for each principal integral extension ring $B = A[c]$ of A, $A \subseteq B$ satisfy GB.*

PROOF. Assume that $A \subseteq B$ satisfy GB, for each princiapl integral extension ring B of A, let C be an integral extension ring of A, and let $P \subset Q$ be prime ideals in C such that height $Q/P = 1$. Let $C^* = C/P$, $Q^* = Q/P$, and $A^* = A/(P \cap A)$. Then, to prove that A is a GB-ring, it clearly suffices to prove that height $Q^* \cap A^* = 1$ (2.2.1). Now, by (3.2), it is readily seen that A^* satisfies the hypothesis on A. Therefore it may be assumed to begin with that A and C are domains and Q is a height one prime ideal in C. Let A' and C' be the integral closure of A and C, respectively, and let Q' be a prime ideal in C' such that $Q' \cap C = Q$, so necessarily height $Q' = 1$. Also, height $Q' \cap A' = 1$ [6, (10.14)]. By [6, (33.10)], let $x \in q' = Q' \cap A'$ such that x is not in any other prime ideal in A' which lies over $Q \cap A$. Let $D = C[x]$, $B = A[x]$, and $Q'' = Q' \cap D$. Then height $Q'' = 1$ and $Q'' \cap B = q' \cap B$. Also, height $q' \cap B = 1$ (by the choice of x), and $Q \cap A = (q' \cap B) \cap A$, so height $Q \cap A =$ height $(q' \cap B) \cap A = 1$ (by hypothesis). Therefore A is a GB-ring.

The converse is clear.

REMARK 4.2. The following statements are equivalent for a Noetherian domain A: (i) A is a GB-ring; (ii) $A \subseteq B$ satisfy GB, for each finitely generated integral extension domain B of A; and, (iii) $A \subseteq B$ satisfy GB, for each principal integral extension domain $B = A[c]$ of A.

PROOF. It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), and the proof that (iii) $\Rightarrow$ (i) is essentially the same as that given for (4.1), q.e.d.

It follows immediately from (4.2) that an integrally closed Noetherian domain A is a GB-ring if and only if, for each *free* principal integral extension domain B of A, $A \subseteq B$ satisfy GB.

In [12, Section 4] the following class $\mathcal{C}$ of local domains was considered: A local domain R is in $\mathcal{C}$ if and only if whenever there exists a maximal chain of prime ideals of length n in an integral extension domain of R, then there exists a maximal chain of prime ideals of length n in R. In [12, (4.8.1)], it was pointed out that (in the terminology of this paper) if R is a GB-ring, then $R \in \mathcal{C}$. However, $\mathcal{C}$ is strictly larger than the class of local domains which are GB-rings, for, in [12, (4.1.7)], it was shown that if p is a prime ideal in $R \in \mathcal{C}$, then R/p need not be in $\mathcal{C}$. (But if p is a prime ideal in a GB-ring R, then R/p is a GB-ring (3.3).) (4.3) shows the relationship between $\mathcal{C}$ and the class of local domains which are GB-rings.

PROPOSITION 4.3. *Let $\mathcal{C}$ be as above, let $\mathfrak{D}$ be the set of local domains which are GB-rings, and let $\mathfrak{D}^* = \{R \in \mathcal{C}; (R/p)_{q/p} \in \mathcal{C}$, for all prime ideals $p \subset q$ in $R\}$. Then $\mathfrak{D} = \mathfrak{D}^*$.*

PROOF. If a local domain R is in $\mathfrak{D}$, then $R \in \mathcal{C}$ [12, (4.8.1)], so $R \in \mathfrak{D}^*$, by (3.3), (3.5), and [12, (4.8.1)].

For the opposite inclusion, let $R \in \mathfrak{D}^*$. Then it will be shown by induction on $a =$ altitude R that $R \in \mathfrak{D}$. If $a = 1$, then clearly $R \in \mathfrak{D}$. Therefore assume that $a > 1$ and that if $R^* \in \mathfrak{D}^*$ is such that altitude $R^* < a$, then $R^* \in \mathfrak{D}$.

Let B be a finite integral extension domain of R, let $P \subset Q$ be prime ideals in B such that height $Q/P = 1$, and let $p = P \cap R$ and $q = Q \cap R$. Then, by (4.2), it suffices to prove that height $q/p = 1$. For this, if $p \neq (0)$, then $R/p \in \mathfrak{D}^*$ (note that $\mathfrak{D}^*$ is closed under passing to factor domains and to local quotient domains) and altitude $R/p < a$, so $R/p \in \mathfrak{D}$ by induction. Therefore, since B/P is integral over R/p and Q/P is a height one prime ideal in B/P, height $((Q/P) \cap (R/p)) = 1$, so height $q/p = 1$. On the other hand, if $q \neq M$ (the maximal ideal in R), then $R_q \in \mathfrak{D}^*$ and altitude $R_q < a$, so $R_q \in \mathfrak{D}$. Therefore, since $S = B_{(R-q)}$ is integral over R_q and height $QS/PS = 1$, height $(QS \cap R_q)/(PS \cap R_q) = 1$, so height $q/p = 1$. Thus it may be assumed that $p = (0)$ and $q = M$. Then $P = (0)$, Q is a maximal ideal in B, and $(0) \subset Q$ is a maximal chain of prime ideals in B. Therefore there exists a maximal chain of prime ideals of length one in R (since $R \in \mathcal{C}$), hence altitude $R = 1$. Therefore height $q/p = 1$, so R is a GB-ring, hence $R \in \mathfrak{D}$.

An additional comment on $\mathcal{C}$ and $\mathfrak{D}$ will be made in § 6.

5. **Noetherian GB-rings and polynomial extensions.** It is known that a Noetherian ring A satisfies the c.c. if and only if $A(X)$ does if and only if $A[X]$ does. In this section we consider the analogous results for GB-rings.

For the first result in this section, recall that if A is a ring and X is an indeterminate, then $A(X)$ is the quotient ring $A[X]_S$ of $A[X]$, where S is the set of $f \in A[X]$ whose coefficients generate the unit ideal in A. (See [**6**, pp. 17–18].)

In [**11**, (2.15)], it is shown that a Noetherian ring A satisfies the c.c. if and only if $A(X)$ does. The author feels quite sure that the analogous result holds for Noetherian GB-rings. However, only the following partial result has been obtained.

PROPOSITION 5.1. *Let A be a Noetherian ring. If $A(X)$ is a GB-ring, then A is a GB-ring.*

PROOF. Assume that $A(X)$ is a GB-ring, let $z \subseteq N$ be prime ideals in A such that z is minimal and N is maximal, and let B be a finite integral extension domain of $R = (A/z)_{N/z}$. Then, by (3.3), (3.5), and (4.2), it suffices to prove that $R \subseteq B$ satisfy GB.

For this, let $P \subset Q$ be prime ideals in B such that height $Q/P = 1$, let $P^* = PB[X]$, $Q^* = QB[X]$, $p^* = P^* \cap R[X]$, and $q^* = Q^* \cap R[X]$. Now $P \cap R = p^* \cap R =$ (say) p and $Q \cap R = q^* \cap R =$ (say) q, and $1 = trd\ (B[X]/P^*)/(B/P) =$ (by integral dependence) $trd\ (R[X]/p^*)/(R/p)$. Therefore $p^* = pR[X]$ and, likewise, $q^* = qR[X]$. Thus $q^* \subseteq MR[X]$, where M is the maximal ideal in R, so $R[X]_{q^*} = R(X)_{q^*R(X)} =$ (say) C, hence C is a GB-ring (by (3.3) and (3.5), since $R(X) \cong (A(X)/zA(X))_{NA(X)/zA(X)})$. Let $D = B[X]_{(R[X]-q^*)}$. Then height $Q^*D/P^*D = 1$, since height $Q^*/P^* = 1$, so height $(Q^*D \cap C)/(P^*D \cap C) = 1$ (since C is a GB-ring), and so height $q^*/p^* = 1$. Thus height $(Q \cap R)/(P \cap R) =$ height $q/p =$ height $q^*/p^* = 1$. Therefore $R \subseteq B$ satisfy GB.

REMARK 5.2. If A is a local domain in (5.1), then, for all analytically independent elements b, c in A, $A(c/b)$ is a GB-ring.

PROOF. If b, c are analytically independent in A, then, by [**8**, Lemma 4.3], $MA[c/b]$ is a prime ideal and $A(c/b) = A[c/b]_{MA[c/b]} \cong A(X)/K$, for some height one prime ideal K such that $K \cap A = (0)$. Therefore, since $A(X)$ is a GB-ring, $A(c/b)$ is (3.3).

By (3.4.3), if $A[X]$ is a GB-ring, then A is. Due to the relationship between GB-rings and rings which satisfy the c.c. (3.8), it seems that the converse should also hold for Noetherian rings (since a Noetherian ring A satisfies the c.c. if and only if $A[X]$ does [8, Theorem 2.6]). However, only the following considerably weaker result has been obtained.

PROPOSITION 5.3. *Let A be a Noetherian ring such that $A(X)$ is a GB-ring. Then, for all rings $B[X]$ such that B is integral over A, $A[X] \subseteq B[X]$ satisfy GB.*

PROOF. Let B be an integral extension ring of A, and let $P' \subset Q'$ be prime ideals in $B[X]$ such that height $Q'/P' = 1$. Let $p' = P' \cap B$, $P = P' \cap A[X]$, $p = P \cap A$, $q' = Q' \cap B$, $Q = Q' \cap A[X]$, and $q = Q \cap A$. Then it must be shown that height $Q/P = 1$.

For this, it may be assumed that p and p' are zero, since the hypotheses continue to hold in $A[X]/pA[X] \subseteq B[X]/p'B[X]$ (since $(A/p)(X) \simeq A(X)/pA(X)$ is a GB-ring (3.3)). Therefore $h =$ height $P =$ height $P' \leqq 1$. If $h = 0$, then height $Q' = 1$, so either $q' = (0)$ or $Q' = q'B[X]$. If $q' = (0)$, then $q = (0)$, hence height $Q = 1$, and so height $Q/P = 1$. If $Q' = q'B[X]$, then (by integral dependence) $Q = qA[X]$. Also, since A is a GB-ring (5.1), height $q =$ height $q' = 1$ (2.2.3), so height $Q = 1$, and so height $Q/P = 1$. Therefore it may be assumed that height $P =$ height $P' = 1$.

It may also be assumed that A is local with maximal ideal q, since the hypotheses continue to hold in $A_q[X] \subseteq B_{(A-q)}[X]$ (since $A_q(X) \cong A(X)_{qA(X)}$ is a GB-ring (3.5)). Therefore $d =$ depth $Q =$ depth $Q' \leqq 1$. If $d = 1$, then $Q = qA[X]$, so $A[X]_Q = A(X)$, hence height $Q/P = 1$ (by hypothesis). Therefore it may be assumed that Q and Q' are maximal ideals.

Thus $(0) \subset P' \subset Q'$ is a maximal chain of prime ideals of length two in $B[X]$, so, by [**11**, (2.14)], there exists an integral extension domain, say C, of A which has a maximal chain of prime ideals of length one. That is, there exists a height one maximal ideal in C. Therefore, since A is a local domain and a GB-ring, altitude $A = 1$ (2.2.3). Thus height $Q = 2$, so height $Q/P = 1$.

6. **Some questions.** The author thinks that each of the following questions has an affirmative answer, but has been unable to show it.

(6.1.1) If $A \subseteq B$ are Noetherian rings which satisfy GB, then do $A(X) \subseteq B(X)$ satisfy GB?

(6.1.2) If $A(X)$ is a Noetherian GB-ring, then is $A(X, Y)$?

(6.1.3) If A is a Noetherian GB-ring, then is $A(X)$?

(6.1.4) If $A[X]$ is a Noetherian GB-ring, then is $A[X, Y]$?

(6.1.5) If A is a Noetherian GB-ring, then is $A[X]$?

(6.1.6) If A is a Henselian local domain, then is $A \in \mathfrak{D}$ of (4.3)?

It is clear that (6.1.5)$\Rightarrow$(6.1.4) and (6.1.3)$\Rightarrow$(6.1.2) and (6.1.1). Also, (6.1.5)$\Rightarrow$(6.1.3), by (3.5).

Finally, to answer (6.1.6) it suffices to show that if A is a Henselian local domain, then, for each prime ideal p in A, $A_p \in \mathcal{C}$ of (4.3). (This follows from (4.3), since every Henselian local domain is in $\mathcal{C}$ [**12**, (4.1.3)], and A/p is Henselian, if A is.) This problem is of some interest, since if some Henselian local domain is not in $\mathfrak{D}$, then the Chain Conjecture (that is, the integral closure of a local domain satisfies the c.c.) is false [**10**, (3.16) (4)].

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