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ON p'-AUTOMORPHISMS OF ABELIAN p-GROUPS

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All groups in this article are finite. Our notation is standard. In particular, let p denote an arbitrary prime integer, let P denote an arbitrary abelian p-group, let F denote the field $Z/(p)$ and let A be a p'-subgroup of Aut(P).

This basic situation is discussed, for example, in [1, § 5.2] and in [2, I, Aufgaben 68–69]. These references show that if P is A-indecomposable, then P is homocyclic. However, it is possible to extend this result to:

THEOREM. *P is A-indecomposable if and only if A acts irreducibly on $\Omega_1(P)$.*

Before describing a proof of this result, we present two applications.
First suppose that P is A-indecomposable. Thus P is homocyclic. Let $\exp(P) = p^n$. Then, for each integer i with $1 \leqq i < n$, the endomorphism $\bar{i}$ of P defined by $\bar{i} : x \rightarrow x^{p^i}$ lies in the center of the ring End(P) and $\bar{i}$ induces an A-isomorphism $\tilde{i}$ of $P/\Phi(P)$ onto $\Omega_{n-i}(P)/\Omega_{n-i-1}(P)$. Hence each of the elementary abelian p-groups $\Omega_j(P)/\Omega_{j-1}(P)$ for $1 \leqq j \leqq n$ is A-isomorphic to $\Omega_1(P)$. Thus [1, 3.2.2, 5.1.4 and 5.3.2] yield:

COROLLARY 1. *If P is A-indecomposable with* $\exp(P) = p^n$, *then:*
(a) *$\{\Omega_i(P) \mid 0 \leqq i \leqq n\}$ is the set of A-invariant subgroups of P;*
(b) *every A-invariant subquotient of P is A-indecomposable;*
(c) *A acts faithfully and irreducibly on $\Omega_1(P)$ and $Z(A)$ is cyclic; and*
(d) *all A-composition factors of P are A-isomorphic to $\Omega_1(P)$.*

Next let P be arbitrary and let $\{V_i \mid 1 \leqq i \leqq s\}$ be a set of representatives for the distinct isomorphism types of irreducible representations of $F[A]$ where A acts trivially on V_1 and let

$$(*) \qquad P = P_1 \times P_2 \times \cdots \times P_r$$

be a direct decomposition of P into A-indecomposable subgroups. For $1 \leqq i \leqq s$, let Q_i be the (direct) sum of all P_j such that $\Omega_1(P_j)$ is $F[A]$-isomorphic to V_i. Clearly $Q_1 \leqq C_P(A)$, $[Q_j, A] = Q_j$ if $j > 1$, $[P, A] = \prod_{j=2}^{s} Q_j$ and $C_P(A) \cap \prod_{j=2}^{s} Q_j = 1$. Hence:

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COROLLARY 2. *Under these conditions, the following hold*:
(a) $P = Q_1 \times Q_2 \times \cdots \times Q_s$ *is a direct decomposition of* P *where for each* i *with* $1 \leqq i \leqq s$, Q_i *is the join of all A-indecomposable A-invariant subgroups* R *of* P *such that* $\Omega_1(R)$ *is* $F[A]$-*isomorphic to* V_i;
(b) Q_i *is independent of the direct decomposition choice* (*) *of* P *into A-indecomposable subgroups*; *and*
(c) $C_P(A) = Q_1$ *and* $[P, A] = \prod_{j=2}^{s} Q_j$.

Note that we have obtained an alternate proof of [1, 5.2.3.].
Finally, we sketch a

PROOF OF THE THEOREM. Assume that $|P|$ is minimal subject to (i) P is A-indecomposable and (ii) $\Omega_1(P)$ is not $F[A]$-irreducible. Thus P is homocyclic and $\bar{P} = P/\Phi(P)$ is not $F[A]$-irreducible. Let $\exp(P) = p^n$ and $|P| = p^{nt}$. Then Maschke's theorem implies that $n > 1$ and that there are $v > 1$ proper A-invariant subgroups X_i with $\Phi(P) < X_i$ and $\bar{P} = \bar{X}_1 \times \cdots \times \bar{X}_v$ such that $\bar{X}_i = X_i/\Phi(P)$ is $F[A]$-irreducible for all $1 \leqq i \leqq v$. Then $X_i = Y_i \times Z_i$ with Y_i, Z_i invariant under A, $\exp(Z_i) < exp(Y_i) = p^n$ and with Y_i homocyclic. Since $Z_i \leqq \Phi(P)$, we have $P = \prod_{i=1}^{v} Y_i$. But $Y_i/\Phi(Y_i)$ is $F[A]$-isomorphic to $\bar{X}_i$ and an easy order calculation shows that $|P| = \prod_{i=1}^{v} |Y_i|$. Thus $P = Y_1 \times \cdots \times Y_v$, which is impossible and we are done.

The theorem of this note has been independently proved in §6 of [D. R. Taunt, *On A-groups*, Proc. Camb. Phil. Soc. **45** (1949), 24–42].

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