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MULTIPLE SUBDIVISIONS OF E^n

GERALD C. MARLEY

1. **Preliminaries.** The notion of an asymptotic subdivision of Euclidean n-space, E^n, was introduced by Groemer [1]. His work was suggested by a result of Schmidt [4]. An asymptotic subdivision may be thought of as a configuration of sets in which the ratio of that part of E^n which is not covered by exactly one set of the configuration to the whole of E^n is 0. In particular, packings and coverings of density 1 are asymptotic subdivisions. (For further discussion of these ideas, the interested reader is referred to Groemer [2] and Rogers [3].) A stronger requirement on this configuration of sets is that almost every point of E^n, lies in exactly one set from the configuration. Such a configuration is called a subdivision of E^n. Under very general hypotheses, Groemer showed that from a given asymptotic subdivision of E^n one can obtain a related subdivision of E^n. In this paper, these ideas will be generalized to the case in which points of E^n lie in exactly k sets (where k is an arbitrary integer greater than 1). We shall refer to such as "k-subdivisions".

We shall consider sequences of sets rather than families of sets in discussing k-subdivisions. This is necessary because some sets may occur repeatedly and should be considered as different. The order in which the sets occur in the sequence is immaterial. (Generalization to uncountable "sequences" of sets may be made, but we shall not do that here.)

If only a finite number of these sets lie in any bounded region of E^n, we say the sets do not accumulate; otherwise, we say the sets *accumulate*.

If $\{X_i\}$, $X_i \subset E^n$, is a (countably) infinite sequence, we shall denote a subsequence of $\{X_i\}$ by $\{X_{ij}\}$ rather than by $\{X_{i_j}\}$. This notation will be employed for the remainder of this paper.

If X is a Lebesgue measurable set in E^n, then $m(X)$ denotes the *measure* of X and $d(X)$ denotes the *diameter* of X.

A sequence S of bounded sets in E^n each of which has positive measure is said to be *compact in the topology of symmetric differences* if to every infinite sequence $\{X_j\}$ in S there exists a subsequence $\{X_{ji}\}$

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and an $X \in S$ such that

$$\lim_{i \to \infty} m((X \cup X_{ji}) - (X \cap X_{ji})) = 0.$$

This will be written as $X_{ji} \to X$.

Some elementary properties of convergent sequences will be employed. In particular, if $\{A_i\}$, $\{B_i\}$ are sequences of measurable sets with

$$A_i \to A \quad \text{and} \quad B_i \to B,$$

then

$$\lim_{i \to \infty} m(A_i) = m(A),$$

$$A_i \cup B_i \to A \cup B, \qquad A_i \cap B_i \to A \cap B,$$

and for any bounded measurable set C

$$C - A_i \to C - A.$$

Let S be a class of measurable sets of E^n for which two positive numbers a and b exist so that for all $X \in S$

$$a \leqq m(X) \quad \text{and} \quad d(X) \leqq b. \tag{1}$$

We consider k-subdivisions of E^n which are translates of sets from S.

2. Definitions.

DEFINITION 1. A sequence $\{M_r\}$ of sets M_r is called a *k-subdivision of E^n generated by* S (briefly, a k-subdivision) whenever the following conditions are satisfied:

(i) Each M_r is a translate of some $X \in S$.

(ii) If $U_k(M_r)$ denotes the set of all points of E^n which lie in exactly k of the sets M_r, then

$$m(E^n - U_k(M_r)) = 0.$$

That is, almost every point of E^n lies in exactly k of the sets M_r. The number k is called the *order* of the subdivision.

We shall assume throughout the remainder of this paper that k denotes the order of the subdivision. In addition, n will denote the dimension of the space.

DEFINITION 2. A sequence $\{M_r\}$ of sets M_r is called an *asymptotic k-subdivision of E^n generated by* S (briefly, an asymptotic k-subdivision) whenever the following conditions are satisfied:

(i) Each M_r is a translate of some $X \in S$.

(ii′) If W_λ is the closed cube with side length λ, faces perpendicular to the coordinate axes and center at the origin, then

$$\liminf_{\lambda\to\infty} m(W_\lambda - U_k(M_r))\lambda^{-n} = 0,$$

where $U_k(M_r)$ again denotes the set of all points of E^n which lie in exactly k of the sets M_r.

It is clear that (ii′) follows directly from (ii); i.e., every k-subdivision is an asymptotic k-subdivision. The converse is not true, however, as we can easily see by adding an arbitrary (finite) number of sets which are translates of sets from S, or by removing an arbitrary (finite) number of sets from a k-subdivision $\{M_r\}$. In neither case is the property of being an asymptotic k-subdivision affected. From the inequalities (1) and condition (ii) of Definition 1, it is readily seen that the sets do not accumulate (and the sequence $\{M_r\}$ is countable even if we permit uncountable sequences) in the case of a k-subdivision. In the case of an asymptotic k-subdivision, however, the sets may accumulate. (If we permit uncountable sequences, then there may be uncountably many sets in the asymptotic k-subdivision.)

Condition (ii′) of Definition 2 can be weakened without losing the essential properties of an asymptotic k-subdivision. It suffices to require that there be arbitrarily large cubes $W_{\lambda(i)}$ $(i = 1, 2, 3, \cdots)$ and corresponding sequences of sets $\{M_j^i\}$ such that

$$\lim_{i\to\infty} m(W_{\lambda(i)} - U_k(M_j^i))(\lambda(i))^{-n} = 0.$$

From a double sequence $\{M_j^i\}$ satisfying this condition one can always select a subsequence $\{M_r\}$ that satisfies condition (ii′). In all that follows, W_λ will denote a cube in E^n as defined in Definition 2.

3. **Statement of the main theorem.**

THEOREM. *Let S be a sequence of measurable sets of E^n for which the inequalities* (1) *are satisfied, and let* S *be compact in the topology of symmetric differences. If there exists an asymptotic k-subdivision of E^n generated by* S, *then there exists a k-subdivision of E^n generated by* S.

4. **Proof of the main theorem.** Before the proof of this theorem is given, two lemmas will be established. The first of these was proved by Groemer [1], and applies unchanged here since the assumption was only that S is compact in the topology of symmetric differences and this lemma is not concerned with subdivisions. The compactness of S is not used in the proof of the second lemma.

LEMMA 2.1. *For each i ($i = 1, 2, 3, \cdots$) let $X_i \in S$ and $p_i \in E^n$ be given. Let*

$$T_i = X_i + p_i \tag{2}$$

(i.e., T_i is the set of all points of the form $x_i + p_i$ with $x_i \in X_i$). If all the T_i lie within a bounded region and S is compact in the topology of symmetric differences then there is a subsequence $\{T_{iv}\}$ of $\{T_i\}$ and a set $T = X + p$ with $X \in S$ and $p \in E^n$ for which $T_{iv} \to T$.

LEMMA 2.2. *Suppose there exists an asymptotic k-subdivision of E^n generated by S. Let $\{\epsilon_i\}$ be a decreasing null sequence. Then for every i there exists a number $m(i)$ and a finite sequence of $m(i)$ sets*

$$S_i = \{M_j{}^i\} = \{M_1{}^i, M_2{}^i, \cdots, M^{\,i}_{m(i)}\}$$

such that the $M_j{}^i$ are translates of sets from S and a number $\lambda(i)$ (where $\lambda(i) \to \infty$ as $i \to \infty$) such that

$$m[W_{\lambda(i)} - U_k(M_j{}^i)] < \epsilon_i. \tag{3}$$

PROOF. Given i and an arbitrarily large N, it suffices to prove that there exists a $\lambda(i)$ with

$$\lambda(i) > N \tag{4}$$

and sets $M_j{}^i$ ($j = 1, 2, 3, \cdots, m(i)$) which fulfill the conclusions of the lemma and in particular, (3). Now, there exists a λ such that

$$\lambda/N - \lambda/(N + 1) \geqq 1, \tag{5}$$

and (by Definition 2(ii′)) such that for the given asymptotic subdivision $\{M_j\}$

$$m[W_\lambda - U_k(M_j)]\lambda^{-n} < \epsilon_i(N + 1)^{-n}. \tag{6}$$

By (5) there exists an integer g with $\lambda/N > g \geqq \lambda/(N + 1)$.

By setting $\lambda(i) = \lambda/g$ we obtain

$$N < \lambda(i) \leqq N + 1. \tag{7}$$

Thus, $\lambda(i) > N$.

Divide W_λ into g^n cubes, each with sides of length $\lambda(i)$, by a family of planes perpendicular to the coordinate axes. For at least one of these cubes, say for $W^1_{\lambda(i)}$, we have that

$$(8) \qquad m[W^1_{\lambda(i)} - U_k(M_j)](\lambda(i))^{-n} < \epsilon_i(N+1)^{-n},$$

since otherwise by adding the g^n inequalities which correspond to (8) (with $\geqq$ instead of $<$), we obtain a contradiction to (6). If σ is that translation for which $\sigma(W^1_{\lambda(i)}) = W_{\lambda(i)}$, then take as the corresponding sequence of sets $\{M_j^i\}$ those from $\{\sigma(M_j)\}$ which have points in common with $W_{\lambda(i)}$. From (7) and (8), we obtain

$$(3') \qquad m[W_{\lambda(i)} - U_k(M_j^i)] < \epsilon_i.$$

It remains to be shown that for each i, the sequence $\{M_j^i\}$ can be replaced by a finite subsequence. Arrange all increasing k-tuples of positive integers in a sequence. If $(j_1, j_2, \cdots, j_k)$ is the pth term of this sequence, let

$$E_p = (M^i_{j_1} \cap M^i_{j_2} \cap \cdots \cap M^i_{j_k}) \cap U_k(M_j^i).$$

The sets E_p $(p = 1, 2, \cdots)$ do not necessarily constitute a partition of $U_k(M_j^i)$ since the M_j^i are not necessarily distinct so that we may have $E_p = E_q$ with $p \neq q$. However, they cover $U_k(M_j^i)$ and so

$$\sum_{p=1}^{\infty} m(W_{\lambda(i)} \cap E_p) \geqq m[W_{\lambda(i)} \cap U_k(M_j^i)].$$

From (3′) we have $m[W_{\lambda(i)} \cap U_k(M_j^i)] > m(W_{\lambda(i)}) - \epsilon_i$ so that

$$\sum_{p=1}^{\infty} m(W_{\lambda(i)} \cap E_p) > m(W_{\lambda(i)}) - \epsilon_i.$$

Hence,

$$\sum_{p=1}^{N} m(W_{\lambda(i)} \cap E_p) > m(W_{\lambda(i)}) - \epsilon_i$$

for some integer N. Let $m(i)$ be the largest integer that appears among the first N k-tuples, and take only the M_j^i for which $j = 1, 2, \cdots, m(i)$. Then (3) is satisfied and the lemma is proved.

In order to prove the main theorem, let $\{\epsilon_i\}$ be an arbitrary but fixed decreasing null sequence. In accordance with Lemma 2.2, corresponding to each ϵ_i, there is a finite sequence $S_i = \{M_j^i\}$ of $m(i)$ translates M_j^i of members of S. For a given i, we arrange the sets M_j^i in such a way that M_u^i precedes M_v^i in the sequence if there is a λ for which $M_u^i \cap W_\lambda \neq \emptyset$, and $M_v^i \cap W_\lambda = \emptyset$. If no such λ exists, M_u^i and M_v^i can be arranged in any order.

For a given j, all the $M_j{}^i$ lie in a bounded region. Otherwise, by (1) and the above ordering, there exists an arbitrarily large λ and an index i so that W_λ contains at most the sets $M_1{}^i, M_2{}^i, \cdots, M_j{}^i$. But this contradicts (3). From (1) and (3) it also follows that the number, $m(i)$, of $M_j{}^i$ as a function of i is not bounded. The hypotheses of Lemma 2.1 are therefore satisfied for the sequence $\{M_j{}^i\}$. Hence, there is a subsequence $\{S_{iv}\}$ of the sequence $\{S_i\}$ such that $M_1{}^{iv} \to N_1$ where N_1 denotes a set of the form $X_1 + p_1$ with $X_1 \in S$ and $p_1 \in E^n$. For a suitable subsequence $\{S_{ivu}\}$ of $\{S_{iv}\}$, $\{M_2{}^{ivu}\}$ also converges, say to N_2, where N_2 is a set of the form $X_2 + p_2$ with $X_2 \in S$ and $p_2 \in E^n$. By means of the well-known diagonalization process we obtain a subsequence $\{T_r\}$ of $\{S_i\}$ with $T_r = \{A_j{}^r\}$ $(r = 1, 2, \cdots)$, and for every j

$$A_j{}^r \to N_j. \tag{9}$$

We shall now show that $\{N_j\}$ produces a k-subdivision. It is obvious that condition (i) of Definition 1 is satisfied so that it suffices to show that condition (ii) is also satisfied.

By (9) we have that for all j, i with $j \neq i$

$$m(N_j \cap N_i) = \lim_{r \to \infty} m(A_j{}^r \cap A_i{}^r). \tag{10}$$

Let an arbitrarily large λ be given. For sufficiently large r all the $A_j{}^r$ which have points in common with W_λ lie in $W_{\lambda+2b}$ by (1). For each such $A_j{}^r$ let $B_j{}^r$ be that subregion of $W_{\lambda+2b} \cap A_j{}^r$ which is contained in more than k bodies of $\{A_j{}^r\}$. By (1) and (3)

$$m(A_j{}^r - B_j{}^r) \geqq a - \epsilon_i \qquad (i \geqq r)$$

for each $A_j{}^r$, so that the number of the $A_j{}^r$ which have points in common with W_λ is below a bound L which is dependent only upon λ. Suppose there is a subsequence $\{N_{ju}\}$ of $\{N_j\}$ such that for each u, $N_{ju} \cap W_\lambda \neq \emptyset$. Now, for each u, $A_{ju}^r \cap W_\lambda \to N_{ju} \cap W_\lambda \neq \emptyset$, so that for sufficiently large r, $A_{ju}^r \cap W_\lambda \neq \emptyset$. But, by the above, there are no more than L values of u to which there corresponds an A_{ju}^r which intersects W_λ. Hence, there are no more than L values of u to which there corresponds an N_{ju} which intersects W_λ. Thus, the subsequence $\{N_{ju}\}$ is finite. We can therefore assume that L is chosen so large that not more than L of the sets N_j intersect W_λ. For fixed r, we order the N_j and the $A_j{}^r$ as before. From (3) and properties of measure and convergence, we conclude

$$(11) \qquad m(E^n - U_k(N_j)) \leqq \sum_{i=1}^{\infty} m(N_i - U_k(N_j)).$$

Given any i we define the sets

$$(12) \qquad \begin{aligned} B_i &= N_i - U_k(N_j), \\ C_i &= B_i \cap \{x\colon x \text{ lies in at least one } A_j^r\}, \\ D_i &= B_i \cap \{x\colon x \text{ does not lie in any } A_j^r\}. \end{aligned}$$

Then for every i,

$$(13) \qquad C_i \cap D_i = \emptyset \quad \text{and} \quad C_i \cup D_i = B_i.$$

Since $D_i \subset N_i - U_k(A_j^r)$ we have by (3) that

$$(14) \qquad m(D_i) = 0.$$

Suppose now that $x \in C_i$. If there is a number R such that for every $r > R$, $x \notin U_k(A_j^r)$, then we obtain (as above)

$$m(C_i \cap \{x\colon x \notin U_k(A_j^r) \text{ for all } r > R\}) = 0.$$

Suppose on the other hand that for every number R there exists an $r > R$ such that $x \in U_k(A_j^r)$. By (9), for every $\epsilon > 0$ there is a number R for which $m((A_j^r - N_j) \cup (N_j - A_j^r)) < \epsilon$ for every $r > R$. Take $\epsilon < \epsilon_r/(k + 1)$ and let the corresponding R be fixed. Let $r > R$ be given and let $A_{j1}^r, A_{j2}^r, \cdots, A_{jk}^r$ be the k bodies in which x lies. Then $x \in Q = \bigcup_{v=1}^{k} (A_{jv}^r - N_{jv})$ so that $C_i \cap \{x : x \in U_k(A_j^r)\} \subset Q$. Thus

$$m(C_i \cap \{x : x \in U_k(A_j^r)\}) \leqq \sum_{v=1}^{k} m(A_{jv}^r - N_{jv}) < \epsilon_r.$$

By letting $r \to \infty$ we see that $m(C_i) = 0$.

This condition along with (13) and (14) establishes that for given i, $m(B_i) = 0$. Hence by (12) and (11) condition (ii) of Definition 1 is satisfied and $\{N_j\}$ is a k-subdivision of E^n.

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California State University, Fullerton, California 92631