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SOME GEOMETRIC PROPERTIES OF LORENTZ SEQUENCE SPACES

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Let $1 \leqq p < \infty$. For any $a = (a_1, a_2, \cdots) \in c_0 \backslash \ell_1$, $1 = a_1 \geqq a_2 \geqq \cdots \geqq 0$, let

$$d(a, p) = \left\{ x = (\alpha_1, \alpha_2, \cdots) \in c_0 : \|x\| = \sup_{\sigma \in \pi} \left(\sum_{i=1}^{\infty} |\alpha_\sigma(\alpha_{\sigma(i)})|^p a_i \right)^{1/p} < \infty \right\}$$

where π is the set of all permutations of the natural numbers N. The Banach space $d(a, p)$ is called a Lorentz sequence space. The Lorentz sequence spaces in some sense are "weighted" ℓ_p-spaces. They possess some common properties with ℓ_p-spaces, but not always. For recent results on Lorentz sequence spaces, see [**1**, **2**, **3**, **4**, **5**].

It is known [**11**] that $d(a, p)$ is reflexive for every $a \in c_0 \backslash \ell_1$ when $1 < p < \infty$. However, in general, $d(a, p)$, $1 < p < \infty$, is not uniformly convex. In fact, it is known [**1**] that in $d(a, p)$, $1 < p < \infty$, uniform convexity, uniform convexifiability, and the condition $\inf_n s_{2n}/s_n > 1$ where $s_n = \sum_{i=1}^n a_i$, $n = 1, 2, \cdots$, are equivalent. In this paper, we show that if $1 < p < \infty$ then for every $a \in c_0 \backslash \ell_1$, $d(a, p)$ is locally uniformly convex.

A Banach space X is said to have the property (H) if X is strictly convex and for any sequence $\{x_n\}$ in X and x in X, $\lim_n \|x_n\| = \|x\|$ and $\{x_n\}$ converges weakly to x imply that $\lim_n \|x_n - x\| = 0$. The space X is said to have property (2R) if for any sequence $\{x_n\}$ in X such that $\|x_n\| = 1$, $n = 1, 2, \cdots$, if $\lim_{n,m} \|x_n + x_m\| = 2$ then $\{x_n\}$ is a Cauchy sequence in X. We show that every $d(a, p)$, $1 \leqq p < \infty$ has property (H) and if $1 < p < \infty$, then every $d(a, p)$ has property $(2R)$. Hence there exist Lorentz sequence spaces with property $(2R)$ but which are not uniformly convexifiable. It is known that Day's spaces [7] also possess these properties. We refer to [9, **10**] for the detailed study of properties (H) and $(2R)$.

A Banach space X is said to be locally uniformly smooth if for any x in X with $\|x\| = 1$ and for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x + y\| + \|x - y\| \leqq 2 + \epsilon \|y\|$ for all y with $\|y\| \leqq \delta$. In § 2, we show that for all $a \in c_0 \backslash \ell_1$ and $1 < p < \infty$, $d(a, p)$ is locally uniform-

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ly smooth. We also prove that if X is a Banach space such that X^* is a locally uniformly smooth space with property $(2R)$ then X is locally uniformly convex. Thus for all $a \in c_0 \backslash \ell_1$ and $1 < p < \infty$, $d(a, p)^*$ is also locally uniformly convex.

For the terminology on basis theory, we refer to Singer's book 13].

1. Throughout the rest of the paper, we shall let $\{e_n\}$ be the unit vector basis of $d(a, p)$. It is easy to see that $\{e_n\}$ is a symmetric basis of $d(a, p)$.

PROPOSITION 1. *Let $x = \sum_{i=1}^{\infty} \alpha_i e_i$ be an element in $d(a, p)$, $1 < p < \infty$. Then for any $n \in N$,*

$$\|x\|^p \leqq \left\| \sum_{i=1}^{n} \alpha_i e_i \right\|^p + \left\| \sum_{i=n+1}^{\infty} \alpha_i e_i \right\|^p.$$

PROOF. Let $\{\hat{\alpha}_i\}$ be the nonincreasing rearrangement of the nonzero terms of $\{|\alpha_i|\}$. Let $\{\beta_i\}$ (respectively, $\{\gamma_i\}$) be a nonincreasing rearrangement of $\{|\alpha_i|\}_{i=1}^{n}$ (respectively, $\{|\alpha_i|\}_{i=n+1}^{\infty}$). Then for some $I = \{k_1, k_2, \cdots, k_n\}$ and $J = N \backslash I$ we have:

$$\begin{aligned} \|x\|^p &= \sum_{i=1}^{\infty} \alpha_i^p a_i \\ &= \sum_{i=1}^{m} \beta_i^p a_{k_i} + \sum_{i \in J} \gamma_i^p a_i \\ &\leqq \sum_{i=1}^{m} \beta_i^p a_i + \sum_{i=1}^{\infty} \gamma_i^p a_i \\ &= \left\| \sum_{i=1}^{n} \alpha_i e_i \right\|^p + \left\| \sum_{i=n+1}^{\infty} \alpha_i e_i \right\|^p \end{aligned}$$

THEOREM 2. *Every $d(a, p)$, $1 < p < \infty$ has property (H).*

PROOF. It is easy to see [e.g., Remark 1; 1] that all $d(a, p)$ are strictly convex.

Let $x = \sum_{i=1}^{\infty} \alpha_i e_i$ and $x_n = \sum_{i=1}^{\infty} \alpha_i^{(n)} e_i$, $n = 1, 2, \cdots$ be such that $\lim_n \|x_n\| = \|x\|$ and $\{x_n\}$ converges weakly to x. Since $\{e_n\}$ is symmetric, without loss of generality, we may assume that $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq 0$ and $\|x\| = 1$.

Given $\epsilon > 0$, choose $k_1 \in N$ such that $\|\sum_{i=k_1+1}^{\infty} \alpha_i e_i\| < \epsilon$. Since $\lim_n \|x_n\| = 1$ and $\lim_n \alpha_i^{(n)} = \alpha_i$, $i = 1, 2, \cdots$, there exists $k \geqq k_1$ such that for all $n \geqq k$,

(1) $$\big|\, \|x_n\|^p - 1 \big| < \epsilon,$$

(2) $$\left\| \sum_{i=1}^{k_1} (\alpha_i - \alpha_i^{(n)}) e_i \right\| < \epsilon,$$

(3) $$\left| \sum_{i=1}^{k_1} (\alpha_i^p - |\alpha_i^{(n)}|^p) a_i \right| < \epsilon,$$

and

(4) $$\left| 1 - \sum_{i=1}^{k} \alpha_i^p a_i \right| < \epsilon.$$

By Proposition 1, for all $n \geqq k$, we have

$$0 \leqq \|x_n\|^p - \left\| \sum_{i=k_1+1}^{\infty} \alpha_i^{(n)} e_i \right\|^p - \sum_{i=1}^{k_1} |\alpha_i^{(n)}|^p a_i$$

$$\leqq \left\| \sum_{i=1}^{k_1} \alpha_i^{(n)} e_i \right\|^p - \sum_{i=1}^{k_1} |\alpha_i^{(n)}|^p a_i$$

$$\leqq \left| \left\| \sum_{i=1}^{k_1} \alpha_i^{(n)} e_i \right\|^p - \left\| \sum_{i=1}^{k_1} \alpha_i e_i \right\|^p \right|$$

$$+ \left| \sum_{i=1}^{k_1} \alpha_i^p a_i - \sum_{i=1}^{k_1} |\alpha_i^{(n)}|^p a_i \right| < 2\epsilon.$$

Hence for all $n \geqq k$,

$$\left\| \sum_{i=k_1+1}^{\infty} \alpha_i^{(n)} e_i \right\|^p \leqq \left| \left\| \sum_{i=k_1+1}^{\infty} \alpha_i^{(n)} e_i \right\|^p \right.$$

$$\left. + \sum_{i=1}^{k_1} |\alpha_i^{(n)}|^p a_i - \|x_n\|^p \right| + \big|\, \|x_n\|^p - 1 \big|$$

$$+ \left| 1 - \sum_{i=1}^{k_1} \alpha_i^p a_i \right|$$

$$+ \left| \sum_{i=1}^{k_1} (\alpha_i^p - |\alpha_i^{(n)}|^p) a_i \right|$$

$$< 5\epsilon.$$

Thus if $n \geqq k$, then

$$\|x_n - x\| \leqq \left\| \sum_{i=1}^{k_1} (\alpha_i^{(n)} - \alpha_i)e_i \right\| + \left\| \sum_{i=k_1+1}^{\infty} \alpha_i^{(n)}e_i \right\| + \left\| \sum_{i=k_1+1}^{\infty} \alpha_i e_i \right\| < \epsilon + (5\epsilon)^{1/p} + \epsilon.$$

This completes the proof that $\lim_n \|x_n - x\| = 0$.

Remark. Let $\{e_n\}$ be the unit vector basis of $d(a, 1)$ and let $\{f_n\}$ be the sequence of coefficient functionals of $\{e_n\}$. Then $\{f_n\}$ is a shrinking basic sequence [cf. 3] and hence $d(a, 1)$ can be identified as the dual space of the closed linear subspace $[f_n]$ spanned by $\{f_n\}$. By the same argument used in Theorem 2, it can be proved that every $d(a, 1)$ possesses property (H^*). That is, for any elements x and x_n, $n = 1, 2, \cdots$ in $d(a, 1)$, if $\lim_n \|x_n\| = \|x\|$ and $\lim_n f(x_n) = f(x)$ for all $f \in [f_n]$, then $\lim_n \|x_n - x\| = 0$.

A Banach space X is called point locally uniformly convex [e.g., **6**] if for any sequence $\{x_n\}$ and any element x in X such that $\lim_n \|x_n\| = 1$, $\|x\| = 1$, and such that $\{x_n\}$ does not have a weak cluster point of norm strictly less than one, then $\lim_n \|x + x_n\| = 2$ implies that $\lim_n \|x_n - x\| = 0$. Since for $1 < p < \infty$, every $d(a, p)$ is strictly convex [e.g., **1**] and reflexive, by a result of Fan and Glicksberg [Theorem 3; **10**], we have the following result.

Corollary 3. *For $1 < p < \infty$, all $d(a, p)$ are point locally uniformly convex.*

Theorem 4. *Let $x = \sum_{i=1}^{\infty} \alpha_i e_i$ and $x_n = \sum_{i=1}^{\infty} \alpha_i^{(n)} e_i$, $n = 1, 2, \cdots$ be elements in $d(a, p)$, $1 < p < \infty$ such that $\|x_n\| \leqq 1$, $n = 1, 2, \cdots$ and $\lim_{n,m} \|x_n + x_m\| = 2$. If x is a weak limit point of $\{x_n\}$ then $\|x\| = 1$.*

Proof. It is clear that $\|x\| \leqq 1$. Suppose that $\|x\| < 1$.

(Case 1): $x = 0$. Choose $\epsilon > 0$ such that $2^{1/p}(1 + \epsilon) < 2$. For any $K \in N$, let $n \geqq K$ and choose k such that $\| \sum_{i=k+1}^{\infty} \alpha_i^{(n)} e_i \| < \epsilon$. Since $\{x_n\}$ converges weakly to 0, choose $m \geqq K$ such that $\| \sum_{i=1}^{k} \alpha_i^{(m)} e_i \| < \epsilon$. Then, by Proposition 1,

$$
\begin{aligned}
\|x_n + x_m\|^p &\leqq \left\| \sum_{i=1}^{k} (\alpha_i^{(n)} + \alpha_i^{(m)})e_i \right\|^p \\
&\quad + \left\| \sum_{i=k+1}^{\infty} (\alpha_i^{(n)} + \alpha_i^{(m)})e_i \right\|^p \\
&\leqq \left(\left\| \sum_{i=1}^{k} \alpha_i^{(n)}e_i \right\| + \left\| \sum_{i=1}^{k} \alpha_i^{(m)}e_i \right\| \right)^p \\
&\quad + \left(\left\| \sum_{i=k+1}^{\infty} \alpha_i^{(n)}e_i \right\| + \left\| \sum_{i=k+1}^{\infty} \alpha_i^{(m)}e_i \right\| \right)^p \\
&< 2(1+\epsilon)^p.
\end{aligned}
$$

Thus for any K, there exist $n, m \geqq K$ such that $\|x_n + x_m\| < 2^{1/p}(1+\epsilon) < 2$, which is a contradiction.

(Case 2): $0 < \|x\| < 1$. Let $\epsilon = 1 - \|x\| > 0$. We may assume that $|\alpha_1| \geqq |\alpha_2| \geqq \cdots$. For any given $\delta > 0$ with $1 - p\delta < (1-\delta)^p$ and for any $K \in N$, choose $n, m \geqq K$ such that

$$\|x_n + x_m\| \geqq 2 - \delta. \tag{1}$$

Choose $k_1 \in N$ such that

$$\left\| \sum_{i=k_1+1}^{\infty} \alpha_i e_i \right\|^p < \delta. \tag{2}$$

Since $\{x_n\}$ converges weakly to x, by choosing n sufficiently large, we may in addition assume that

$$\left| \sum_{i=1}^{k_1} (|\alpha_i|^p - |\alpha_i^{(n)}|^p)a_i \right| < \delta. \tag{3}$$

Next, choose $k_2 > k_1$ such that

$$\left\| \sum_{i=k_2+1}^{\infty} \alpha_i^{(n)}e_i \right\|^p < \delta. \tag{4}$$

Finally, let $m \geqq K$ satisfy (1) and

$$\left| \sum_{i=1}^{k_2} (|\alpha_i|^p - |\alpha_i^{(m)}|^p)a_i \right| < \delta. \tag{5}$$

Choose $\sigma \in \pi$ such that

$$\|x_n + x_m\| = \left(\sum_{i=1}^{\infty} |\alpha_i^{(n)} + \alpha_i^{(m)}|^p a_{\sigma(i)} \right)^{1/p}$$

$$\leqq \left(\sum_{i=1}^{\infty} |\alpha_i^{(n)}|^p a_{\sigma(i)} \right)^{1/p} + \left(\sum_{i=1}^{\infty} |\alpha_i^{(m)}|^p a_{\sigma(i)} \right)^{1/p}$$

$$\leqq \|x_n\| + \|x_m\| \leqq 2.$$

By (1),

$$(1 - \delta)^p \leqq \sum_{i=1}^{\infty} |\alpha_i^{(j)}|^p a_{\sigma(i)} \leqq 1, \quad j = n, m. \tag{6}$$

Also, by (3) and (5), we have $\sum_{i=1}^{k_1} (|\alpha_i^{(n)}|^p - |\alpha_i^{(m)}|^p) a_{\sigma(i)} < 2\delta$. Hence by (4) and (6),

$$\left| \sum_{i=k_1+1}^{k_2} |\alpha_i^{(n)}|^p a_{\sigma(i)} - \sum_{i=k_2+1}^{\infty} |\alpha_i^{(m)}|^p a_{\sigma(i)} \right|$$

$$\leqq \left| \sum_{i=1}^{\infty} (|\alpha_i^{(n)}|^p - |\alpha_i^{(m)}|^p) a_{\sigma(i)} \right|$$

$$+ \left| \sum_{i=1}^{k_1} (|\alpha_i^{(n)}|^p - |\alpha_i^{(m)}|^p) a_{\sigma(i)} \right|$$

$$+ \sum_{i=k_2+1}^{\infty} |\alpha_i^{(n)}|^p a_{\sigma(i)} + \sum_{i=k_1+1}^{k_2} |\alpha_i^{(m)}|^p a_{\sigma(i)}$$

$$\leqq [1 - (1 - \delta)^p] + 2\delta + \delta + \left| \sum_{i=k_1+1}^{k_2} (|\alpha_i^{(m)}|^p - |\alpha_i|^p) a_{\sigma(i)} \right|$$

$$+ \sum_{i=k_1+1}^{k_2} |\alpha_i|^p a_{\sigma(i)}$$

$$\leqq p\delta + 3\delta + \delta + \delta = (5 + p)\delta. \tag{7}$$

Now

$$\|x_n + x_m\|^p \leqq \Bigg[\Big(\sum_{i=1}^{k_1} |\alpha_i^{(n)}|^p a_{\sigma(i)} \Big)^{1/p}$$

$$+ \Big(\sum_{i=1}^{k_1} |\alpha_i^{(m)}|^p a_{\sigma(i)} \Big)^{1/p} \Bigg]^p$$

$$+ \Bigg[\Big(\sum_{i=k_1+1}^{k_2} |\alpha_i^{(n)}|^p a_{\sigma(i)} \Big)^{1/p}$$

$$+ \Big(\sum_{i=k_1+1}^{k_2} |\alpha_i^{(m)}|^p a_{\sigma(i)} \Big)^{1/p} \Bigg]^p$$

$$+ \Bigg[\Big(\sum_{i=k_2+1}^{\infty} |\alpha_i^{(n)}|^p a_{\sigma(i)} \Big)^{1/p}$$

$$+ \Big(\sum_{i=k_2+1}^{\infty} |\alpha_i^{(m)}|^p a_{\sigma(i)} \Big)^{1/p} \Bigg]^p .$$

Let

$$\tau = \sum_{i=k_1+1}^{k_2} |\alpha_i^{(n)}|^p a_{\sigma(i)}. \tag{8}$$

Then

$$\sum_{i=1}^{k_1} |\alpha_i^{(n)}|^p a_{\sigma(i)} = \sum_{i=1}^{k_2} |\alpha_i^{(n)}|^p a_{\sigma(i)}$$

$$- \sum_{i=k_1+1}^{k_2} |\alpha_i^{(n)}|^p a_{\sigma(i)} \leqq 1 - \tau.$$

Hence, by (3) and (5),

$$\sum_{i=1}^{k_1} |\alpha_i^{(m)}|^p a_{\sigma(i)} \leqq \sum_{i=1}^{k_1} |\alpha_i^{(n)}|^p a_{\sigma(i)} + \left| \sum_{i=1}^{k_1} (|\alpha_i^{(n)}|^p - |\alpha_i^{(m)}|^p) a_{\sigma(i)} \right| \leqq (1-\tau) + 2\delta.$$

By (2) and (5), we have

$$\sum_{i=k_1+1}^{k_2} |\alpha_i^{(m)}|^p a_{\sigma(i)} \leqq \left| \sum_{i=k_1+1}^{k_2} (|\alpha_i^{(m)}|^p - |\alpha_i|^p) a_{\sigma(i)} \right| + \sum_{i=k_1+1}^{k_2} |\alpha_i|^p a_{\sigma(i)} < 2\delta.$$

From (4),

$$\sum_{i=k_2+1}^{\infty} |\alpha_i^{(n)}|^p a_{\sigma(i)} \leqq \delta^{1/p}$$

and from (7),

$$\sum_{i=k_2+1}^{\infty} |\alpha_i^{(m)}|^p a_{\sigma(i)} \leqq \tau + (5+p)\delta.$$

Hence,

$$\begin{aligned} \|x_n + x_m\| &\leqq [(1-\tau)^p + (1-\tau+2\delta)^p]^p \\ &\quad + [\tau^{1/p} + (2\delta)^{1/p}]^p + [\delta^{1/p} + (\tau + 5\delta + p\delta)^{1/p}]^p \\ &\leqq 2^p(1-\tau+2\delta) + [\tau^{1/p} \\ &\quad + (2\delta)^{1/p}]^p + [\delta^{1/p} + (\tau+5\delta+p\delta)^{1/p}]^p. \end{aligned} \tag{8}$$

Now, by (5),

$$\sum_{i=1}^{k_1} |\alpha_i^{(n)}|^p a_{\sigma(i)} = \left| \sum_{i=1}^{k_1} (|\alpha_i^{(n)}|^p - |\alpha_i|^p) a_{\sigma(i)} \right| + \sum_{i=1}^{k_1} |\alpha_i|^p a_{\sigma(i)} < \delta + (1-\epsilon).$$

Hence, from (6) and (4), we have

$$\tau = \sum_{i=1}^{\infty} |\alpha_i^{(n)}|^p a_{\sigma(i)} - \sum_{i=1}^{k_1} |\alpha_i^{(n)}|^p a_{\sigma(i)} - \sum_{i=k_2+1}^{\infty} |\alpha_i^{(n)}|^p a_{\sigma(i)} \tag{10}$$
$$\geqq (1-\delta)^p - (1+\delta-\epsilon) - \delta.$$

On the other hand.

$$\tau \leqq \sum_{i=1}^{\infty} |\alpha_i^{(n)}|^p a_{\sigma(i)} - \sum_{i=1}^{k_1} |\alpha_i^{(n)}|^p a_{\sigma(n)} \leqq 1 - |\alpha_1^{(n)}|^p a_{\sigma(1)}. \tag{11}$$

By (3), if $\delta \leqq |\alpha_1|/2$, then $|\alpha_1^{(n)}| \geqq |\alpha_1| - \delta \geqq |\alpha_1|/2$. Choose $j \in N$ such that $|\alpha_1|^p s_j > 2^p$ where $s_j = \sum_{i=1}^{j} a_i$. Note that $|\alpha_1^{(n)} + \alpha_1^{(m)}| \geqq 2|\alpha_1| - |\alpha_1^{(n)} - \alpha_1| - |\alpha_1^{(m)} - \alpha_1| \geqq 2|\alpha_1| - 2\delta \geqq |\alpha_1|$. Hence if $\sigma(1) \geqq j$ then $2^p \geqq \|x_n + x_m\|^p \geqq \sum_{i=1}^{j} |\alpha_1^{(n)} + \alpha_1^{(m)}|^p a_i = |\alpha_1^{(n)} + \alpha_1^{(m)}|^p s_j \geqq |\alpha_1|^p s_j > 2^p$, which is a contradiction. Thus $a_{\sigma(1)} \geqq a_j$ and by (10) and (11), we conclude that

$$(1-\delta)^p - 1 - 2\delta + \epsilon < \tau < 1 - |\alpha_1^{(n)}|^p a_{\sigma(1)}$$
$$< 1 - \left(\frac{|\alpha_1|}{2}\right)^p a_j.$$

Since $\lim_{\delta\to 0}[(1-\delta)^p - 1 - 2\delta + \epsilon] = \epsilon$, we have proved that there exist $\epsilon_0 > 0$ and constants a,b with $0 < a < b < 1$ such that if $0 < \delta < \epsilon_0$ then for any τ defined as in (8) we have $a < \tau < b$. We may also choose δ sufficiently small so that $a < \tau - (5+p)\delta < r + (5+p)\delta < b$ and let $c = \max(a, 1-b)$. The from (1) and (9), we get

$$(2-\delta)^p \leqq \|x_n + x_m\|^p \leqq 2^p(1-c)$$
$$+ [c^{1/p} + (2\delta)^{1/p}]^p + [\delta^{1/p} + c^{1/p}]^p. \tag{12}$$

That is, for any $\delta > 0$ sufficiently small and for any integer K there exist $n, m \geqq K$ such that (12) holds. But

$$\lim_{\delta\to 0} \{2^p(1-c) + [c^{1/p} + (2\delta)^{1/p}]^p$$
$$+ [\delta^{1/p} + c^{1/p}]^p\} = 2^p(1-c) + c + c < 2^p$$

and $\lim_{\delta\to 0}(2-\delta)^p = 2^p$, which is a contradiction.

From Corollary 3 and Theorem 4, we get immediately,

Corollary 5. *For $1 < p < \infty$, all $d(a,p)$ are locally uniformly convex. That is, for each $0 < \epsilon \leqq 2$ and $\|x\| = 1$ in $d(a,p)$, there exists $\delta > 0$ such that $\|x+y\| \leqq 2 - \delta$ for all $\|y\| \leqq 1$ with $\|x - y\| \geqq \epsilon$.*

THEOREM 6. *If* $1 < p < \infty$, *then every* $d(a, p)$ *possesses property* (2*R*). *That is, for any sequence* $\{x_n\}$ *in* $d(a, p)$ *with* $\|x_n\| = 1$, $n = 1, 2, \cdots$, *if* $\lim_{n,m}\|x_n + x_m\| = 2$ *then* $\{x_n\}$ *is a Cauchy sequence in* $d(a, p)$.

PROOF. Since $d(a, p)$ is reflexive when $1 < p < \infty$, there exist $x \in d(a, p)$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to x. By Theorem 10, $\|x\| = 1$. Since $d(a, p)$ has property (H), we conclude that $\lim_i\|x_{n_i} - x\| = 0$.

Suppose $\{x_n\}$ does not converge to x in norm. Then there exists an $\epsilon > 0$ and a subsequence $\{x_{k_j}\}$ of $\{x_n\}$ such that $\|x_{k_j} - x\| \geqq \epsilon$, $j = 1, 2, \cdots$. Since $d(a, p)$ is locally uniformly convex, there exists a $\delta > 0$ such that $\|x + x_{k_j}\| < 2 - \delta$, $j = 1, 2, \cdots$. Since $\lim_i\|x_{n_i} - x\| = 0$ and $\lim_{n,m}\|x_n + x_m\| = 2$, choose n_i, k_j such that $\|x - x_{n_i}\| < \delta/2$ and $\|x_{n_i} + x_{k_j}\| \geqq 2 - \delta/2$. Then

$$2 - \delta > \|x + x_{k_j}\| \geqq \|x_{k_j} + x_{n_i}\| - \|x_{n_i} - x\|$$

$$\geqq 2 - \frac{\delta}{2} - \frac{\delta}{2} = 2 - \delta,$$

which is a contradiction.

2. DEFINITION. A Banach space X is said to be locally uniformly smooth if for any element x in X with $\|x\| = 1$ and for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x + y\| + \|x - y\| \leqq 2 + \epsilon\|y\|$ for all y with $\|y\| \leqq \delta$.

It is clear that every uniformly smooth space is locally uniformly smooth.

THEOREM 7. *If* X *is a Banach space and if* X^* *is a locally uniformly smooth space with property* (2*R*) *then* X *is locally uniformly convex.*

PROOF. Assume not. Then there exists an $\epsilon > 0$ and elements $\|x\| = \|x_n\| = 1$, $n = 1, 2, \cdots$ in X such that $\|x - x_n\| \geqq \epsilon$, $n = 1, 2, \cdots$ and $\lim_n\|x + x_n\| = 2$. Let f_n, g_n be elements in X^* such that $\|f_n\| = \|g_n\| = 1$, $\|x + x_n\| = f_n(x + x_n)$ and $\|x - x_n\| = g_n(x - x_n)$, $n = 1, 2, \cdots$. Then $2 = \lim_n\|x + x_n\| = \lim_n f_n(x + x_n)$ and so $1 = \lim_n f_n(x) = \lim_n f_n(x_n)$. Now since $\lim_{n,m}\|f_n + f_m\| \geqq \lim_{n,m}|f_n(x) + f_m(x)| = 2$ and X^* possesses property (2*R*), there exists f in X^* such that $\lim_n\|f_n - f\| = 0$. Hence $\lim_n f(x_n) = \lim_n f_n(x_n) = 1$ and $f(x) = \lim_n f_n(x) = 1$. By switching to a subsequence if necessary, we may assume that $f(x_n) \geqq 1 - \epsilon/2n$, $n = 1, 2, \cdots$. Now for $n = 1, 2, \cdots$,

$$\left\| f+\frac{1}{n}g_n \right\| + \left\| f-\frac{1}{n}g_n \right\|$$
$$\geqq \left(f+\frac{1}{n}g_n\right)(x) + \left(f-\frac{1}{n}g_n\right)(x_n)$$
$$= f(x+x_n) + \frac{1}{n}g_n(x-x_n)$$
$$\geqq 1+\left(1-\frac{\epsilon}{2n}\right)+\frac{\epsilon}{n} = 2+\frac{\epsilon}{2}\left\|\frac{g_n}{n}\right\|.$$

This shows that X^* is not locally uniformly smooth, which is a contradiction.

We now prove three technical lemmas for our next main result.

LEMMA 8. *For any given* $1 < p < \infty$, $a > 0$ *and* $\epsilon > 0$, *there exist positive real numbers* b *and* c *such that* $[\alpha + (\beta+\gamma)^p]^{1/p} \leqq (\alpha + \beta^p)^{1/p} + \epsilon\gamma$ *for all* $\alpha \geqq a$, $b \geqq \beta \geqq 0$ *and* $c \geqq \gamma \geqq 0$.

PROOF. Since $\lim_{\beta,\gamma\to 0} (\beta+\gamma)^{p-1}/[a+(\beta+\gamma)^p]^{1-1/p} = 0$, there exist $b > 0$, $c > 0$ such that $(\beta+\gamma)^{p-1} \leqq \epsilon/2[a+(\beta+\gamma)^p]^{1-1/p}$ for all β and γ with $b \geqq \beta \geqq 0$, $c \geqq \gamma \geqq 0$. Fix α and β where $\alpha \geqq a$ and $b \geqq \beta \geqq 0$ and define for each γ with $c \geqq \gamma \geqq 0$,

$$f(\gamma) = (\alpha+\beta^p)^{1/p} + \epsilon\gamma - [\alpha+(\beta+\gamma)^p]^{1/p}.$$

Then $f(0) = 0$ and for all γ with $c \geqq \gamma \geqq 0$, $f'(\gamma) = \epsilon - (\beta+\gamma)^{p-1}/[\alpha+(\beta+\gamma)^p]^{1-1/p} \geqq \epsilon/2 > 0$. Hence $f(\gamma) \geqq 0$.

LEMMA 9. *Let* $x = \sum_{n=1}^{\infty} \alpha_n e_n$ *be an element in* $d(a,p)$, $1 \leqq p < \infty$ *such that* $|\alpha_1| \geqq |\alpha_2| \geqq \cdots$ *and let* $k \in N$ *such that* $|\alpha_k| > |\alpha_{k+1}|$. *Then for any* $0 < \delta < (|\alpha_k| - |\alpha_{k+1}|)/3$ *and for any element* $y = \sum_{i=1}^{\infty} \beta_n e_n$ *in* $d(a,p)$ *such that* $\|y\| \leqq \delta$ *there exists* $\sigma \in \pi$ *such that* $\|x+y\|^p = \sum_{n=1}^{\infty} |\alpha_n+\beta_n|^p a_{\sigma(n)}$, $\sigma(I_k) = I_k$ *and* $\sigma(J_k) = J_k$ *where* $I_k = \{1, 2, \cdots, k\}$ *and* $J_k = N\backslash I_k$.

PROOF. For any $i \in I_k$, $|\alpha_i+\beta_i| \geqq |\alpha_i| - |\beta_i| \geqq |\alpha_k| - \|y\| \geqq |\alpha_k| - \delta \geqq |\alpha_k| - (|\alpha_k| - |\alpha_{k+1}|)/3 > (|\alpha_k| + |\alpha_{k+1}|)/2$. On the other hand, if $j \in J_k$ then $(|\alpha_k| + |\alpha_{k+1}|)/2 > |\alpha_{k+1}| + \delta \geqq |\alpha_{k+1}| + \|y\| \geqq |\alpha_j| + |\beta_j| \geqq |\alpha_j+\beta_j|$. Hence $|\alpha_i+\beta_i| > |\alpha_j| + |\beta_j|$ for all $i \in I_k$ and $j \in J_k$. Since the norm $\|x+y\|^p$ is assumed when both sequence $\{|\alpha_n+\beta_n|\}_{n=1,2,\cdots}$ and $\{a_n\}$ are in non-decreasing order. Hence there exists $\sigma \in \pi$ such that $\|x+y\|^p = \sum_{n=1}^{\infty} |\alpha_n+\beta_n|^p a_{\sigma(n)}$, $\sigma(I_k) = I_k$ and $\sigma(J_k) = J_k$.

LEMMA 10. *Let* $x = \sum_{i=1}^{n} \alpha_i e_i$ *be an element in* $d(a, p)$, $1 \leqq p < \infty$. *If* $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_n > 0$ *then there exists* $\delta > 0$ *such that* $(\sum_{i=1}^{n} \alpha_i^p a_{\sigma(i)})^{1/p} \leqq \|x\| - \delta$ *for all* $\sigma \in \pi$ *such that* $a_{\sigma(i)} \neq a_i$ *for some* $i = 1, 2, \cdots, n$.

PROOF. Let $I_n = \{1, 2, \cdots, n\}$ and let π_n be the set of all permutations σ of I_n such that $a_{\sigma(i)} \neq a_i$ for some $i \in I_n$. Then $\sum_{i=1}^{n} \alpha_i^p a_{\sigma(i)} < \|x\|^p$ for all $\sigma \in \pi_n$. Since π_n is finite, there exists $\delta_1 > 0$ such that $\sup (\sum_{i=1}^{n} \alpha_i^p a_{\sigma(i)})^{1/p} \leqq \|x\| - \delta_1$.

Let m be the smallest integer such that $m \geqq n$ and $a_m > a_{m+1}$. Let $\epsilon = \min\{a_m - a_{m+1},\ a_i - a_n : i \in I_n \text{ and } a_i \neq a_n\}$. It is clear that $\epsilon > 0$. Choose $0 < \delta_2 < \delta_1$ such that $\delta_2 \leqq \epsilon \alpha_n^p$. Then for any $\sigma \in \pi$ such that $a_{\sigma(j)} \neq a_{\sigma(k)}$ for some $j > n$, $k \leqq n$, $\sigma(j) \leqq n$ and $\sigma(k) > n$, it is easy to see that $a_{\sigma(j)} - a_\sigma(k) \geqq \epsilon$. Finally, let $0 < \delta \leqq \delta_2$ satisfy $\|x\|^p - \delta_2 \leqq (\|x\| - \delta)^p$.

Now for any $\sigma \in \pi$ such that $a_{\sigma(i)} \neq a_i$ for some $i \in I_n$, let σ_n be the restriction of σ to I_n. If $\sigma_n \in \pi_n$ then $(\sum_{i=1}^{n} \alpha_i^p a_{\sigma(i)})^{1/p} \leqq \|x\| - \delta_1 \leqq \|x\| - \delta$. Otherwise, there exist $k \in I_n$ and $j > n$ such that $\sigma(j) \in I_n$ and $\sigma(k) > n$. Then

$$\begin{aligned}\sum_{i=1}^{n} \alpha_i^p a_{\sigma(i)} &= \sum_{\substack{i=1\\ i \neq k}}^{n} \alpha_i^p a_{\sigma(i)} + \alpha_k^p a_{\sigma(j)} + \alpha_k^p (a_{\sigma(k)} - a_{\sigma(j)}) \\ &\leqq \sum_{i=1}^{n} \alpha_i^p a_i - \alpha_k^p (a_{\sigma(j)} - a_{\sigma(k)}) \\ &\leqq \|x\|^p - \delta_2 \leqq (\|x\| - \delta)^p.\end{aligned}$$

THEOREM 11. *Every* $d(a, p)$, $1 < p < \infty$ *is locally uniformly smooth.*

PROOF. Given any $\epsilon > 0$ and $x = \sum_{n=1}^{\infty} \alpha_n e_n$ in $d(a, p)$ with $\|x\| = 1$, we may assume that $|\alpha_1| \geqq |\alpha_2\| \geqq \cdots$. Let $\epsilon_1 = \epsilon/2 + 2^{1/p}$ and $a = 1/2^p$. By Lemma 8, there exist numbers $b > 0$ and $c > 0$ such that for all $\alpha \geqq a$, $b \geqq \beta \geqq 0$ and $c \geqq \gamma \geqq 0$,

$$[\alpha + (\beta + \gamma)^p]^{1/p} \leqq (\alpha + \beta^p)^{1/p} + \epsilon_1 \gamma. \tag{1}$$

Choose an integer k such that $|\alpha_k| > |\alpha_{k+1}|$,

$$\left\| \sum_{n=1}^{k} \alpha_n e_n \right\| \geqq \frac{3}{4} \quad \text{and} \quad \left\| \sum_{n=k+1}^{\infty} \alpha_n e_n \right\| \leqq b. \tag{2}$$

Let $I_k = \{1, 2, \cdots, k\}$ and $J_k = N \backslash I_k$. By Lemma 10, there exists $\delta_1 > 0$ such that for all $\sigma \in \pi$ with $a_\sigma(n) \neq a_n$ for some $n \in I_k$. Then

$$(3) \qquad \sum_{n=1}^{k} |\alpha_n|\, a_{\sigma(n)} < \left\| \sum_{n=1}^{k} \alpha_n e_n \right\| - \delta_1.$$

Since ℓ_p, $1 < p < \infty$ is uniformly smooth, there exists $\delta_2 > 0$ such that

$$(4) \qquad \|x + y\|_p + \|x - y\|_p \leqq 2 + \epsilon_1 \|y\|_p$$

for all $\|x\|_p = 1$ and $\|y\|_p \leqq \delta_2$ where $\|\cdot\|_p$ is the usual norm in ℓ_p.

Finally, let $\delta_3 > 0$ and $1 - \delta_1 < (1 - \delta_3)^p$ and let

$$\delta = \begin{cases} \min \left\{ \dfrac{\delta_2}{2^{1/p}}, \dfrac{\delta_3}{2}, \dfrac{|\alpha_k| - |\alpha_{k+1}|}{3}, b, \dfrac{c}{2} \right\} & \text{if } \sum_{n=k+1}^{\infty} \alpha_n e_n = 0 \\[2ex] \min \left\{ \dfrac{\delta_1}{2}, \dfrac{\delta_2}{2^{1/p}}, \dfrac{|\alpha_k| - |\alpha_{k+1}|}{3}, b, \dfrac{c}{2}, \left\| \sum_{n=k+1}^{\infty} \alpha_n x_n \right\|, \dfrac{1}{4} \right\} & \\ \qquad \text{if } \sum_{n=k+1}^{\infty} \alpha_n e_n \neq 0. \end{cases}$$

We shall show that for any $y = \sum_{n=1}^{\infty} \beta_n e_n$ in $d(a, p)$ with $\|y\| \leqq \delta$ then $\|x + y\| + \|x - y\| \leqq 2 + \epsilon \|y\|$.

Choose $\sigma_i \in \pi$, $i = 1, 2$ such that $\|x + y\|^p = \sum_{n=1}^{\infty} |\alpha_n + \beta_n|^p a_{\sigma_1(n)}$ and $\|x - y\|^p = \sum_{n=1}^{\infty} |\alpha_n - \beta_n|^p a_{\sigma_2(n)}$. By Lemma 9, we may assume that $\sigma_i(I_k) = I_k$ and $\sigma_i(J_k) = J_k$, $i = 1, 2$.

(Case I). $a_{\sigma_1(n)} \neq a_n$ (resp., $a_{\sigma_2(n)} \neq a_n$) for some $n \in I_k$. Since $\delta < (|\alpha_k| - |\alpha_{k+1}|)/3$, by (3),

$$\begin{aligned} \|x + y\| &\leqq \left(\sum_{n=1}^{\infty} |\alpha_n|^p a_{\sigma_1(n)} \right)^{1/p} + \|y\| \\ &< \left(\sum_{n=1}^{k} |\alpha_n|^p a_n - \delta_1 + \sum_{n=k+1}^{\infty} |\alpha_n|^p a_n \right)^{1/p} + \delta \\ &= (1 - \delta_1)^{1/p} + \delta. \end{aligned}$$

Hence $\|x + y\| + \|x - y\| \leqq (1 - \delta_1)^{1/p} + \delta + 1 + \delta < (1 - \delta_3) + 1 + 2\delta \leqq 2 \leqq 2 + \epsilon \|y\|$.

(Case II). $a_{\sigma_1(n)} = a_{\sigma_2(n)} = a_n$, $n = 1, 2, \cdots, k$. We consider two subcases.

$$(\text{i}) \qquad \sum_{n=k+1}^{\infty} \alpha_n e_n = 0.$$

Let $x_1 = (\alpha_1 a_1{}^{1/p}, \alpha_2 a_2{}^{1/p}, \cdots, \alpha_k a_k{}^{1/p}, 0, 0, \cdots)$ and $y_1 = (\beta_1 a_1{}^{1/p}, \beta_2 a_2{}^{1/p}, \cdots, \beta_k a_k{}^{1/p}, \|y\|, 0, \cdots)$. Then by (4),

$$\left(\sum_{n=1}^{k} |\alpha_n{}^p + \beta_n|^p a_n + \|y\|^p\right)^{1/p} + \left(\sum_{n=1}^{k} |\alpha_n{}^p - \beta_n|^p a_n + \|y\|^p\right)^{1/p}$$
$$= \|x_1 + y_1\|_p + \|x_1 - y_1\|_p \leqq 2 + \epsilon_1\|y_1\|_p$$
$$= 2 + \epsilon_1\left(\sum_{n=1}^{k} |\beta_n|^p a_n + \|y\|^p\right)^{1/p}$$
$$\leqq 2 + \epsilon_1(2\|y\|^p)^{1/p} \leqq 2 + \epsilon\|y\|.$$

Thus

$$\|x + y\| + \|x - y\| = \left(\sum_{n=1}^{k} |\alpha_n + \beta_n|^p a_n + \sum_{n=k+1}^{\infty} |\beta_n|^p a_{\sigma_1(n)}\right)^{1/p}$$
$$+ \left(\sum_{n=1}^{k} |\alpha_n - \beta_n|^p a_n + \sum_{n=k+1}^{\infty} |\beta_n|^p a_{\sigma_2(n)}\right)^{1/p}$$
$$\leqq \|x_1 + y_1\|_p + \|x_1 - y_1\|_p \leqq 2 + \epsilon\|y\|.$$

(ii) $\sum_{n=k+1}^{\infty} \alpha_n e_n \neq 0$. Then

$$\|x + y\| + \|x - y\|$$
$$\leqq \left[\sum_{n=1}^{k} |\alpha_n + \beta_n|^p a_n + \left\{\left(\sum_{n=k+1}^{\infty} |\alpha_n|^p a_{\sigma_1(n)}\right)^{1/p} + \left(\sum_{n=k+1}^{\infty} |\beta_n|^p a_{\sigma_1(n)}\right)^{1/p}\right\}^p\right]^{1/p}$$
$$+ \left[\sum_{n=1}^{k} |\alpha_n - \beta_n|^p a_n + \left\{\left(\sum_{n=k+1}^{\infty} |\alpha_n|^p a_{\sigma_2(n)}\right)^{1/p} + \left(\sum_{n=k+1}^{\infty} |\beta_n|^p a_{\sigma_2(n)}\right)^{1/p}\right\}^p\right]^{1/p}$$

$$\leqq \left[\sum_{n=1}^{k} |\alpha_n + \beta_n|^p a_n + \left(\left\| \sum_{n=k+1}^{\infty} \alpha_n e_n \right\| + \|y\| \right)^p \right]^{1/p}$$

$$+ \left[\sum_{n=1}^{k} |\alpha_n - \beta_n| a_n + \left(\left\| \sum_{n=k+1}^{\infty} \alpha_n e_n \right\| + \|y\| \right)^p \right]^{1/p} .$$

By (2),

$$\left(\sum_{n=1}^{k} |\alpha_n - \beta_n|^p a_n \right)^{1/p} \geqq \left(\sum_{n=1}^{k} |\alpha_n|^p a_n \right)^{1/p} - \left(\sum_{n=1}^{k} |\beta_n|^p a_n \right)^{1/p}$$

$$\geqq \frac{3}{4} - \delta \geqq \frac{1}{2} .$$

Also from (2), we have $b \geqq \|\sum_{n=k+1}^{\infty} \alpha_n e_n\| - \|y\| \geqq \|\sum_{n=k+1}^{\infty} \alpha_n e_n\| - \delta \geqq 0$ and $c \geqq 2\delta \geqq 2\|y\| \geqq 0$. Hence by (1), we conclude that

$$\left[\sum_{n=1}^{k} |\alpha_n - \beta_n|^p a_n + \left(\left\| \sum_{n=k+1}^{\infty} \alpha_n e_n \right\| + \|y\| \right)^p \right]^{1/p}$$

$$\leqq \left[\sum_{n=1}^{k} |\alpha_n - \beta_n|^p a_n + \left(\left\| \sum_{n=k+1}^{\infty} \alpha_n e_n \right\| \right.\right.$$

$$\left.\left. + \|y\| \right)^p \right]^{1/p} + 2\epsilon_1 \|y\|.$$

Now let $x_2 = (\alpha_1 a_1{}^{1/p}, \cdots, \alpha_k a_k{}^{1/p}, \|\sum_{n=k+1}^{\infty} \alpha_n e_n\|, 0, \cdots)$ and $y_2 = (\beta_1 a_1{}^{1/p}, \cdots, \beta_k a_k{}^{1/p}, \|y\|, 0, \cdots)$. Then $\|x_2\|_p = 1$, $\|y_2\| \leqq (2\|y\|^p)^{1/p} \leqq \delta_1$. Hence by (4),

$$\left[\sum_{n=1}^{k} |\alpha_n + \beta_n|^p a_n + \left(\left\| \sum_{n=k+1}^{\infty} \alpha_n e_n \right\| + \|y\| \right)^p \right]^{1/p}$$

$$+ \left[\sum_{n=1}^{k} |\alpha_n - \beta_n|^p a_n + \left(\left\| \sum_{n=k+1}^{\infty} \alpha_n e_n \right\| - \|y\| \right)^p \right]^{1/p}$$

$$= \|x_2 + y_2\|_p + \|x_2 - y_2\|_p$$

$$\leqq 2 + \epsilon_1 \|y_2\|_p \leqq 2 + \epsilon_1 (2^{1/p} \|y\|).$$

Thus

$$\|x + y\| + \|x - y\| \leqq \|x_2 + y_2\|_p + \|x_2 - y_2\|_p + 2\epsilon_1 \|y\|$$

$$\leqq 2 + (2 + 2^{1/p})\epsilon_1 \|y\| = 2 + \epsilon \|y\|.$$

COROLLARY 12. *For* $1 < p < \infty$, *all* $d(a,p)^*$ *are locally uniformly convex and so are strictly convex.*

REFERENCES

1. Z. Altshuler, *Uniform convexity in Lorentz sequence spaces,* to appear.

2. Z. Altshuler, P. G. Casazza and B. L. Lin, *On symmetric basic sequences in Lorentz sequence spaces,* Israel J. Math. **15**(1973), 140-155.

3. J. R. Calder and J. B. Hill, *A collection of sequence spaces,* Trans. Amer. Math. Soc. **152** (1970), 107-118.

4. P. G. Casazza and B. L. Lin, *On symmetric basic sequences in Lorentz sequence spaces* II, Israel J. Math. **17** (1974), 191-218.

5. P. G. Casazza and B. L. Lin, *On Lorentz sequence spaces,* Bull. Acad. Sinica **2** (1974), 233-240.

6. D. F. Cudia, *Rotundity,* Proc. Symp. Pure Math., Amer. Math. Soc. **7** (1963), 73-97.

7. M. M. Day, *Reflexivity Banach spaces not isomorphic to uniformly convex spaces,* Bull. Amer. Math. Soc. **47** (1941), 313-317.

8. ——, *Normed linear spaces,* Third Ed., Springer-Verlag, 1973.

9. K. Fan and I. Glicksberg, *Fully convex normed linear spaces,* Proc. Nat. Acad. Sci. **41** (1955), 947-953.

10. K. Fan and I. Glicksberg, *Some geometric properties of the spheres in a normed linear space,* Duke Math. J. **25** (1958), 553-568.

11. D. J. H. Garling, *A class of reflexive symmetric BK spaces,* Canad. J. Math. **21** (1969), 602-608.

12. A. R. Lovaglia, *Locally uniformly convex Banach spaces,* Trans. Amer. Math. Soc. **78** (1955), 225-238.

13. I. Singer, *Bases in Banach spaces* I. Springer-Verlag, 1970.

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