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A WEAK HARTMAN'S THEOREM FOR HOMOMORPHISMS AND SEMI-GROUPS IN A BANACH SPACE*

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In this article we examine the extent to which Hartman's Theorem holds for homomorphisms and semi-groups in a Banach space. The technique used here for the main theorem is a modification of the technique of Moser's used by Pugh [4] to prove Hartman's Theorem for isomorphisms and groups in a Banach space.

Let E be a Banach space and let $L : E \to E$ be linear on E; possibly 0 is in the spectrum of L. A basic assumption throughout the paper is that L is hyperbolic; that is, $E = E^u \oplus E^s$ where $LE^u \subset E^u$ and $LE^s \subset E^s$, and $L^s \equiv L \mid E^s$ is a contraction while $L^u \equiv L \mid E^u$ is invertible and $(L^u)^{-1}$ is also a contraction. We let $k \equiv \max\{|L^s|, |(L^u)^{-1}|\} < 1$. It is not hard to prove that if the spectrum of L has no points on the unit circle, then L is hyperbolic in some norm on E. Assume that E is given the norm $|x + y| = \max\{|x|, |y|\}$ for $x \in E^u$, $y \in E^s$.

Let $\beta(a)$ denote the set of bounded maps $\lambda : E \to E$ such that $|\lambda(x) - \lambda(y)| \leqq a|x - y|$ and $\lambda(0) = 0$. We use $\Lambda = L + \lambda$ and $\Lambda' = L + \lambda'$ for $\lambda, \lambda' \in \beta(a)$. We use 1 to denote an identity map.

We now state Pugh's version of Hartman's Theorem for isomorphisms for reference purposes:

THEOREM 1. *If L is an isomorphism and a is small enough, then for each Λ there is a unique bounded, uniformly continuous map $g : E \to E$ such that if $h = 1 + g$, then*

$$hL = \Lambda h. \tag{1}$$

Furthermore h is a homeomorphism depending continuously on λ.

Equation (1) implies that h maps orbits of L into orbits of Λ and vice versa.

Hale gives the example [1]

$$\dot{x}(t) = 2\alpha x(t) + N(x_t) \tag{2}$$

where $\alpha > 0$, $N(0) = 0$, and the Lipschitz constant of N in the ϵ-ball at 0 goes to 0 as $\epsilon \to 0$. Considered as a delay equation, (2) generates a strongly continuous semi-group $T(t)$ defined on $C([-r, 0], \mathbf{R}^n)$. If $N = 0$, the range of $T(r)$ is one dimensional. It is not hard to con-

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vince yourself that for the perturbed equation, $T(t)$ has much larger range for all t. Thus no continuous map on E could map orbits of (2) with $N = 0$ to orbits of (2) for some other choices of N.

The difficulty exposed here is that the linear map is not injective. One could still ask whether (1) might hold for homomorphisms L which are injective but not isomorphisms, or whether (1) might hold on subsets where L is injective. The following simple examples show that even this should not be expected.

EXAMPLE 1. Let $E = \ell_2$, the Hilbert space of square summable real sequences, and let $L : E \to E$ be defined by $L\{a_i\} = \{2^{-i}a_i\}$. For each i, L has an eigenvalue 2^{-i} with eigenvector $e_i = \{0, \cdots, 0, 1, 0, \cdots\}$, the 1 being in the i-th place. Let $\lambda = \lambda_\epsilon : E \to E$ be defined in the unit ball by $\lambda\{a_i\} = \epsilon\{a_i^2\}$, and elsewhere preserve the eigenspaces through each e_i. Notice that L is injective.

Now suppose that there exists a unique $h = 1 + g_\epsilon$ satisfying (1) and g_ϵ is bounded and varies continuously with ϵ. If h is continuous, then $hLe_i \to h(0) = 0$. On the other hand, it follows from uniqueness (see Cor. 1) that h must also preserve the eigenspaces through the e_i's. The boundedness of g then implies $Lhe_i \to 0$. Thus if h is continuous we would have $0 = \lim hLe_i = \lim \Lambda he_i = \lim(Lhe_i + \lambda he_i) = 0 + \lim \lambda he_i$. Since g_ϵ varies continuously with ϵ, we can choose ϵ small enough that h is bounded away from 0 on the unit circle Σ. But then λ is bounded away from 0 on $h\Sigma$, and it follows that $\lim \lambda he_i \not\to 0$. This contradiction indicates that even if (1) were to hold, we could not expect h to be continuous.

EXAMPLE 2. Again, $E = \ell_2$. Let $L\{a_i\} = \{0, a_1/2, a_2/2, \cdots,\}$. Notice that no non-zero point of E has an infinite backward orbit. Let $h_0 : \mathbf{R} \to \mathbf{R}$ be any homeomorphism such that $h_0(0) = 0$. Define $h : \ell_2 \to \ell_2$ by $h\{a_i\} = \{2^{-i}h_0(2^i a_i)\}$. Then h is a homeomorphism and if $h_0 = 1 + g_0$ for g_0 bounded and uniformly continuous, then $h = 1 + g$ for $g\{a_i\} = \{2^{-i}g_0(2^i a_i)\}$, which is also bounded and uniformly continuous. Furthermore, it is not hard to check that $hL = Lh$. Since $1 \cdot L = L \cdot 1$, this example indicates that the uniqueness of h does not hold in the presence of points with no infinite backward orbits.

We continue with a few more definitions in preparation for the main theorem: $F \subset E$ is *Λ-invariant* if $\Lambda F = F$, and *Λ-injective* if Λ is injective on F. A sequence $\{x_i\}$ in E is a *bi-infinite Λ-orbit* if $i = 0, \pm 1, \cdots$ and $\Lambda x_i = x_{i+1}$ for all i. Notice that each element of a Λ-invariant set has a bi-infinite Λ-orbit.

If F is Λ-invariant, then F_1 will denote a maximal Λ-injective subset of F. Then there is exactly one way to define Λ^{-1} on F such that

$\Lambda^{-1}\Lambda x = x$ on F_1 and $\Lambda\Lambda^{-1}x = x$ on F. Note that the maximality of F_1 implies $\Lambda F_1 = F$.

For a pair (F, F_1) as above, we say $F_2 \subset F$ is Λ-*compatible* if it is Λ-invariant, $\Lambda^{-1}F_2 \subset F_2$, and $\Lambda^{-1} \mid F_2$ is uniformly continuous. We define $C = C(L, \lambda, \lambda', F, F_1) \equiv \{g : F \to E \mid g$ is bounded and $g \mid F_2$ is uniformly continuous whenever F_2 is Λ-compatible.$\}$. Note that C with the sup norm is a Banach space.

THEOREM 2. *Let $L : E \to E$ be a hyperbolic linear homomorphism of a Banach space and $k = \max\{|L^s|, |(L^u)^{-1}|\}$. Suppose $\lambda, \lambda' \in \beta(a)$ where $a + k < 1$. Let $\Lambda = L + \lambda$ and $\Lambda' = L + \lambda'$, $F \subset E$ be Λ-invariant, and F_1 be a maximal Λ-injective subset of F. Then there is a unique bounded function $g = g(\lambda, F, F_1; \lambda') : F \to E$ such that if $h = h(\lambda, F, F_1; \lambda') \equiv 1 + g$, then*

$$h\Lambda = \Lambda' h \text{ on } F_1. \tag{3}$$

Furthermore, $g \in C$ and g varies continuously with $\lambda' \in \beta(a')$ (given the sup norm) for any a' with $a' + k < 1$.

REMARK. The uniqueness of h depends on the fact that we have restricted to a Λ-invariant set F, which has the property that every one of its points has a bi-infinite orbit in F.

PROOF. (3) is equivalent to the equation $g\Lambda - Lg = \lambda'(1 + g) - \lambda$ on F_1, which when expanded in $E^u \oplus E^s$ coordinates, as in [4], is equivalent on F_1 to

$$g_u = L_u^{-1}[g_u\Lambda + \lambda_u - \lambda_u'(1 + g)] \equiv Ug \tag{4a}$$

$$g_s = [L_s g + \lambda_s'(1 + g) - \lambda_s]\Lambda^{-1} \equiv Sg. \tag{4b}$$

(We restrict attention to F_1 since the derivation of (4b) requires the use of $\Lambda\Lambda^{-1} =$ identity on F_1.)

It is a trivial consequence of the facts that $\Lambda F = F$ and $\Lambda^{-1}F \subset F$ that the operator $T \equiv (U, S)$ maps the Banach space of bounded functions on F (with the sup norm) into itself. To check that T maps C into C, observe the following: Suppose $g \in C$. Thus Tg is bounded, and Tg is uniformly continuous on those sets where g, $g_u\Lambda$, Λ^{-1}, and $g\Lambda^{-1}$ are all uniformly continuous; in particular, if F_2 is Λ-compatible then Tg is uniformly continuous on F_2. Thus $T : C \to C$.

It is also easy to check that T is a contraction:

$$\begin{aligned} |Ug_1 - Ug_2| &\leqq |L_u^{-1}|(|g_1 - g_2| + |\lambda_u'(1 + g) - \lambda_u'(1 + g_2)|) \\ &\leqq |L_u^{-1}|(|g_1 - g_2| + a|g_1 - g_2|) \\ &\leqq (k + ka)|g_1 - g_2|. \end{aligned}$$

$$|Sg_1 - Sg_2| \leqq |L_s|\,|g_1 - g_2| + |\lambda'(1 + g_1) - \lambda'(1 + g_2)|$$
$$\leqq k|g_1 - g_2| + a|g_1 - g_2| \leqq (k + a)|g_1 - g_2|.$$

Since $k + ka < k + a < 1$, T is a contraction.

Therefore, T has a fixed point in C which is unique even in the space of bounded functions on F. Since T varies continuously in λ', when the contraction constant of T is bounded away from 1, its fixed point also varies continuously.

The fixed point of T will satisfy (4) on all of F and hence satisfy (3) on F_1. It is not immediate, however, that any solution of (3) on F_1 must be a fixed point of T. However, if g_1 and g_2 satisfy (3) on F_1, then they satisfy (4) on F_1. Thus on F_1, $|g_1 - g_2| = |Tg_1 - Tg_2| \leqq (a + k)\ |g_1 - g_2|$. This implies that $g_1 = g_2$ on F_1. But since $\Lambda F_1 = F$, the values on all of F of any function satisfying (3) are determined completely by the values on F_1. Thus $g_1 = g_2$ on all of F. This completes the proof.

The following corollary indicates that not much improvement can be expected on Theorem 2.

Corollary 1. *Suppose $L = L_s$, $g = g(\lambda, F, F_1; \lambda')$, and $\lambda - \lambda'$ is supported on a set G. Then $g = (\lambda' - \lambda)\Lambda^{-1}$ on the set $F_0 = F - \Lambda^2 G$. In particular, if $h = 1 + g$ is continuous at some point of F_0, then $(\lambda - \lambda')\Lambda^{-1}$ is also.*

Proof. Since g is the unique fixed point of the contraction $Sg = (Lg + \lambda'(1 + g) - \lambda)\Lambda^{-1}$, the S iterates $\{g_n\}$ of $g_0 \equiv 0$ converge uniformly to g. But $g_1 = (\lambda' - \lambda)\Lambda^{-1}$, so $g_1\Lambda^{-1} \mid F_0 \equiv 0$. Induction implies that $g_n\Lambda^{-1} \mid F_0 \equiv 0$ for all n, which implies that $g_{n+1} \mid F_0 = (\lambda' - \lambda)\Lambda^{-1} \mid F_0$ for all n. Corollary 1 now follows.

Although h is not necessarily continuous or invertible (as shown by Example 2), it does have some injective and surjective properties.

Corollary 2. *If $L = \Lambda$, then h has the following injective property on orbits: suppose $x_0, y_0 \in E$ with bi-infinite L-orbits $\{x_i\}$ and $\{y_i\}$ respectively. If $x_0 \neq y_0$ and $h(x_0) = h(y_0)$, then there is a negative integer n such that $h(x_n) \neq h(y_n)$.*

Proof. Otherwise, for all n, $0 = h(x_n) - h(y_n) = x_n - y_n + g(x_n) - g(y_n)$. Since g is bounded, this implies that $\{x_n - y_n\}$, the bi-infinite L-orbit of $x_0 - y_0$, is also bounded. The following lemma finishes the proof.

Lemma 1. *The only bounded, bi-infinite orbit of L is $\{0\}$.*

PROOF. The lemma follows from the following inequalities:
If $x_0 \in E$ and $x_n \equiv L^n x_0$ for $n > 0$, then

(5a) $$|x_i| \geqq |x_i^u| = |L_u x_{i-1}^u| \geqq k^{-1}|x_{i-1}^u| \geqq \cdots \geqq k^{-i}|x_0^u|.$$

If x_0 has a bi-infinite L-orbit $\{x_i\}$, then for $i > 0$,

$$|x_{i+1}^s| = |L_s x_i^s| \leqq k|x_i^s| \qquad \text{and hence}$$

(5b) $$|x_i^s| \geqq k^{-i}|x_0^s|.$$

COROLLARY 3. *If $L = \Lambda$, and F' is a bounded Λ'-invariant subset of E, then $F' = \{0\}$.*

PROOF. Suppose F' is a bounded Λ'-invariant subset of E and $h' = h(\lambda', F', F_1'; 0)$ for some F_1'. Then $h'\Lambda' = Lh'$ on F_1'. Since F' is bounded, $h' = 1 + g'$ is bounded on F'. It follows from (3) and the fact that every point in F' has an infinite backward orbit in F_1', that every point in $h'(F')$ has an infinite backward orbit in $h'(F_1')$. Since $h'(F')$ is bounded, it follows from (5a) that $h'(F') \subset E^u$. It is not yet immediate that $h'(F')$ is invariant, so let F be the union of all L-iterates of $h'(F')$. Then $LF = F$, and $F \subset E^u$, and since L^u is invertible, L is injective on F. Thus we can define $h = h(0, F, F; \lambda')$ and then

$$hh'\Lambda' = hLh' = \Lambda' hh' \quad \text{on} \quad F_1' \cap h'^{-1}(F) = F_1'.$$

Since $hh' = 1 + (g' + gh')$ and $g' + gh'$ is bounded on F_1', it follows from uniqueness that $hh' = 1$ on F_1'. It follows that if $\{x_i'\}$ is a bi-infinite Λ'-orbit in F', then $\{h'(x_i')\}$ is a bi-infinite L-orbit in E^u which is bounded. It follows from (5b) that $\{h'(x_i')\} = 0$, and then from Cor. 2 that $x_i' = 0$ for all i. This finishes the proof of Corollary 3.

COROLLARY 4. *Let $L = \Lambda$. Then the following surjective property holds: If $\{x_i'\}$ is a bi-infinite Λ'-orbit, then there is a pair (F, F_1) such that if $h = h(0, F, F_1; \lambda')$, then $hF \supset \{x_i'\}$ unless x_i' remains bounded as $i \to \infty$. In this case, if L is not injective on F, it is possible that hF contains only $\{x_i'\}$, $i \leqq N$, for some N.*

PROOF. Let $F' = \{x_i'\} \neq \{0\}$, and let $h' = h(\lambda', F', F_1'; 0)$ for some choice of F_1'. Let $x_0 = h'(x_0')$ where $x_0' \in F_1'$. (Renumber if necessary.) Let $x_i = h'(x_i')$ for $i < 0$, and $x_i = L^i(x_0)$ for $i > 0$. Then, since $h'\Lambda' = Lh'$ on F_1', we have $Lx_i = x_{i+1}$ for all i. Let $F = \{x_i\}$ and $h = h(0, F, F_1; \lambda')$ for some choice of F_1.

Case 1. Λ' is injective on F' and L on F. In this case $F' = F_1'$ and $F = F_1$; then $hh'\Lambda' = hLh' = \Lambda' hh'$ on $F_1' \cap h'^{-1}F_1 = F'$ and

$h'hL = Lh'h$ on $F_1 \cap h^{-1}F_1' = F$. Uniqueness implies that h and h' are mutual inverses.

Case 2. L is injective on F but Λ' is not injective on F' $(F' \neq F_1')$. If Λ' is not injective on an orbit, it follows that the orbit must properly contain a periodic orbit. Corollary 3 implies this periodic orbit is actually a fixed point. Thus by renumbering if necessary we can assume that $x_1' \neq 0$ but $x'_{1+i} = 0$ for all $i > 0$. Now the first equation of case 1 implies h maps F onto $\{x_0, x_{-1}, \cdots\}$, and since $hL = \Lambda'h$, it follows that $h(x_1) = x_1'$ and $h(x_{1+i}) = 0$ for $i > 0$. Thus $hF = F'$. Note that boundedness of g forces $F \subset E^s$.

Case 3. L is not injective on F. If this is the case, then we can assume by renumbering if necessary that $x_1 \neq 0$ but $x_{1+i} = 0$ for $i > 0$. Let $F_1 = \{0, x_0, x_{-1}, \cdots\}$. (The boundedness of g' now implies that x_i' is bounded as $i \to \infty$.) Uniqueness implies that $h(x_i) = x_i'$ for $i \leqq 0$, since $g = h - 1$ must be bounded if $g' = 1 - h'$ is. Thus the image of h contains a negative Λ'-half orbit. This completes the proof.

Corollary 5. *Suppose $\Lambda = L$, F is the set of all points with bi-infinite L-orbits and F' the set of points with bi-infinite Λ'-orbits.*

(a) *If L is injective on F, then $hF \supset F'$.*

(b) *If Λ' is injective on F', then $h(0, F, F_1; \lambda')$ is injective for any choice of F_1.*

Proof. (a) follows from Cor. 4 and (b) from Cor. 2.

Remark. The injectivity of L on F is a very reasonable hypothesis; for example, it is implied by the condition that the kernel of L^{N+n} = kernel of L^N for some N and all positive n. Henry [3] has shown this to be true of any L arising from a functional differential equation.

The hypothesis in (b) can also be verified in certain cases; for example see Chapter 6 in Hale [1].

Now let L_t be a linear hyperbolic strongly continuous semi-group on E. Let $\lambda_t, \lambda_t' : E \to E$ be for each $t \geqq 0$ a bounded Lipschitz continuous map such that $\Lambda_t \equiv L_t + \lambda_t'$, $\Lambda_t' \equiv L_t + \lambda_t'$ satisfy the hypotheses of Theorem 2. Suppose F is Λ_t-invariant for all $t \geqq 0$. Let $\{F_t\}_{t \geqq 0}$ be a family of sets with the property that F_t is a maximal Λ_t-injective subset of F such that $\Lambda_t\ F_{t+\tau} = F_\tau$ for all $\tau \geqq 0$, and $F_t \subset F_\tau$ if $t > \tau$. (The existence of such families follows from Zorn's Lemma.) It follows that Λ_{-t} is uniquely defined on F such that $\Lambda_{-t}\Lambda_t x = x$ for all $x \in F_\tau$ whenever $\tau > t$.

THEOREM 3. (Conjugacy theorem for semi-groups). *Let* $L = L_1$. *The function* $h = h(\lambda_1, F, F_1, \lambda_1{}')$ *from Theorem* 2 *satisfies*

$$h\Lambda_t \equiv \Lambda_t{}'h \text{ on } F_t \text{ for each } t \geqq 0.$$

h is the only function of the form $1 + g$ *for g bounded which satisfies this equation for any t. Furthermore,* $h \in C(L_t, \lambda_t, \lambda_t{}', F, F_t)$ *for all t.*

PROOF. From Theorem 2, we have $h\Lambda_1 = \Lambda_1{}'h$ on F_1. Now let $1 \geqq t \geqq 0$ and $\bar{h} \equiv \Lambda_t{}'h\Lambda_{-t}$ defined on F. Then $\bar{h}\Lambda_1 = \Lambda_t{}'h\Lambda_{-t}\Lambda_1 = \Lambda_t{}'h\Lambda_{1-t} = \Lambda_t{}'h\Lambda_1\Lambda_{-t}$ on F_1. Since $\Lambda_{-t}F_1 \subset F_1$, we have on F_1,

$$\bar{h}\Lambda_1 = \Lambda_t{}'\Lambda_1{}'h\Lambda_{-t} = \Lambda_1{}'\Lambda_t{}'h\Lambda_{-t} = \Lambda_1{}'\bar{h}.$$

It is easy to check that $\bar{g} = \bar{h} - 1$ is bounded, so the uniqueness part of Theorem 2 implies that $h = \bar{h}$ and hence

$$h\Lambda_t = \Lambda_t{}'h\Lambda_{-t}\Lambda_t = \Lambda_t{}'h \quad \text{on} \quad F_t.$$

The rest of Theorem 3 follows from an induction and Theorem 2.

EXAMPLE 3. Suppose L, λ' are as in Theorem 2 and for $\epsilon > 0$, M_ϵ is the eigenspace associated to eigenvalues $\geqq \epsilon$. Then L is injective on $F = \bigcup\{M_\epsilon : \epsilon > 0\}$ and $L \mid M_\epsilon$ has a bounded linear inverse. If $\Lambda' = L + \lambda'$, then Theorem 2 provides a function h defined on F such that $hL = \Lambda'h$ on F, and $h \mid M_\epsilon$ is uniformly continuous for each $\epsilon > 0$. Now suppose further that Λ' is injective on some neighborhood V of 0, and that $L = L_s$. Then $V \subset F'$ for some Λ'-injective F'. If $h = h(\lambda', F', F'; 0)$ then $h'hL = \Lambda'h'h$ on $F \cap h^{-1}(F')$. Thus $h'h$ is the identity on this set, and since $h \mid M_\epsilon$ is continuous, it follows that for each $\epsilon > 0$, there is a neighborhood U_ϵ of 0 such that $h \mid M_\epsilon \cap U_\epsilon$ is injective. However, it is not clear that $h^{-1} = h'$ is continuous or that h takes M_ϵ into the associated invariant manifold of Λ' (if $L = L_s$, Cor. 1 implies that $h = 1$ outside G). In many cases however, M_ϵ is finite dimensional; then $h \mid M_\epsilon \cap U_\epsilon$ is a homeomorphism since it is injective and $M_\epsilon \cap U_\epsilon$ is compact.

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