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ON DECOMPOSITIONS OF $E(G)$[1]

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1. **Introduction.** The theory of near rings has been studied in some detail by several authors. In a paper that briefly summarized the elementary theory of near rings Berman and Silverman [1] generalized the Peirce Decomposition Theorem to obtain a decomposition theorem for near rings. Fröhlich [2], [3] studied the class of distributively generated near rings, and Malone [4] has emphasized the class of endomorphism near rings.

For an arbitrary group G the set of endomorphisms of G, denoted by $\mathrm{End}(G)$, form a distributive generating set (d.g. set) for the endomorphism near ring $E(G)$. The convention of writing functions on the right (i.e., $f: G \to G$ sends g to $(g)f$) makes $E(G)$ a left near ring. Therefore, all of the results in this paper are stated for left near rings.

The decomposition of Berman and Silverman provides a starting point for the investigation of two basic problems related to endomorphism near rings. First, by examining the decomposition theorem and using a construction technique of Malone and Lyons [6] one is able to construct classes of groups for which the endomorphism near ring decomposes in a predictable manner. Secondly, one is able to supply a sufficient condition on the relationship between groups G and H so that $E(H)$ embeds in $E(G)$. This provides an embedding result for endomorphism near rings that parallels the results of Malone and Heatherly [5] for the embedding of transformation near rings.

2. **The decomposition.** The statement of the Berman and Silverman decomposition theorem is

THEOREM 2.1 [1, p. 27]. *Let e be an idempotent in the near ring R. For each $r \in R$, $r = er + (-er + r) = (r - er) + er$. Thus $R = A_e + M_e = M_e + A_e$ where $A_e = \{r - er : r \in R\} = \{t \in R : et = 0\}$, $M_e = \{er : r \in R\}$, and $A_e \cap M_e = \{0\}$.*

When no confusion can arise the summands A_e and M_e will be designated by A and M respectively.

Theorem 2.1 says that the group structure of any near ring with

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nontrivial idempotent is a semidirect sum with normal summand A. The group morphism $f: R \to M$ via $(r)f = er$ is associated with the sum $R = A + M$ and motivates the following characteristics of the summand A.

THEOREM 2.2. *The following statements are equivalent.*
(1) $f: R \to M$ *via* $(r)f = er$ *is a near ring morphism.*
(2) A *is an ideal.*
(3) $MA = \{0\}$.
(4) $eres = ers$ *for each* $r, s \in R$.

PROOF. (1)$\Rightarrow$(2) is obvious as is (4)$\Rightarrow$(1). It remains to be shown that (2)$\Rightarrow$(3)$\Rightarrow$(4).
(2)$\Rightarrow$(3) Since $M = eR$, $MA = eRA \subseteq eA = (0)$.
(3)$\Rightarrow$(4) Let $r, s \in R$, then, $er \in M$, $s - es \in A$, and $0 = er(s - es) = ers - eres$. □

The equivalence of (1) and (2) guarantees that if A is an ideal M is a homomorphic image of R. Thus M inherits all structural properties that are preserved by homomorphisms.

If the near ring R contains a right identity one obtains another equivalence condition that A be an ideal.

COROLLARY 2.3. *Let R be a near ring with right identity* 1. *A is an ideal if and only if e is a right identity for M.*

PROOF. ($\Rightarrow$) Let A be an ideal. Then for any $r \in R$, $ere = ere(1) = er(1) = er$ by equivalence (4). Thus, e is a right identity for M.
($\Leftarrow$) Suppose that e is a right identity for M and let $r, s \in R$. Then $eres = (ere)s = (er)s = ers$ and equivalence (4) provides the result. □

It is clear that e is a left identity for M. Thus, if R has a right identity, A is an ideal if and only if M has an identity. The condition that M have an identity is not as restrictive as it may seem. For example, consider the near ring $E(G)$ with idempotent e. If the image of G under e, $(G)e$, is a fully invariant subgroup of G, then e is a right identity for M and A is an ideal.

3. **D. g. near rings.** It is convenient at this point to make a definition.

DEFINITION 3.1. Let R be a distributively generated (d.g.) near ring. The set $S \subseteq R$ is called a *d.g. set* provided that S is a subsemigroup of $(R, \cdot)$ and that S additively generates $(R, +)$.

Throughout this section the near ring R will be d.g. with d.g. set S, idempotent e, and decomposition $R = A + M$. Both M and A are subnear rings of R and have additive generating sets $\{es : s \in S\}$ and $\{s - es : s \in S\}^M = \{m + (s - es) - m : m \in M, s \in S\}$ respectively [**6**, Theorem 2.3]. The problem of constructing a d.g. near ring R from a d.g. set S and an idempotent e reduces to the construction of the summands M and A.

Conditions under which the additive generating sets are in fact d.g. sets follow.

THEOREM 3.2. *A is an ideal if and only if M is d.g. with d.g. set eS and eses′ = ess′ for each s, s′ ∈ S.*

PROOF. ($\Rightarrow$) Statements (1) and (2) of Theorem 2.2 imply that M is d.g. with d.g. set eS. Statements (2) and (4) conclude the proof in this direction.

($\Leftarrow$) This implication will be proved by showing that $eret = ert$ for each $r, t \in R$. For $r, t \in R$, $r = \sum_{i=1}^{q} n_i s_i$ and $t = \sum_{j=1}^{p} n_j' s_j'$ where $n_i, n_j' \in Z$ and $s_i, s_j' \in S$ for $i = 1, 2, \cdots, q$ and $j = 1, 2, \cdots, p$. It is clear that for any $x \in R$, $s \in S$ and $n \in Z$

$$(*) \qquad x(ns) = n(xs) = (nx)s.$$

It follows from equation (*) and left distributivity that

$$eret = ere \sum_{j=1}^{p} n_j' s_j' = \sum_{j=1}^{p} ere(n_j' s_j')$$

$$= \sum_{j=1}^{p} n_j'(eres_j') = \sum_{j=1}^{p} n_j' \left(e \left(\sum_{i=1}^{q} n_i s_i \right) es_j' \right).$$

But since eS is a d.g. set for M and every element of R is left distributive

$$\sum_{j=1}^{p} n_j' \left(e \left(\sum_{i=1}^{q} n_i s_i \right) es_j' \right) = \sum_{j=1}^{p} n_j' \left(\left(\sum_{i=1}^{q} e(n_i s_i) \right) es_j' \right)$$

$$= \sum_{j=1}^{p} n_j' \left(\left(\sum_{i=1}^{q} n_i(es_i) \right) es_j' \right)$$

$$= \sum_{j=1}^{p} n_j' \left(\sum_{i=1}^{q} n_i(es_i es_j') \right).$$

By the hypothesis of the theorem, the fact that S is a d.g. set for R, and the validity of equation (*),

$$\sum_{j=1}^{p} n_j'\left(\sum_{i=1}^{q} n_i es_i es_j'\right) = \sum_{j=1}^{p} n_j'\left(\sum_{i=1}^{q} n_i es_i s_j'\right)$$

$$= \sum_{j=1}^{p} n_j'\left(\left(\sum_{i=1}^{q} n_i es_i\right) s_j'\right) = \sum_{j=1}^{p} n_j'\left(e\left(\sum_{i=1}^{q} n_i s_i\right) s_j'\right)$$

$$= \sum_{j=1}^{p} n_j'(ers_j') = \sum_{j=1}^{p} er(n_j's_j') = er\sum_{j=1}^{p} n_j's_j' = ert. \ \square$$

Theorem 3.3. *If $AM = \{0\}$ then A is d.g. with d.g. set*

$$\{s - es : s \in S\}^M.$$

Proof. The set $S' = \{s - es : s \in S\}^M$ is an additive generating set for A [**6**, Theorem 2.3]. To be a d.g. set for A each element of S' must distribute from the right over A and S' must be a multiplicative semigroup of A. Let $a, b \in A$ and $er + (s - es) - er = (s - es)^{er} \in S'$. Then

$$\begin{aligned}(a + b)(s - es)^{er} &= (a + b)er + (a + b)(s - es) - (a + b)er \\ &= (a + b)s - (a + b)es = as + bs \\ &= (aer + as - aes - aer) + (ber + bs - bes - ber) \\ &= a(s - es)^{er} + b(s - es)^{er}\end{aligned}$$

so that the elements of S' are right distributive over A. Now let $(s - es)^{er}, (s' - es')^{et} \in S'$ and consider

$$\begin{aligned}(s - es)^{er}(s' - rs')^{et} &= (s - es)^{er}et \\ &\quad + (s - es)^{er}(s' - es') - (s - es)^{er}et \\ &= (s - es)^{er}s' - (s - es)^{er}es' = (ss' - ess')^{ers'}\end{aligned}$$

which is in the generating set. $\square$

The $AM = \{0\}$ condition is not necessary. The following example, which is due to Willhite [7], demonstrates this fact. Let the additive structure for the near ring R be the dihedral group of order eight. The addition table is included for reference.

$+$	0	a	$2a$	$3a$	b	$a+b$	$2a+b$	$3a+b$
0	0	a	$2a$	$3a$	b	$a+b$	$2a+b$	$3a+b$
a	a	$2a$	$3a$	0	$a+b$	$2a+b$	$3a+b$	b
$2a$	$2a$	$3a$	0	a	$2a+b$	$3a+b$	b	$a+b$
$3a$	$3a$	0	a	$2a$	$3a+b$	b	$a+b$	$2a+b$
b	b	$3a+b$	$2a+b$	$a+b$	0	$3a$	$2a$	a
$a+b$	$a+b$	b	$3a+b$	$2a+b$	a	0	$3a$	$2a$
$2a+b$	$2a+b$	$a+b$	b	$3a+b$	$2a$	a	0	$3a$
$3a+b$	$3a+b$	$2a+b$	$a+b$	b	$3a$	$2a$	a	0

The multiplication table that follows (Table 6(4), p. 34–35 of [7]) defines the unique d.g. near ring with identity on the dihedral group of order eight.

$\cdot$	0	a	$2a$	$3a$	b	$a+b$	$2a+b$	$3a+b$
0	0	0	0	0	0	0	0	0
a	0	a	$2a$	$3a$	b	$a+b$	$2a+b$	$3a+b$
$2a$	0	$2a$	0	$2a$	0	$2a$	0	$2a$
$3a$	0	$3a$	$2a$	a	b	$3a+b$	$2a+b$	$a+b$
b	0	b	0	b	b	0	b	0
$a+b$	0	$a+b$	0	$a+b$	0	$a+b$	0	$a+b$
$2a+b$	0	$2a+b$	0	$2a+b$	b	$2a$	b	$2a$
$3a+b$	0	$3a+b$	0	$3a+b$	0	$3a+b$	0	$3a+b$

It is clear that the set $S = \{a, b\}$ forms a d.g. set for R. Let $a + b$ decompose R, then $M = \{0, a + b\} = eS$ is d.g. Furthermore, $A = \{0, b, 2a, 2a + b\}$ is d.g. with d.g. set $\{s - es : s \in S\}^M = \{b, 2a + b\}$, but $AM = \{0, 2a\}$.

If the group sum $R = A + M$ is direct then the summand M is normal and addition in R is componentwise. Conversely, if the addition in R is componentwise then the sum is direct. But, the normality of M is not enough to guarantee that M is an ideal.

Componentwise multiplication in R implies that A is an ideal and that both summands are d.g. near rings with d.g. sets as described in Theorems 3.2 and 3.3. However, componentwise multiplication in R does not imply that M is normal.

The link between componentwise addition and multiplication in R is provided by the condition that both summands are in fact ideals.

THEOREM 3.4. *R is the direct sum of ideals A and M if and only if both operations in R are componentwise.*

PROOF. ($\Rightarrow$) Suppose that both A and M are ideals, then $AM = MA = A \cap M = \{0\}$. Since $M \lhd R$, addition is componentwise. It remains to be shown that $(a_1 + m_1)(a_2 + m_2) = a_1a_2 + m_1m_2$ for each $a_1, a_2 \in A$ and $m_1, m_2 \in M$. Let $a_2 = \sum_{i=1}^{q} n_i s_i$ and $m_2 = \sum_{j=1}^{p} n_j' s_j'$

where $n_i, n_j' \in Z$ and $s_i, s_j' \in S$ for $i = 1, 2, \cdots, q$ and $j = 1, 2, \cdots, p$. Now, the left distributive law and equation(*) provide that

$$(a_1 + m_1)(a_2 + m_2) = (a_1 + m_1)a_2 + (a_1 + m_1)m_2$$

$$= \sum_{i=1}^{q} (n_i(a_1 + m_1))s_i + \sum_{j=1}^{p} (n_j'(a_1 + m_1))s_j'.$$

But, since addition is componentwise, $n(a_1 + m_1) = na_1 + nm_1$ for any $n \in Z$. Thus,

$$(a_1 + m_1)(a_2 + m_2)$$

$$= \sum_{i=1}^{q} (n_i a_1)s_i + \sum_{i=1}^{q} (n_i m_1)s_i + \sum_{j=1}^{p} (n_j' a_1)s_j' + \sum_{j=1}^{p} (n_j' m_1)s_j'.$$

Equation (*) and the left distributive property applied to the last equality give

$$(a_1+m_1)(a_2 + m_2) = a_1\left(\sum_{i=1}^{q} n_i s_i\right) + m_1\left(\sum_{i=1}^{q} n_i s_i\right)$$

$$+ a_1\left(\sum_{j=1}^{p} n_j' s_j'\right) + m_1\left(\sum_{j=1}^{p} n_j' s_j'\right)$$

$$= a_1a_2 + m_1a_2 + a_1m_2 + m_1m_2 = a_1a_2 + m_1m_2.$$

($\Leftarrow$) Suppose now that the operations in R are componentwise. Then $AM = MA = \{0\}$, so that A is an ideal. Also, the group sum is direct, so M is normal and hence a right ideal. It remains to be shown that $RM \subseteq M$. Let $r = a + m \in R$ and $m' \in M$. Then $rm' = (a + m)m' = (a + m)(0 + m') = a(0) + mm' = mm'$ which is certainly contained in M. □

If the conditions of Theorem 3.4 are satisfied then both M and A are d.g. with d.g. sets as described in Theorems 3.2 and 3.3 respectively.

4. **Applications.** For an arbitrary group G the endomorphism near ring $E(G)$ is not easily found. Specifically, d.g. sets are elusive and any known construction technique requires at least an additive generating set. However, the results of §2 and §3 provide some insight into the structure of $E(G)$ for certain groups.

Suppose, for example, that G is a semidirect sum with normal summand K. The endomorphism $e: G \to H \subseteq G$ having Ker $e = K$ with $e^2 = e$ yields a decomposition of $E(G)$. Let $S = \text{End}(G)$ so that eS is an additive generating set for M. Since $e \in S$, $eS \subseteq S$ is a multiplicative semigroup of right distributive elements and hence eS is a d.g. set for M.

If the sum is direct and the summand H is fully invariant then e is a right identity for M and A is an ideal by Corollary 2.3. Consider the slightly more general case in

THEOREM 4.1. *Let e be any idempotent in $E(G)$ such that $(G)e = H$ is fully invariant. Suppose also that if $f \in \text{End}(H)$, $f = f'|_H$ for some $f' \in E(G)$. Then $E(H)$ is isomorphic to M where e decomposes $E(G)$ into $A + M$.*

PROOF. Let $i: \text{End}(h) \to M$ via $(f)i = ef'$. Now, H is fully invariant and e fixes H elementwise, thus e is a right identity for M. It follows that i is a semigroup morphism which extends to a near ring epimorphism $i': E(H) \to M$. Suppose that $(f)i' = 0$. Then $ef' = 0$ and for $h \in H$, $(h)ef' = (h)f' = (h)f = 0$, so that f is the zero map of H. Thus i' is an isomorphism. □

In a paper by Malone and Heatherly, [5], it is shown that if H is a direct summand of G then $T_0(H)$ embeds as a direct summand in $T_0(G)$, where $T_0(G)(T_0(H))$ is the near ring of transformations from G to G (H to H) that send 0 to 0. A similar result holds for endomorphism near rings whereby $E(H)$ embeds as a direct summand in $E(G)$.

Let $G = K \oplus H$ with H fully invariant and abelian. If $e: G \to H$ is the projection map, the decomposition $E(G) = A + M$ has M in the additive center of $E(G)$ and the sum $A + M$ is direct. This fact along with Theorem 4.1 provide the following embedding result.

THEOREM 4.2. *Let H be a fully invariant abelian summand of the group G. Then $E(H)$ embeds in $E(G)$ as a direct summand.*

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