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SINGULARLY PERTURBED CONSERVATIVE SYSTEMS

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1. **Introduction.** Singular perturbation theory has been applied to a wide variety of physical problems, cf. [1], [2], [3]. In these applications, the problem is stated in the form of a system of ordinary differential equations.

$$
\begin{aligned}
dx/dt &= f(x, y), x(0) = \alpha \\
\epsilon dy/dt &= g(x, y), y(0) = \beta
\end{aligned}
\tag{1}
$$

where x is an m-vector and y is an n-vector. It is then assumed that the algebraic equation $0 = g(x, y)$ has a solution $y = \Phi(x)$. Another assumption is that the m-dimensional system

$$
dx_0/dt = f(x_0, \Phi(x_0)), \ x_0(0) = \alpha
\tag{2}
$$

has a unique solution over an interval $[0, T]$. The solution of system (1) is then compared to $(x_0(t), \Phi(x_0(t))$ over the interval $[0, T]$. Most theorems dealing with these singularly perturbed initial-value problems, cf. [4], [5], [6], and [7], assume that, over at least some portion of an open interval containing $[0, T]$, the real parts of the eigenvalues of

$$
\frac{\partial g}{\partial y}(x_0(t), \Phi(x_0(t)))
$$

are negative and bounded away from zero.

To understand this condition physically, we convert time scales by letting $\tau = t/\epsilon$; system (1) is transformed into

$$
\begin{aligned}
d\tilde{x}/d\tau &= \epsilon f(\tilde{x}, \tilde{y}), \ \tilde{x}(0) = \alpha \\
d\tilde{y}/d\tau &= g(\tilde{x}, \tilde{y}), \tilde{y}(0) = \beta.
\end{aligned}
\tag{3}
$$

System (3) is a regular perturbation problem and has a reduced problem,

$$
\begin{aligned}
d\tilde{x}_0/d\tau &= 0, \ \tilde{x}_0(0) = \alpha \\
d\tilde{y}_0/d\tau &= g(\alpha, \tilde{y}_0), \ \tilde{y}(0) = \beta.
\end{aligned}
\tag{4}
$$

The condition on the eigenvalues of $\partial g/\partial y$ implies that $(\alpha, \Phi(\alpha))$ is an asymptotically stable critical point of system (4).

The asymptotic behavior of $(x, \Phi(x))$ implies, in physical terms, that the process described by the y-variables tends very rapidly to equilibrium. However, in some physical problems, cf. [8] and [9], the process modeled by the singularly perturbed variables oscillates rapidly about the equilibrium solution. The current paper deals with this type of physical behavior.

Other authors have examined similar problems. Mitropolski [10] considered a system of the form

$$\begin{aligned} &\epsilon^2 d^2x_k/dt^2 + \omega_k{}^2 x_k = \epsilon F_k(x_1, \cdots, x_n, \epsilon dx_1/dt, \epsilon dx_n/dt) \\ &x_k(0) = \alpha_k, \epsilon dx_k/dt(0) = \beta_k;\ k = 1, \cdots, n. \end{aligned} \tag{5}$$

where each ω_k is a constant. The boundedness of $\{x_k(t)\}$ and $\{\epsilon dx_k/dt\}$ over an interval $[0, T]$ is determined by examining the functions F_k when its arguments are replaced by the solutions of

$$\begin{aligned} &\epsilon^2 d^2\bar{x}_k/dt^2 + \omega_x{}^2 \bar{x}_k = 0 \\ &\bar{x}_k(0) = \alpha_k, \epsilon \frac{d\bar{x}_k}{dt}(0) = \beta_k,\ k = 1, \cdots, n. \end{aligned} \tag{6}$$

In this paper, we examine a more general problem than system (5). We will consider vector differential equations where ω_k is a function of t. Scalar problems with ω, a function of t, have previously been examined (cf. [11] and [12]); the results of these works are also expressed in terms of conditions on the function $F(x, \epsilon dx/dt)$. We will state boundedness results in terms of conditions on the set $\omega_k(t)$; $k = 1, \cdots, n$.

In the next section, we will state the problem to be examined and the conditions on the set $\{\omega_k(t) \mid k = 1, \cdots, n\}$. § 3 will contain certain preliminary lemmas needed to prove the main result. In § 4, the main theorem will be proved.

2. **Formulation.** Consider the system of differential equations

$$\begin{aligned} dx/dt &= f(x, y, \epsilon^2 dy/dt) + \epsilon^2 \hat{f}(x, y, \epsilon dy/dt) \\ \epsilon^2 d^2y/dt^2 &= g(x, y, \epsilon^2\, dy/dt) + \epsilon^2 \hat{g}(x, y, \epsilon dy/dt) \end{aligned} \tag{7}$$

where x is an m-vector, y is an n-vector, and ϵ is a small positive parameter. Let the "reduced problem"

$$\begin{aligned} dx_0/dt &= f(x_0, y_0, 0) \\ 0 &= g(x_0, y_0, 0) \\ x_0(0) &= \alpha, \end{aligned} \tag{8}$$

where α is an arbitrary initial-vector, have a unique solution on the interval $[0, T]$.

We suppose

H 1. *If* $\Phi(x_0)$ *is the solution of* $0 = g(x_0, \Phi(x_0), 0)$, *then* $y_0(t) = \Phi(x_0(t))$ *is twice continuously differentiable on* $[0, T]$ *with bounded second derivative.*

Because we are examining systems without asymptotic stability, the initial-vectors for equations (7) must be restricted. We will consider two possible scalings and suppose

H 2.
$$x(0) = \alpha,$$
$$y(0) = y_0(0) + \epsilon^{\nu}\beta \tag{9}$$
$$\frac{dy}{dt}(0) = dy_0/dt(0) + \epsilon^{\nu-1}\gamma, \ \nu = 1 \text{ or } 2$$

where β *and* γ *are arbitrary n-vectors.*

We assume

H 3. f *and* g *are of class* C^2 *and* C^3, *respectively, in an open set* $E \subseteq R^{m+n+n}$ *containing* $(x_0(t), y_0(t), dy_0/dt(t))$ *for* $t \in [0, T]$. $\hat{f}$ *and* $\hat{g}$ *are of class* C^0 *and* C^1, *respectively, in* E.

To insure that we obtain oscillatory behavior, we may suppose

H 4. *The eigenvalues of*

$$\frac{\partial g}{\partial y}(x_0(t), y_0(t), 0)$$

are negative. Denote them by $-\omega_k{}^2(t)$ *where* $\omega_k(t) > 0$ *for* $t \in [0, T]$.

To avoid problems of internal resonance, we assume

H 5. *If, for any* $t \in [0, T]$, $\omega_i(t) = \omega_j(t)$, *then* $i = j$.

A technical assumption that is needed when $\nu = 1$ is

H 6. *Let* ω_i, ω_j, ω_ℓ *be any three elements of* $\{\omega_k \mid k = 1, \cdots, n\}$. *Then there exists no* $t \in [0, T]$ *where* $\omega_i(t) = \omega_j(t) + \omega_\ell(t)$.

The principal theorem is:

THEOREM 1. *Given systems* (7) *and* (8), *suppose* H 1–H 6 *hold. Then there exists* $C > 0$ *and* $\epsilon^* > 0$ *such that the solution of system* (7) *with initial conditions satisfying equations* (9) *exists on* $[0, T]$ *and satisfies*

$$|x(t,\epsilon) - x_0(t)| + |y(t,\epsilon) - y_0(t)| + \epsilon\left|\frac{dy}{dt}(t,\epsilon) - dy_0/dt(t)\right| \leqq C\epsilon^{\nu}, \tag{10}$$

there, provided $0 < \epsilon \leqq \epsilon^*$.

This theorem will be proved in § 4.

3. **Preliminaries.** The proof of Theorem 1 will crucially use a theorem on averaging proved by Sethna and Balachandra [13]. Their result examines problems involving multiple time scales t, ϵt, $\epsilon^2 t$, etc. For example, consider

$$\begin{aligned} dz/d\tau &= \epsilon F(z, \tau, \epsilon\tau) \\ z(0) &= z_0, \end{aligned} \tag{11}$$

where z is an n-vector. If we define the "average" of $F(z, \tau, \epsilon\tau)$ to be

$$\bar{F}(z, \tau_1) = \lim_{M\to\infty} 1/M \int_0^M F(z, s, \tau_1)\, ds \tag{12}$$

and we define the "averaged problem" to be

$$\begin{aligned} dz/d\tau &= \epsilon \bar{F}(z, \epsilon\tau) \\ z(0) &= z_0, \end{aligned} \tag{13}$$

the theorem of Sethna and Balachandra gives the relationship between the solutions of systems (11) and (13) over an interval of the form $[0, T/\epsilon]$.

We will state a modified version of their result without proof. Their proof of Theorem 1 [13] with minor modifications will prove the theorem stated below.

Consider the system of differential equations

$$\begin{aligned} dz/d\tau = {} & \epsilon f(\tau, \epsilon\tau, \alpha_1(\epsilon)\tau, \cdots, \alpha_r(\epsilon)\tau, z, \epsilon) \\ & + \epsilon^2 \hat{f}(\tau, \epsilon\tau, \alpha_1(\epsilon)\tau, \cdots, \alpha_r(\epsilon)\tau, z, \epsilon) \end{aligned} \tag{14}$$

where z, f, and $\hat{f}$ are n-vectors and ϵ is a small positive parameter.

A 1. *The scalar functions* $\alpha_i(t)$ *are such that*

$\alpha_i(\epsilon) \geqq 0$, *monotone increasing in* ϵ,

$\alpha_i(\epsilon) \to 0$ *as* $\epsilon \to 0$,

$\alpha_{i+1}(\epsilon)/\alpha_i(\epsilon) \to 0$ *as* $\epsilon \to 0$, $i = 1, \cdots, r-1$, *and*

$\epsilon/\alpha_1(\epsilon)$ *is bounded as* $\epsilon \to 0$.

Denote the interval $[0, T]$ by I.

A 2. *The functions f and $\hat{f}$ are mappings into R^n from*

$$\mathcal{L} = \{(\tau, \epsilon\tau, \tau_1, \cdots, \tau_r, z, \epsilon) \mid 0 \leqq \tau, \epsilon\tau \in I, \tau_i \in I, i = 1, \cdots, r, z \in G,$$

a bounded domain in R^n, $0 \leqq \epsilon \leqq \epsilon_1\}$.

The function f is continuously differentiable in τ_i, $i = 1, \cdots, r$ and z_j and there exists a positive constant M_1 such that

$$|f| \leqq M_1, \left| \frac{\partial f}{\partial \tau_i} \right| \leqq M_1, \left| \frac{\partial f}{\partial z} \right| \leqq M_1, \text{ and}$$

$$|\hat{f}| \leqq M_1 \text{ on } \mathcal{L}.$$

The function $\hat{f}$ is continuous in τ and $\epsilon\tau$ in $\mathcal{L}$.

A 3. *The limit*

$$f_0(\tau_1, \cdots, \tau_r, z, \epsilon) \equiv \lim_{M\to\infty} 1/M \int_0^M f(s, \epsilon s, \tau_i, \cdots, \tau_r, \epsilon)\, ds$$

exists uniformly for $\tau_i \in I$, $i = 1, \cdots, r$, $z \in G$, and $0 \leqq \epsilon \leqq \epsilon_1$.

A 4. *The limits*

$$h_{ij}(\tau_1, \cdots, \tau_r, z, \epsilon) \equiv \lim_{M\to\infty} 1/M \int_0^M \frac{\partial f_i}{\partial \tau_j}(s, \epsilon s, \tau_1, \cdots, \tau_r, z, \epsilon)\, ds$$

$$i = 1, \cdots, n; j = 1, \cdots, r$$

$$g_{ij}(\tau_1, \cdots, \tau_r, z, \epsilon) \equiv \lim_{M\to\infty} 1/M \int_0^M \frac{\partial f_i}{\partial x_j}(s, \epsilon s, \tau_1, \cdots, \tau_r, z, \epsilon)\, ds$$

$$i, j = 1, \cdots, n$$

exist uniformly for $\tau_i \in I$, $i = 1, \cdots, r$, $z \in G$, and $0 \leqq \epsilon \leqq \epsilon_1$.

A 5. *The differential equation*

$$dz/d\tau = \epsilon f_0(\alpha_1(\epsilon)\tau, \cdots, \alpha_r(\epsilon)\tau, z, \epsilon) \tag{15}$$

has a unique solution for $0 \leqq \tau$.

THEOREM 2. *Let* A 1–A 5 *hold. Then given any* $\delta > 0$, *there exists* $\epsilon_2 > 0$ *such that if* Ψ *and* ϕ *are respectively the solutions of equations* (14) *and* (15) *with* $\Psi(0, q_0, \epsilon) = \phi(0, q_0, \epsilon) = q_0$, *and if* ϕ *with its* Q-*neighborhood for* $Q > 0$ *lies inside* G *for* $0 \leqq \tau \leqq T/\alpha_1(\epsilon)$ *and* $0 < \epsilon \leqq \epsilon_1$, *then for all* $0 < \epsilon \leqq \epsilon_2$ *and* $0 \leqq \tau \leqq T/\alpha_1(\epsilon)$, $|\Psi(\tau) - \phi(\tau)| < \delta$.

To show how this theorem is applied, we prove the following lemma.

LEMMA 1. *Let* A *and* B *be* n-*vectors where* $A = \mathrm{col}(A_1, \cdots, A_n)$ *and* $B = \mathrm{col}(B_1, \cdots, B_n)$. *Let* $\bar{B}$ *be the sum of some subset of* $\{B_i\}$. *Let* $\theta(s)$ *be a scalar function continuous in* s. *Let* ξ *be an* m-*vector and* $F(\tau_1, \xi, A)$ *be a* p-*vector. Let*

$$f(\tau, \epsilon\tau, \tau_1, \xi, A, B, \epsilon)$$
$$= F(\tau_1, \xi, A)\, \theta(\epsilon\tau) \cos\left(\int_0^\tau \theta(\epsilon u)\, du + \bar{B} \right)$$

where F *satisfies* A 2 *of Theorem* 2 *for* $0 \leqq \tau_1 \leqq T$ *and* (ξ, A) *in a bounded domain* $G \subseteq R^{m+n}$. *Then*

(a) $$\lim_{M\to\infty} 1/M \int_0^M f(s, \epsilon s, \tau_1, \xi, A, B, \epsilon)\, ds = 0$$

(b) $$\lim_{M\to\infty} 1/M \int_0^M \frac{\partial f_i}{\partial \tau_1}(s, \epsilon s, \tau_1, \xi, A, B, \epsilon)\, ds = 0$$
$$i = 1, \cdots, p$$

(c) $$\lim_{M\to\infty} 1/M \int_0^M \frac{\partial f_i}{\partial \xi_j}(s, \epsilon s, \tau_1, \xi, A, B, \epsilon)\, ds = 0$$
$$i = 1, \cdots, p; j = 1, \cdots, m$$

(d) $$\lim_{M\to\infty} 1/M \int_0^M \frac{\partial f_i}{\partial A_j}(s, \epsilon s, \tau_1, \xi, A, B, \epsilon)\, ds = 0$$
$$i = 1, \cdots, p; j = 1, \cdots, n$$

(e) $$\lim_{M\to\infty} 1/M \int_0^M \frac{\partial f_i}{\partial B_j}(s, \epsilon s, \xi, A, B, \epsilon)\, ds = 0$$
$$i = 1, \cdots, p; j = 1, \cdots, n$$

uniformly for $0 \leqq \tau_1 \leqq T$ *and* $(\xi, A) \in G$ *and for arbitrary* B.

PROOF. Integrating, we find

$$\int_0^M \theta(\epsilon s) \cos\left(\int_0^s \theta(\epsilon u)\, du + \bar{B}\right) ds$$

$$= \sin\left(\int_0^M \theta(\epsilon u)\, du + \bar{B}\right),$$

because

$$\left| 1/M \sin\left(\int_0^M \theta(\epsilon u)\, du + \bar{B}\right)\right| \leqq 1/M,$$

$$\lim_{M\to\infty} \left| 1/M \sin\left(\int_0^M \theta(\epsilon u)\, du + \bar{B}\right)\right| = 0$$

uniformly in B. Hence

$$\lim_{M\to\infty} \left| 1/M \int_0^M \theta(\epsilon s) \cos\left(\int_0^s \theta(\epsilon u)\, du + \bar{B}\right) ds \right| = 0$$

uniformly in B. Since $F(\tau_1, \xi, A)$ along with all of its partial derivatives are uniformly bounded for $0 \leqq \tau_1 \leqq T$ and $(\xi, A) \in G$, conclusions (a), (b), (c), and (d) are proved. Conclusion (e) is proved by noting that the argument used in part (a) is equally true if f is defined in terms of $\sin(\int_0^T \theta(\epsilon s)\, ds + \bar{B})$. This proves lemma 1.

Theorem 2 implies that the solution of the origin problem is close to the solution of the "averaged problem." Lemma 1 simplifies the "averaged problem" by implying that terms of a particular form, i.e., the function f of Lemma 1, "average" to zero. This procedure will be used in the next section to prove Theorem 1.

4. **Proof of Theorem 1.** Let $\epsilon\xi(t) = x(t) - x_0(t)$, $\epsilon\,\mathfrak{z}(t) = y(t) - y_0(t)$, and $\mathfrak{z}_2(t) = dy/dt - dy_0/dt$. Differential equations (7) transform into

$$
\begin{aligned}
d\xi/dt &= f_1(t)\xi + f_2(t)\,\mathfrak{z}_1 + \epsilon F(\xi, \mathfrak{z}_1, \mathfrak{z}_2, t, \epsilon),\\
\epsilon\, d\mathfrak{z}_1/dt &= \mathfrak{z}_2,\\
\epsilon\, d\mathfrak{z}_2/dt &= g_1(t)\xi + g_2(t)\mathfrak{z}_1 + \epsilon[g^{(2)}(t, \xi, \mathfrak{z}_1)\\
&\quad + \hat{g}(x_0, y_0, 0)\\
&\quad + g_3(t)(\mathfrak{z}_2 + dy_0/dt - d^2y_0/dt^2]\\
&\quad + \epsilon^2 G(\xi, \mathfrak{z}_1, \mathfrak{z}_2, t, \epsilon),\\
\xi(0) &= 0, \quad \mathfrak{z}_1(0) = \epsilon^{\nu-1}\beta, \quad \mathfrak{z}_2(0) = \epsilon^{\nu-1}\gamma,
\end{aligned}
\tag{16}
$$

where $f_1(t) = \partial f/\partial x(x_0(t), y_0(t), 0)$, etc.; where

$$\epsilon^2 F(\xi, \ \mathfrak{z}_1, \ \mathfrak{z}_2, t, \epsilon) = f(x, y, \epsilon^2\, dy/dt) - f(x_0(t), y_0(t), 0)$$
$$- \epsilon f_1(t)\xi - \epsilon f_2(t)\ _1 + \epsilon^2 \hat{f}(x, y, \epsilon\, dy/dt);$$

where $g^{(2)}(t, \xi, \ \mathfrak{z}_1)$ consists of all quadratic terms in ξ and $\mathfrak{z}_1$; and where

$$\epsilon^3 G(\xi, \ \mathfrak{z}_1, \ \mathfrak{z}_2, t, \epsilon) = g(x, y, \epsilon^2\, dy/dt) -$$
$$\epsilon g_1(t)\xi - \epsilon g_2(t)\ \mathfrak{z}_1 - \epsilon^2[g^{(2)} +$$
$$g_3(\ \mathfrak{z}_2 - d^2y_0/dt^2)] + \epsilon^2 \hat{g}(x, y, \epsilon\, dy/dt),$$
$$- \epsilon^2 g(x_0, y_0, 0).$$

Because $g(x, y, \epsilon^2\, dy/dt)$ is C^4 and $(x_0(t), y_0(t))$ is C^1 on $[0, T]$, $g_2(t)$ is C^1 on $[0, T]$. From H 5 and a theorem of Gingold [14], there exists a non-singular matrix $S(t) \in C^1$ such that

$$S^{-1}(t) g_2(t)\, S(t) = D(t) \tag{17}$$

where

$$D(t) = \text{diagonal}\,(-\,\omega_k{}^2(t)).$$

Equation (17) implies that $\omega_k(t) \in C^1$ on $[0, T]$ for $k = 1, \cdots, n$.

Using the matrix $S(t)$, we apply the transformation

$$\xi = \xi, \ \ \mathfrak{z}_1 = S\zeta_1 - g_2{}^{-1} g_1 \xi, \text{ and } \ \mathfrak{z}_2 = S\zeta_2$$

to system (16) and obtain

$$d\xi/dt = (f_1 - f_2 g_2{}^{-1})\xi + f_2 S\zeta_1$$
$$+ \epsilon \bar{F}(\xi, \zeta_1, \zeta_2, t, \epsilon),$$
$$\epsilon\, d\zeta_1/dt = \zeta_2 - \epsilon S^{-1}\, \frac{dS}{dt}\, \zeta_1 + \epsilon S^{-1}\, \frac{d(g_2{}^{-1} g_1 \xi)}{dt},$$
$$\epsilon\, d\zeta_2/dt = D(t)\zeta_1 - \epsilon S^{-1} dS/dt \zeta_2 \tag{18}$$
$$+ \epsilon S^{-1}[\bar{g}^{(2)}(t, \xi, \zeta_1) + g_3\,(t)(S\zeta_2 + dy_0/dt)$$
$$- d^2y_0/dt^2 + \hat{g}(x_0, y_0, 0)] \ + \epsilon^2 \bar{G}(\xi, \zeta_1, \zeta_2, t, \epsilon),$$
$$\xi(0) = 0, \zeta_1(0) = \epsilon^{\nu-1} S^{-1}(0)\beta, \zeta_2(0) = \epsilon^{\nu-1} S^{-1}(0)\gamma,$$

where $\bar{F}(\xi, \zeta_1, \zeta_2, t, \epsilon) =$

$$F(\xi, S\zeta_1 - g_2{}^{-1} g_1 \xi, S\zeta_2, t, \epsilon), \text{ etc.}$$

Let $(V)_i$ denote the ith component of the vector V. We now apply the additional transformation,

$$
\begin{aligned}
\theta_k(t) &= 1/\epsilon \int_0^t \omega_k(s)\,ds + B_k(t) \\
(\zeta_1)_k &= A_k(t)\sin(\theta_k(t)) \qquad (19)\\
(\zeta_2)_k &= \omega_k(t)A_k(t)\cos(\theta_k(t)).
\end{aligned}
$$

If we let

$$
\begin{aligned}
\epsilon\, \mathcal{G}_1(t,\xi,\zeta_1,\zeta_2,\epsilon) &= \epsilon d\zeta_1/dt - \zeta_2 \\
\epsilon\, \mathcal{G}_2(t,\xi,\zeta_1,\zeta_2,\epsilon) &= \epsilon d\zeta_2/dt - D(t)\zeta_1,
\end{aligned}
$$

system (18) is transformed into the following system of differential equations.

$$
\begin{aligned}
d\xi/dt &= (f_1 - f_2 g_2^{-1} g_1)\xi + \\
&\quad + f_2 S \operatorname{col}(A_k \sin\theta_k(t)) + \epsilon \bar{F} \\
dA_k/dt &= -\frac{d\omega_k}{dt}\cdot\frac{A_k}{\omega_k}\cos^2\theta_k + \\
&\quad + (\mathcal{G}_1)_k \sin\theta_k + \frac{1}{\omega_k}(\mathcal{G}_2)_k\cos\theta_k \qquad (20)\\
&\quad k = 1, \cdots, n, \\
dB_k/dt &= (A_k\omega_k)^{-1}\left(\frac{d\omega_k}{dt}A_k \cos\theta_k \sin\theta_k + \right.\\
&\quad \left. + (\mathcal{G}_2)_k \sin\theta_k + (\mathcal{G}_1)_k\omega_k\cos\theta_k\right) \\
&\quad k = 1, \cdots, n, \\
\xi(0) &= 0,\ A_k(0) = \epsilon^{\nu-1}[(S^{-1}(0)\beta)_k^2 \\
&\quad + \omega_k^{-2}(0)(S^{-1}(0)\gamma)_k^2]^{1/2}, \\
&\quad \text{and } 0 \leqq B_k(0) \leqq 2\pi.
\end{aligned}
$$

To apply Theorem 2 to system (20), we need to change time scales by letting $\epsilon\tau = t$. In an expression of the form $f_1(t) - f_2(t)g_2^{-1}(t)g_1(t)$, we let $t = \alpha_1(\epsilon)\tau$ where $\alpha_1(\epsilon) = \epsilon$. In terms containing $\theta_k(t)$, we convert to τ,

$$
\begin{aligned}
\theta_k(\epsilon\tau) &= 1/\epsilon \int_0^{\epsilon\tau} \omega_k(s)\,ds + B_k(\epsilon\tau) \\
&= \int_0^{\tau} \omega_k(\epsilon s)\,ds + \tilde{B}_k(\tau) = \tilde{\theta}_k(\tau),
\end{aligned}
$$

where

$$\tilde{B}_k(\tau) = B_k(\epsilon\tau), \text{ etc.}$$

To define the domain G of Theorem 2, we examine the linear differential equations

$$\begin{aligned} dz_k/dt &= 1/2 \left[-\frac{d\omega_k}{dt}\omega_k^{-1} + \left[S^{-1}\left(-2\frac{dS}{dt} + \right.\right.\right. \\ (21) \qquad &\left.\left.\left. + g_2^{-1}g_1f_2S + g_3S\right)e_k\right]_{k}, \right] z_k, \\ z_k(0) &= A_k(0), \; k = 1, \cdots, n, \end{aligned}$$

where e_k is the n-vector with 1 in the kth component and zeros elsewhere. Let

$$(22) \qquad \sigma = \sup_{t \in [0,T]} z_k(t) \qquad k = 1, \cdots, n$$

and

$$(23) \qquad \mu = \inf_{t \in [0,T]} z_k(t), \qquad k = 1, \cdots, n,$$

where we note that $\mu > 0$.

Define

$$\begin{aligned} G = \{(\xi, \tilde{A}, \tilde{B}) \in R^{m+2n} | \, |\xi| < \mu/2, \\ 1/2\,\mu < (\tilde{A})_k < \sigma + \mu/2, |(\tilde{B})_k - B_k(0)| < \mu/2\}. \end{aligned}$$

To determine the "averaged problem" associated with system (2), we first note that because $(\zeta_1)_k^2 + (\zeta_2)_k^2\omega_k^2(t) = A_k^2$, $\bar{F}$ and $\bar{G}$ are uniformly bounded for $(\xi, A, B) \in G$. Hence, the terms arising from $\bar{F}$ and $\bar{G}$ can be set equal to zero in the "averaged problem". This simplifies $\mathcal{G}_1$ and $\mathcal{G}_2$ in system (20) to

$$\begin{aligned} \mathcal{G}_1 &= S^{-1}\left(-\frac{dS}{dt} \cdot \operatorname{col}(A_k \sin\theta_k) \right. \\ &\quad + \frac{d(g_2^{-1}g_1)}{dt}\xi + \\ (24) \qquad &\quad + g_2^{-1}g_1[(f_1 - f_2g_2^{-1}g_1)\xi + f_2S \cdot \\ &\quad \left. \operatorname{col}(A_k \sin\theta_k)] \right) \\ \mathcal{G}_2 &= S^{-1}[-dS/dt \cdot \operatorname{col}(\omega_kA_k\cos\theta_k) \\ &\quad + \bar{g}^{(2)} + g_3(S \cdot \operatorname{col}(\omega_kA_k\cos\theta_k) \\ &\quad + dy_0/dt) - d^2y_0/dt^2 + \hat{g}(x_0, y_0, 0)]. \end{aligned}$$

To use integral averaging, we note that terms involving only one trigonometric function like $\sin(\int_0^\tau \omega_k(\epsilon s)\,ds + \tilde{B}_k)$ average to zero after multiplying by $\omega_k(\epsilon\tau)/\omega_k(\alpha_1(\epsilon)\tau)$ and applying Lemma 1. Terms involving products of $\sin\tilde{\theta}_k(\tau)$ and $\cos\tilde{\theta}_j(\tau)$ can be written as a sum of $\sin(\tilde{\theta}_k(\tau) + \tilde{\theta}_j(\tau))$ and $\sin(\tilde{\theta}_k(\tau) - \tilde{\theta}_j(\tau))$. Multiplying by $(\omega_k(\epsilon\tau) + \omega_j(\epsilon\tau))(\omega_k(\alpha_1\tau) + \omega_j(\alpha_1\tau))^{-1}$ and, if $k \neq j$ by $(\omega_k(\epsilon\tau) - \omega_j(\epsilon\tau))\cdot(\omega_k(\alpha_1\tau) - \omega_j(\alpha_1\tau))^{-1}$, respectively, we find by Lemma 1 that the average is zero. Products of $\sin\tilde{\theta}_k(\tau)$ and $\sin\tilde{\theta}_j(\tau)$ in the same manner average to zero if $k \neq j$. If $k = j$, $\sin^2\tilde{\theta}_k = 1/2(1 - \cos(2\tilde{\theta}_k))$, which averages to 1/2. The same procedure works for products of $\cos\tilde{\theta}_k(\tau)$ and $\cos\tilde{\theta}_j(t)$ with the average of $\cos^2\theta_k(\tau)$ being 1/2. Using this information, we note that from System (24)

$(\mathcal{G}_1)_k \sin\tilde{\theta}_k$ averages to

$$\left[S^{-1}\left(-\frac{dS}{dt}e_k\frac{A_k}{2} + g_2{}^{-1}g_1f_2Se_k\frac{A_k}{2}\right)\right]_k,$$

$(\mathcal{G}_1)_k \cos\tilde{\theta}_k$ averages to zero,

$(\mathcal{G}_2 - S^{-1}g^{(2)})_k \sin\tilde{\theta}_k$ averages to zero,

and

$(\mathcal{G}_2 - S^{-1}g^{(2)})_k \cos\tilde{\theta}_k$ averages to

$$\left[S^{-1}\left(-\frac{dS}{dt}\cdot e_k\frac{\omega_k A_k}{2} + g_3S\cdot e_k\frac{\omega_k A_k}{2}\right)\right]_k.$$

At this point, the value of ν becomes important. If $\nu = 2$, the terms quadratic in $\{A_k\}$ are multiplied by ϵ^2 and can be set equal to zero in the "averaged problem". If $\nu = 1$, these terms are only multiplied by ϵ; and H 6 must be applied. Terms quadratic in $\{A_k\}$ are multiplied by products of three trigonometric functions; such products can be written as sums of either sines or cosines with arguments that are either $\tilde{\theta}_i(\tau) + \tilde{\theta}_j(t) + \tilde{\theta}_k(t)$ or $\tilde{\theta}_i(\tau) + \tilde{\theta}_j(t) - \tilde{\theta}_k(t)$ for some $i, j, k = 1, \cdots, n$.

The first type averages to zero after multiplying by

$$\frac{\omega_i(\epsilon\tau) + \omega_j(\epsilon\tau) + \omega_k(\epsilon\tau)}{\omega_i(\alpha_1\tau) + \omega_j(\alpha_1\tau) + \omega_k(\alpha_1\tau)}.$$

By H 6, $\omega_i(\epsilon\tau) + \omega_j(\epsilon\tau) - \omega_k(\epsilon\tau) \neq 0$; hence, with the same procedure we can average the other terms to zero. Therefore, the quadratic terms in $\{A_k\}$ are not present in the "averaged problem".

The "averaged problem" for ξ, therefore, is

$$
\begin{aligned}
d\xi_0/d\tau &= \epsilon\,[(f_1(\epsilon\tau) - f_2(\epsilon\tau)g_2^{-1}(\epsilon\tau) \\
&\qquad g_1(\epsilon\tau)]\,\xi_0 \\
\xi_0(0) &= 0.
\end{aligned} \tag{25}
$$

System (25) has $\xi_0(\tau) = 0$ as its solution; this information further simplifies the "averaged problem" for $\tilde{A}$ and $\tilde{B}$.

$$
\begin{aligned}
d\tilde{A}_0/d\tau &= \epsilon/2 \text{ diagonal } \Bigg(-\frac{d\omega_k}{dt}(\epsilon\tau)\,\omega_k(\epsilon\tau)^{-1} \\
&\quad + \Bigg[S^{-1}(\epsilon\tau) \Bigg(-2\frac{dS}{dt}(\epsilon\tau) + \\
&\quad + g_2^{-1} g_1 f_2 S(\epsilon\tau) + g_3 S(\epsilon\tau)\Bigg) e_k \Bigg]_k \Bigg) \cdot \tilde{A}_0, \\
(\tilde{A}_0)_k &= A_k(0),\ k = 1, \cdots, n, \\
d\tilde{B}_0/dt &= 0, \\
(\tilde{B}_0)_k &= B_k(0),\ k = 1, \cdots, n.
\end{aligned} \tag{26}
$$

Comparing the solution of system (26), with that of system (21); if $Q = \mu/4$, the solution of system (25) and (26) and a Q-neighborhood of the solution is contained in G for all $0 < \epsilon \leqq 1$ for $0 \leqq \tau \leqq T/\epsilon$. Hence, Theorem 2 applies.

In Theorem 2, if we set $\delta = \mu/4$, we find that

$$
\begin{aligned}
&|\xi(\tau)| \leqq \mu/4 \\
&|\tilde{A}_k(\tau) - z_k(\epsilon\tau)| \leqq \mu/4, \\
&\qquad k = 1, \cdots, n, \\
&|\tilde{B}_k(\tau) - \tilde{B}_k(0)| \leqq \mu/4, \\
&\qquad k = 1, \cdots, n.
\end{aligned} \tag{27}
$$

for all $0 < \epsilon \leqq \epsilon^*$ for $0 \leqq \tau \leqq T/\epsilon$ where ϵ^* is sufficiently small. The middle n inequalities imply that

$$
|A_k(t)| \leqq |z_k(t)| + \mu/4 \leqq \sigma + \mu/4. \tag{28}
$$

When we convert back to the origin time scale t and back to the $(\xi, \mathfrak{z}_1, \mathfrak{z}_2)$ system, we see that inequalities (27) and (28) prove Theorem 1.

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