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BIFURCATION IN SINGULAR SELFADJOINT BOUNDARY VALUE PROBLEMS[1]

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1. **Introduction.** In recent years considerable progress has been made in the study of the bifurcation phenomenon associated with nonlinear perturbations of Sturm-Liouville problems. For example,

$$(1.1)\qquad \begin{aligned} Ly &= \lambda[y + f(t, y)], \\ \alpha y(0) + \beta y'(0) &= 0, \\ \gamma y(1) + \delta y'(1) &= 0, \end{aligned}$$

where $Ly = -(py')' + qy$ and f is small in the appropriate sense as $y \to 0$. A typical result is: if λ_k is an eigenvalue of the linearized problem (i.e. $f \equiv 0$), there exist solutions to the nonlinear problem (1.1) for small $y \not\equiv 0$ and $(\lambda - \lambda_k)$. The result is considered as a branching or bifurcation from the point $(\lambda_k, 0)$ relative to the subspace $\{(\lambda, y) : y = 0\}$. For example, see [1] and [2].

More recently it has been shown that, although the bifurcation phenomenon is usually considered to be a local result, it often is global in the sense that the solution pair to (1.1), (λ, y), can be extended indefinitely; i.e. $\|(\lambda, y)\| \to \infty$. Some results along this line have been established using topological techniques by Crandall and Rabinowitz [3] and Turner [4].

Our purpose here is to establish a *local* bifurcation property for a generalization of (1.1) which will include singular problems on intervals of the form $[0, \omega)$ where $\omega \leqq \infty$. The right boundary condition of (1.1) is replaced by: $y \in D$, D a Banach space. This condition is motivated by singular conditions like $D = L^\infty(0, \omega)$ or $D = L^2(0, \omega)$. We consider

$$(1.2a)\qquad L_0 y = -(ay')' + b_0 y = \lambda[y + f(t, y)],$$

$$(1.2b)\qquad m_0 y(0) - y'(0) = 0,$$

$$(1.2c)\qquad y \in D,$$

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where a, a' and b_0 are continuous real-valued functions on $[0, \omega)$, $a > 0$, D is a Banach space of real-valued functions on $(0, \omega)$, and f is appropriately small as $y \to 0$ in the D norm. We are interested in real eigenvalues and eigenfunctions for (1.2). In what follows we normally think of (1.2) as a singular problem (i.e. $\omega = \infty$, or $a(w^-)$ and/or $b_0(\omega^-)$ undefined, or $a(\omega^-) = 0$); however, the regular Sturm-Liouville problem is included if D is chosen to be the appropriate set of functions satisfying $\gamma y(\omega^-) + \delta y'(\omega^-) = 0$.

If λ_0 is an eigenvalue of the linearization of (1.2) (more specifically an isolated point of the spectrum; see (H1) below) we establish conditions on L_0 and f which guarantee that, for some interval $[-\gamma, \gamma]$, for each $\xi \in [-\gamma, \gamma]$ there exists a unique $\lambda = \lambda(\xi)$ such that the corresponding solution to (1.2a) with $(y(0), y'(0)) = (\xi, m_0\xi)$ lies in D; hence is an eigenfunction for (1.2). So $\lambda(\xi)$, $-\gamma \leqq \xi \leqq \gamma$, is a set of eigenvalues for (1.2) with $\lambda(\xi) \approx \lambda_0$.

We will be making use of the approach and some of the results of Hartman [5]. There fixed point methods are used to study nonlinear perturbations of systems

$$(1.3) \qquad \mathrm{y}'(t) = A(t)\mathrm{y}(t)$$

for $0 \leqq t < \omega$, $\omega \leqq \infty$. If (1.3) has a k-dimensional subset of D-solutions (i.e. solutions to (1.3) which lie in a Banach space D) it is shown that, under the proper assumptions, the nonlinear problem

$$\mathrm{y}' = A(t)\mathrm{y} + f(t, \mathrm{y})$$

has, near $\mathrm{y} = 0$, a k-dimensional manifold of D-solutions. (The reader can probably detect a logical connection between this behavior and the bifurcation phenomenon we study.)

Our approach, like that of Hartman and others, is to use a linear theory in order to construct a mapping from some convex set into itself. The fixed point of the mapping turns out to solve the nonlinear problem. For each $\xi \in [-\gamma, \gamma]$ we will construct a mapping $T = T_\xi$ from a subset of $R \times D$ into itself as follows. For $(\eta, x) \in R \times D$ we show that the inhomogeneous linear initial-value problem

$$L_0 y = (\lambda_0 + \mu)y + (\lambda_0 + \eta)f(t, x(t)) \equiv \lambda y + g(t),$$
$$(y(0), y'(0)) = (\xi, m_0\xi)$$

has a unique D-solution for some $\mu = \mu(\xi)$. This defines the map $T_\xi(\eta, x) = (\mu, y)$ which is shown to be a contraction on the appropriate spheres in $R \times D$. The resulting fixed point clearly is the desired solution to our nonlinear problem for this ξ. The linear theory necessary for the construction of T_ξ is developed in the next section.

2. **The linear theory.** The linearization of (1.2) gives

$$L_0 y = -(ay')' + b_0 y = \lambda y, \tag{2.1a}$$

$$m_0 y(0) - y'(0) = 0, \tag{2.1b}$$

$$y \in D. \tag{2.1c}$$

Our first hypothesis regards the linear problem (2.1).

(H1) Let $a > 0$, a' and b_0 be continuous and real-valued on $[0, \omega)$, $\omega \leqq \infty$. Let λ_0 be an eigenvalue of (2.1); moreover, let λ_0 be an isolated point of the spectrum in the classical sense (see remark below). For some open interval, I, containing λ_0 let $L_0 y = \lambda y$ have a one-dimensional set of D-solutions for each $\lambda \in I$.

REMARK. Since $a(0) > 0$ we can, without loss of generality, assume $a(0) = 1$. By "spectrum" we mean that as defined in the classical limit point-limit circle analysis for the problem (2.1a), (2.1b). (For example, see [6] or [7].) The situation in this analysis is, briefly, as follows. In the limit circle case (the unusual case) by definition *all* solutions to $L_0 y = \lambda y$ are in $L^2(0, \omega)$ for *all* λ. In this case the spectrum consists entirely of isolated points (called eigenvalues). In the more important limit point case (in which $L_0 y = \lambda y$ has, for each λ, at *most* one independent $L^2(0, \omega)$ solution) the spectrum may be more complicated, containing continuous and/or point spectrum. In this case a value λ_1 in the point spectrum generates an $L^2(0, \omega)$ solution to (2.1a), (2.1b) (called an eigenfunction). It follows that in all (both) cases if λ_0 is an eigenvalue for (2.1) as assumed in (H1) that the corresponding eigenfunction is in $L^2(0, \omega)$ as well as in D. This fact will be used extensively in §4 below. Finally, we remark that the assumption of the last sentence of (H1) frequently is satisfied a priori (e.g. in the limit point case when $D = L^2(0, \omega)$).

From the remark above it would perhaps seem that the natural selection for D is $L^2(0, \omega)$. While this is true for the linear analysis, frequently L^2 is *not* a convenient choice for the nonlinear problems associated with (2.1) (see the example and discussion early in §3). Often in nonlinear applications $D = L^\infty(0, \omega)$ (hence one seeks bounded solutions as $t \to \omega$) or $D = L_0{}^\infty(0, \omega)$ (giving solutions which $\to 0$ as $t \to \omega$). It is helpful to think of the D-solutions to (2.1a) as the "small" solutions (even though "small" is not necessarily meaningful in the limit circle case).

Since we will be centering our attention on λ near λ_0, it is convenient to define $\mu = \lambda - \lambda_0$, $L = L_0 - \lambda_0$, and $b = b_0 - \lambda_0$. Then

(2.1) becomes

(2.2a) $$Ly \equiv -(ay')' + by = \mu y,$$

(2.2b) $$m_0 y(0) - y'(0) = 0,$$

(2.2c) $$y \in D.$$

We are interested in solutions to (2.2) for small values of μ. We will also be considering the inhomogeneous equation

(2.3) $$(L - \mu)y = g$$

and will seek D-solutions to (2.3) for $g \in B$, B a second (possibly) Banach space. A key concept is that of the admissibility of the pair (B, D); we say the pair (B, D) is *admissible* relative to (2.3) iff for every $g \in B$ there exists at least one D-solution to (2.3). The major hypothesis regarding the relationship between (2.3) and the spaces B and D is

(H2) Let B be $L^p(0, \omega)$, or a subspace, for $1 \leqq p \leqq \infty$ and let D be a subset of B with $\|y\|_D \geqq \|y\|_B = \|y\|_p$ for $y \in D$ (where $\|y\|_p = (\int_0^\omega |y|^p)^{1/p}$). Let the D-solutions, $v(t, \mu)$, to $(L - \mu)y = 0$ be in $L^q(0, \omega)$, $1/p + 1/q = 1$. Let the following pairs be admissible relative to (2.3): (B, D), (L^q, L^q), (L^2, L^2).

REMARK. For *regular* two-point problems the admissibility requirements are normally trivially satisfied. In singular problems it turns out that admissibility depends upon the relative sizes, as $t \to \omega$, of the D-solutions and the "large" solutions to $(L - \mu)y = 0$; and, of course, the spaces B and D. This will be seen explicitly later. The reader can verify that, for the problem $Ly = -y'' + y = \mu y$ with $-y(0) - y'(0) = 0$, the pairs (L^s, L^s) are admissible for all $1 \leqq s \leqq \infty$.

Let Y_μ denote the subspace of R^2 corresponding to the initial values of the D-solution to $(L - \mu)y = 0$. By (H1) it follows that Y_μ is one-dimensional and further that, for $\mu \neq 0$ and small,

(2.4) $$Y_\mu \neq Y_0 = \{(\tau, m_0\tau) : -\infty < \tau < \infty\}.$$

The following is simply a version of Lemma 6.3 of Hartman [5] which is given for the *system* $y' = A(t)y$. The proof is rather long so it is not repeated here.

LEMMA 2.1. *Let (B, D) be admissible for $Ly = g$ and let $(y_0, m_0 y_0) \in Y_0$. Then for $g \in B$, $Ly = g$ has a unique D-solution such that $P(y(0), y'(0)) = (y_0, m_0 y_0)$, where P is a projection of R^2 onto Y_0. Moreover, for a fixed P, there exist constants C_0 and K_0, independent of g, such that the unique D-solution satisfies*

$$\|y\| \equiv \|y\|_D \leqq C_0|y_0| + K_0\|g\|_p. \tag{2.5}$$

REMARK. Note that we are henceforth denoting the D-norm simply by $\|\cdot\|$. Frequently the first part of the lemma will also be applied to $(L-\mu)y = g$ where $\mu \neq 0$. However, the inequality (2.5) is only valid, so will only be used, when $\mu = 0$ as stated.

We apply Lemma 2.1 to $(L-\mu)y = g$ where $\mu \neq 0$, $y_0 = 0$ and $P = Q_\mu$ is the projection of R^2 onto Y_μ along Y_0. Recall that (H1) guarantees $Y_0 \neq Y_\mu$ for $\mu \neq 0$ and small. We are guaranteed a unique D-solution to $(L-\mu)y = g$ with initial values on Y_0. This solution is the key to our approach and we now obtain a Green's operator representation of it.

Let $P_\mu = I - Q_\mu$ be the projection of R^2 onto Y_0 along Y_μ. We put $(L-\mu)y = g$ in the form of a system by letting $\mathrm{y} = \binom{y}{y'}$ and $\tilde{\mathrm{g}} = \binom{0}{g/a}$. Let $U(t,\mu)$ denote the fundamental matrix solution to $(L-\mu)y = 0$ with $U(0,\mu) = I$. Then any solution to (2.3), and in particular that solution with initial values on Y_0, can be expressed as follows.

$$\begin{aligned}
\mathrm{y}(t) &= U(t,\mu)\left[\mathrm{y}(0) + \int_0^t U^{-1}(s,\mu)\,\tilde{\mathrm{g}}(s)\,ds\right] \\
&= U(t,\mu)\left[\mathrm{y}(0) + \int_0^t Q_\mu U^{-1}\tilde{\mathrm{g}}\,ds + \int_0^t P_\mu U^{-1}\tilde{\mathrm{g}}\,ds\right] \\
&= U(t,\mu)\left[\mathrm{y}(0) + \int_0^\omega P_\mu U^{-1}\tilde{\mathrm{g}}\,ds\right. \\
&\qquad \left. + \int_0^t Q_\mu U^{-1}\tilde{\mathrm{g}}\,ds - \int_t^\omega P_\mu U^{-1}\tilde{\mathrm{g}}\,ds\right],
\end{aligned}$$

$$\begin{aligned}
\mathrm{y}(t) &= U(t,\mu)\left[\mathrm{y}(0) + \int_0^\omega P_\mu U^{-1}\tilde{\mathrm{g}}\,ds\right] \\
&\quad + \int_0^\omega G(t,s,\mu)\,\tilde{\mathrm{g}}\,ds,
\end{aligned} \tag{2.6}$$

where

$$G(t,s,\mu) = \begin{cases} U(t,\mu)Q_\mu U^{-1}(s,\mu), & 0 \leqq s \leqq t, \\ -\,U(t,\mu)P_\mu U^{-1}(s,\mu), & 0 \leqq t < s. \end{cases}$$

Clearly only the second of the above steps requires justification and that is valid iff $\int_0^\omega P_\mu U^{-1}(s,\mu)\tilde{\mathrm{g}}(s)\,ds$ exists. This follows easily from the fact that $v \in L^q$ and $g \in L^p$, $1/p + 1/q = 1$, and an application of the Hölder inequality (see (2.10) below). In order to proceed, however, it is necessary to make one further assumption regarding B and D and $(L - \mu)y = g$.

(H3) For each $g \in B$ let the first component of the vector function $\int_0^\omega G(t, s, \mu)\tilde{g}(s)\, ds$ be in D.

REMARK. This hypothesis is closely related to (H2). In fact, in some cases (H3) is superfluous (e.g. if $D = L^\infty$, (H3) is necessary and sufficient for the admissibility of (B, D); see Hartman [5, Theorem 7.1 and Corollary 7.1]). As mentioned earlier these admissibility requirements are met when, for a given pair (B, D), the relative size of the "small" solution and the "large" solution to $(L - \mu)y = 0$ is acceptable. This can be checked using (2.12) and (2.13) below.

Using (H3) we can simplify somewhat our expression, (2.6), for the D-solution to $(L - \mu)y = g$ with initial values in Y_0. Since $\int_0^\omega G(t, s, \mu)\tilde{g}(s)\, ds$ is in D, it is the vector solution to $(L - \mu)y = g$ we are looking for because

$$\int_0^\omega G(0, s, \mu)\tilde{g}(s)\, ds = -\int_0^\omega P_\mu U^{-1}(s, \mu)\tilde{g}(s)\, ds, \in Y_0.$$

By Lemma 2.1 this D-solution is unique, hence

$$\mathrm{y}(t) = \int_0^\omega G(t, s, \mu)\tilde{g}(s)\, ds, \tag{2.7}$$

$$\boldsymbol{y}(0) = -\int_0^\omega P_\mu U^{-1}(s, \mu)\tilde{g}(s)\, ds. \tag{2.8}$$

The fact expressed in (2.8) will be of great use to us below so we put it in a more readable form. For each μ we denote by $v(t, \mu)$ the D-solution to $(L - \mu)y = 0$ satisfying

$$v(0, \mu) = \cos\alpha(\mu), \qquad v'(0, \mu) = \sin\alpha(\mu), \tag{2.9}$$

where $m = m(\mu) = \tan\alpha(\mu)$ is the slope of Y_μ. The solutions satisfying

$$w(0, \mu) = -\sin\alpha(\mu), \qquad w'(0, \mu) = \cos\alpha(\mu)$$

are clearly independent of $v(t, \mu)$, hence $w(t, \mu)$ represent the "large" solutions (i.e. $w \notin D$). A simple calculation shows

$$P_\mu = \frac{1}{m - m_0}\begin{bmatrix} m & -1 \\ m_0 m & -m_0 \end{bmatrix}.$$

REMARK. We will show later, Lemma 4.1, that $\alpha(\mu)$ is C^1 for small μ. Since $\tan \alpha(0) = m_0 \neq \infty$, it follows that, for small μ, $\cos \alpha(\mu) \neq 0$.

Expressing U and U^{-1} in terms of v and w one can compute

$$P_\mu U^{-1}(s, \mu) = \frac{(\cos \alpha(\mu))^{-1}}{m - m_0} a(s) \begin{bmatrix} v'(s, \mu) & -v(s, \mu) \\ m_0 v'(s, \mu) & -m_0 v(s, \mu) \end{bmatrix}. \tag{2.10}$$

Note that P_μ annilhilates the w terms in U^{-1}. Using this in (2.8) and selecting the first component gives

$$\begin{aligned} y(0) &= r(\mu) \int_0^\omega v(s, \mu) g(s)\, ds, \\ r(\mu) &\equiv [\cos \alpha(\mu)(m(\mu) - m_0)]^{-1}. \end{aligned} \tag{2.11}$$

Similarly expressing $U(t, \mu)$ in terms of v and w one can obtain

$$\begin{aligned} \mathrm{y}(t) &= \int_0^\omega G(t, s, \mu) \tilde{\mathrm{g}}(s)\, ds \\ &= r(\mu) \left\{ \begin{bmatrix} -m_0 v & v \\ -m_0 v' & v' \end{bmatrix} (t) \int_0^t U^{-1}(s, \mu) \tilde{g}\, ds \right. \\ &\qquad \left. - U(t, \mu) \int_t^\omega a(s) \begin{bmatrix} v' & -v \\ m_0 v' & -m_0 v \end{bmatrix} \tilde{g}\, ds \right\}. \end{aligned}$$

In examining the behavior of y as $t \to \omega$ it is clear that the dominant terms in the two integrals above are of the form $v(t) \int_0^t w(s) g(s)\, ds$ and $w(t) \int_t^\omega v(s) g(s)\, ds$. Applying L'Hospital's rule gives

$$\lim_{t \to \omega} v(t) \int_0^t wg\, ds = \lim_{t \to \omega} \left(\frac{-v^2 w}{v'} \right)(t) g(t), \tag{2.12}$$

$$\lim_{t \to \omega} w(t) \int_t^\omega vg\, ds = \lim_{t \to \omega} \left(\frac{w^2 v}{w'} \right)(t) g(t). \tag{2.13}$$

Clearly, if the asymptotic behavior, as $t \to \omega$, of the solutions v and w is known (2.12) and (2.13) provide a convenient way of checking the admissibility requirements in (H2) and (H3). For example, if $(v^2 w / v')$ and $(w^2 v / w')$ remain bounded as $t \to \omega$, then the D-solution to $(L - \mu) y = g$ behaves like g (or is smaller) at ω.

3. **Solving the nonlinear problem.** We now turn to the nonlinear equation

$$L_0 y = \lambda[y + f(t, y)].$$

As before we let $\mu = \lambda - \lambda_0$ and, for small μ, we seek solutions to

$$(L - \mu)y = (\lambda_0 + \mu)f(t, y), \tag{3.1a}$$

$$m_0 y(0) - y'(0) = 0, \tag{3.1b}$$

$$y \in D. \tag{3.1c}$$

Solutions to (3.1) will be called eigenfunctions, the values of μ eigenvalues, and we denote the solution pair by (μ, y). The assumptions on f are

(H4) For small $\rho > 0$, $x_1, x_2 \in D$, and $\|x_i\| \leqq \rho$ assume: f continuous and $\partial f/\partial x$ exists for $t \geqq 0$, $g(t) \equiv f(t, x_1(t))$ and $(\partial f/\partial x)(t, x_2(t))x_1(t)$ are in B. The following are $o(\|x_1\|)$ as $x_i \to 0$: $\|f(\cdot, x_1)\|_p$, $\|f(\cdot, x_1 + x_2) - f(\cdot, x_2)\|_p$, $\|(\partial f/\partial x)(\cdot, x_2 + \tau x_1)x_1\|_p$ where $|\tau| \leqq 1$. Finally, assume

$$\lim_{\epsilon \to 0} \|\epsilon^{-1}[f(\cdot, x_2 + \epsilon x_1) - f(\cdot, x_2)]\|_p = \left\| \frac{\partial f}{\partial x}(\cdot, x_2)x_1 \right\|_p .$$

As an example, let $D = B = L^\infty(0, \omega)$ and $f(t, x) = g(t)x^\alpha$ where $|g(t)| \leqq M$. It is easily verified that f satisfies (H4) iff $\alpha > 1$. For B and D in $L^p(0, \omega)$ for $p < \infty$, the situation is more complex since $x \in L^p$ does *not* imply $x^\alpha \in L^p$ for $\alpha > 1$. If one is interested in, say, L^2 solutions (as in the classical linear problem) this difficulty can frequently be resolved by letting $B = L^2$ and $D = W^{2,2}$, the Sobolev space with the norm

$$\|x\|_{2,2} \equiv \left[\int_0^\omega (|x(t)|^2 + |x'(t)|^2 + |x''(t)|^2)\, dt \right]^{1/2}.$$

Under this norm it is easily seen (e.g. see Kato [8]) that $\|x\|_{2,2} \leqq \rho$ implies $|x(t)| \leqq K\rho$ for all t; consequently, it follows that $x \in W^{2,2}$ implies x^α is in L^2, for $\alpha > 1$, and indeed $x^\alpha = o(\|x\|_{2,2})$ as $x \to 0$.

To make this section somewhat more readable some of the more technical results are put in §4 as lemmas.

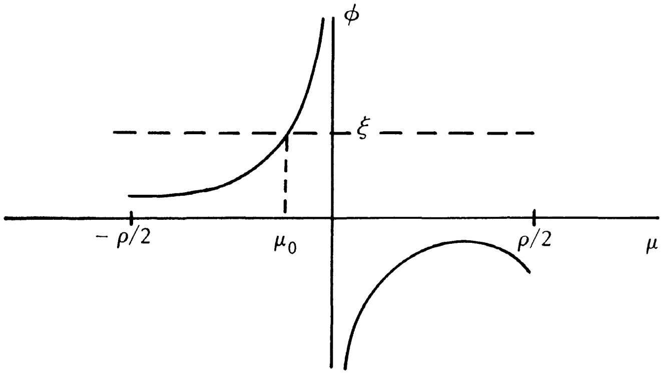
We now proceed to construct the mappings, $T = T_\xi$, and their domains in such a way that the resulting fixed points solve (3.1). The domains are spheres in $R \times D$ of the form $S_\sigma(\rho) = \{(\eta, x) : \|(\eta, x)\|_\sigma \leqq \rho\}$ where $\|(\eta, x)\|_\sigma \equiv |\eta| + \sigma\|x\|$ for $\sigma \geqq 1$.

It will be shown that for some $\rho > 0$, there exists a $\gamma > 0$ and a function $\sigma = \sigma(\xi)$ with the property that for each $\xi \in [-\gamma, \gamma]$ the mapping $T = T_\xi$ is a contraction of the sphere $S_\sigma(\rho)$ into itself. The idea of the mapping is as follows. Let $\xi \in [-\gamma, \gamma]$ and $(\eta, x) \in S_\sigma(\rho)$. Define $g(t) \equiv (\lambda_0 + \eta)\ f(t, x(t))$ and consider the family (with parameter μ) of initial value problems

$$(3.2a) \qquad (L - \mu)y = g(t),$$

$$(3.2b) \qquad (y(0), y'(0)) = (\xi, m_0\xi).$$

We will show that for each ξ there exists a unique $\mu \in [-\rho/2, \rho/2]$ such that the solution to the linear nonhomogeneous problem (3.2) is in D. The resulting point (μ, y) is shown to be in $S_\sigma(\rho)$ and this defines $T_\xi(\eta, x) = (\mu, y)$. Clearly a fixed point of T_ξ solves (3.1) with $y(0) = \xi$. This gives the mapping of the interval $[-\gamma, \gamma]$ onto a bifurcation curve $\{(\mu_\xi, y_\xi) : -\gamma \leqq \xi \leqq \gamma\}$ in $R \times D$.



The main tool for the construction of T_ξ is (2.11), which we must study in detail. Recall that the unique D-solution to (3.2a) with initial values on Y_0 is given by (2.7) and $y(0)$ is given by (2.11). Our goal is therefore to show that, for some $-\rho/2 \leqq \mu \leqq \rho/2$, $y(0) = \xi$. Since it will be convenient to think of $y(0)$ as a function of μ we make the definition

$$(3.3a) \qquad \phi(\mu) \equiv r(\mu) \int_0^\omega v(s, \mu) g(s)\, ds,$$

$$(3.3b) \qquad r(\mu) = [\cos \alpha(\mu)(m(\mu) - m_0)]^{-1}$$

and seek to solve $\phi(\mu) = \xi$ for $-\rho/2 \leqq \mu \leqq \rho/2$. Roughly speaking, the reason it has a solution on $[-\rho/2, \rho/2]$ is as follows. A cursory examination of (3.3b) suggests that $r(\mu) \to \pm\infty$ as $\mu \to 0^-$ and $r(\mu) \to \mp\infty$ as $\mu \to 0^+$. Hence, if $\int_0^\omega v(s, \mu)g\, ds \to \int_0^\omega v(s, 0)g\, ds \neq 0$, it would follow that ϕ likewise approaches both $+\infty$ and $-\infty$. If, in addition, $|\phi(\pm\rho/2)| \leqq \xi$ and ϕ is continuous it would follow $\phi(\mu) = \xi$ for at least one $\mu \in [-\rho/2, \rho/2]$. It would then remain to show that $\phi(\mu) = \xi$ has only one solution to guarantee a unique $\mu = \mu(\xi)$ and the proper definition of T_ξ. Much of what follows is the establishing of these facts.

In what follows $\rho > 0$ will be considered "small". Our initial restriction is $\rho < 1$ and such that $|\mu| \leqq \rho$ and $\|x\| \leqq \rho$ implies

(3.4a) $\qquad \lambda = \lambda_0 + \mu$ is in the interval, I, of (H1),

(3.4b) $\qquad 0 < 1/(1 - K_0|\mu|) \leqq 2$, K_0 as in Lemma 2.1,

(3.4c) $\qquad \|f(\cdot, x)\|_p \leqq k\|x\|$, where k is sufficiently small that $kK_0|\lambda_0 + \mu| \leqq \frac{1}{8}$.

THEOREM 3.1. *There exists a $\rho > 0$ and a $\gamma > 0$ such that for each nonzero $\xi \in [-\gamma, \gamma]$ and $\sigma = \sigma(\xi) \equiv \rho/(8C_0|\xi|)$, C_0 the constant in Lemma* 2.1, *the mapping T_ξ defined above maps $S_\sigma(\rho)$ into itself.*

PROOF. For the moment we think of $\rho > 0$ as small but fixed and find $\gamma = \gamma(\rho)$. Suppose $|\eta| \leqq \rho$ and $\|x\| \leqq \epsilon \leqq \rho$ and consider $|\phi(\pm\rho/2)|$. Applying the Hölder inequality we get

$$
\begin{aligned}
|\phi(\pm\rho/2)| &= \left| r(\pm\rho/2) \int_0^\omega (\lambda_0 + \eta)v(s, \pm\rho/2)f(s, x(s))\, ds \right| \\
&\leqq |r(\pm\rho/2)|(|\lambda_0| + \rho)\|v(\cdot, \pm\rho/2)\|_q \|f(\cdot, x)\|_p \\
&\leqq \max_{\mu = \pm\rho/2} \{|r(\mu)|(|\lambda_0| + \rho)\|v(\cdot, \mu)\|_q\} \|f(\cdot, x)\|_p \\
&\equiv K_p\|f(\cdot, x)\|_p.
\end{aligned}
\tag{3.5}
$$

Since $f = o(\|x\|)$ we can pick $\epsilon > 0$ sufficiently small that $K_p\|f(\cdot, x)\|_p \leqq \|x\|/8C_0$ for all $\|x\| \leqq \epsilon$. We therefore define $\gamma \equiv \epsilon/8C_0$ and claim that for $\xi \in [-\gamma, \gamma]$ and $\sigma \equiv \rho/(8C_0|\xi|)$ the mapping T_ξ maps $S_\sigma(\rho)$ into itself.

First we show that for each $(\eta, x) \in S_\sigma(\rho)$, $(\mu, y) = T_\xi(\eta, x)$, if defined, is in $S_\sigma(\rho)$. By construction of T_ξ we have a μ value, $|\mu| \leqq \rho/2$, such that the solution to

$$
\begin{gathered}
Ly = \mu y + (\lambda_0 + \eta)f(t, x(t)) \equiv h(t), \\
(y(0), y'(0)) = (\xi, m_0\xi),
\end{gathered}
$$

lies in D. Again applying Lemma 2.1 to $Ly = h$ and using (H2) and (3.4) we get

$$\begin{aligned}\|y\| &\leqq C_0|\xi| + K_0\|\mu y + (\lambda_0 + \eta)f(t, x(t))\|_p \\ &\leqq C_0|\xi| + K_0|\mu|\,\|y\|_p + K_0|\lambda_0 + \eta|\,\|f(\,\cdot\,, x)\|_p \\ &\leqq C_0|\xi| + K_0|\mu|\,\|y\| + K_0|\lambda_0 + \eta|\,\|f(\,\cdot\,, x)\|_p \\ &\leqq C_0|\xi| + K_0|\mu|\,\|y\| + \|x\|/8.\end{aligned}$$

Hence, again using (3.4)

$$\|y\|(1 - K_0|\mu|) \leqq C_0|\xi| + \|x\|/8 \leqq \rho/8\sigma + \rho/8\sigma = \rho/4\sigma,$$

$$\|y\| \leqq \frac{1}{1 - K_0|\mu|}\frac{\rho}{4\sigma} \leqq 2\frac{\rho}{4\sigma} = \frac{\rho}{2\sigma}.$$

It follows that $\sigma\|y\| \leqq \rho/2$, hence $\|(\mu, y)\|_\sigma = |\mu| + \sigma\|y\| \leqq \rho/2 + \rho/2 = \rho$ and $(\mu, y) \in S_\sigma(\rho)$.

We now show that for each $(\eta, x) \in S_\sigma(\rho)$, $T_\xi(\eta, x) = (\mu, y)$ is uniquely defined. Note that by our parameter selection above $\rho/\sigma \leqq \epsilon$. Referring to (3.5), for $(\eta, x) \in S_\sigma(\rho)$,

$$|\phi(\pm\rho/2)| \leqq K_p\|f(\,\cdot\,, x)\|_p \leqq \frac{1}{8C_0}\|x\| \leqq \frac{1}{8C_0}\frac{\rho}{\sigma} = |\xi|. \tag{3.6}$$

Since $\phi(\pm\rho/2) \leqq |\xi|$, if we can show that ϕ approaches both $+\infty$ and $-\infty$ as $\mu \to 0$ and is continuous on $[-\rho/2, 0) \cup (0, \rho/2]$, it will follow that $\phi(\mu) = \xi$ for some $-\rho/2 \leqq \xi \leqq \rho/2$. In seeking the solution to $\phi(\mu) = \xi$ there are two possible cases to consider.

Case 1. $\int_0^\omega v(s, 0)g\,ds \neq 0$, where $g(s) \equiv (\lambda_0 + \eta)f(s, x(s))$. By Lemma 4.2, $\|v(\,\cdot\,, \mu)\|_q$ is continuous in μ; it follows immediately from the Hölder inequality that

$$\lim_{\mu\to 0}\int_0^\omega v(s, \mu)g\,ds \to \int_0^\omega v(s, 0)g\,ds \neq 0.$$

According to Lemma 4.6 $r(\mu) = [c + o(1)]/\mu$ as $\mu \to 0$ where $c \neq 0$. Hence as $\mu \to 0$

$$\begin{aligned}\phi(\mu) &= r(\mu)\int_0^\omega v(s, \mu)g\,ds \\ &= \frac{1}{\mu}\int_0^\omega v(s, 0)g\,ds[c + o(1)].\end{aligned} \tag{3.7}$$

Lemma 4.1 gives the continuity of $\alpha(\mu)$ and the continuity of $r(\mu)$ for

$\mu \neq 0$ follows immediately. This fact together with (3.6) and (3.7) shows that for $-\rho/2 \leqq \mu \leqq \rho/2$, ϕ takes on all values in $(-\infty, -\xi] \cup [\xi, \infty)$ and in particular $\phi(\mu_0) = \xi$ for some μ_0. It remains to show that there is only one ξ-value for ϕ. In Lemma 4.7 we have that if $\phi(\mu_0) = \xi$ then $\mathrm{sgn}(\phi'(\mu_0)) = \mathrm{sgn}(-\xi\mu_0)$. A simple geometric argument now shows that there is exactly one μ_0 such that $\phi(\mu_0) = \xi$. This now guarantees that $T_\xi(\eta, x)$ is uniquely defined for $(\eta, x) \in S_\sigma(\rho)$ if $\int_0^\omega v(s, 0)g \neq 0$.

Case 2. $\int_0^\omega v(s, 0)g\, ds = 0$. In this case we can *define* $\mu = 0$ and let y be the resulting solution to

$$Ly = g, \tag{3.8a}$$

$$(y(0),\ y'(0)) = (\xi, m_0\xi). \tag{3.8b}$$

We first must verify that such a y is indeed in D. Once again Lemma 2.1 is applied where P is chosen as the projection along $Y_0{}^\perp$ onto Y_0 ($Y_0{}^\perp$ the orthogonal complement of Y_0). We can use the Green's function as was done in §2 to construct the unique D-solution to (3.8a) with initial values on $Y_0{}^\perp$. In particular at $t = 0$ we get

$$\begin{aligned} y(0) &= -\int_0^\omega PU^{-1}(s, 0)\tilde{g}(s)\, ds \\ &= \frac{\cos^{-1}\alpha(0)}{m_0 + m_0{}^{-1}} \int_0^\omega v(s, 0)g(s) \begin{pmatrix} 1 \\ -1/m_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

That is, this D-solution has zero initial values. It follows that all D-solutions to (3.8a) have initial values in Y_0 and conversely that for $(a, b) \in Y_0$ there exists a D-solution with initial values (a, b). In particular, the solution to (3.8a), (3.8b) is in D.

Finally, we must show that if $\int_0^\omega v(s, 0)g = 0$ then there are no values $\mu \in [-\rho/2, 0) \cup (0, \rho/2]$ such that $\phi(\mu) = \xi$. It will follow then that T_ξ is well defined for Case 2. We apply L'Hospital's rule to the limit of ϕ, the derivatives being justified by Lemmas 4.5 and 4.6.

$$\begin{aligned} \lim_{\mu\to 0} \phi(\mu) &= \lim_{\mu \to 0} r(\mu) \int_0^\omega v(s, \mu)g\, ds \\ &= \lim_{\mu\to 0} \frac{\int_0^\omega (\partial v/\partial \mu)g\, ds}{-r^{-2}(\mu)r'(\mu)}. \end{aligned}$$

Again referring to Lemma 4.6, $-r^{-2}(\mu)r'(\mu) = (1 + o(1))/c$ as $\mu \to 0$ for $c \neq 0$. Lemma 4.4 shows that $\partial v/\partial\mu$ is in D and also continuous in the L^q norm. Hence, again referring to (H2),

$$\left| \int_0^\omega \frac{\partial v}{\partial \mu} g \, ds \right| \leqq \left\| \frac{\partial v}{\partial \mu}(\,\cdot\,,\mu) \right\|_q \|g\|_p$$
$$= \left\| \frac{\partial v}{\partial \mu} \right\|_q |\lambda_0 + \eta| \, \|f(\,\cdot\,,x)\|_p.$$

Since $\|f(\,\cdot\,,x)\|_p = o(x)$ as $x \to 0$ it follows that, for ρ sufficiently small and $\sigma \geqq 1$ (recall that $\xi = \pm\, \rho/8C_0\sigma$),

$$c\|\partial v/\partial \mu\|_q \, |\lambda_0 + \eta| \, \|f(\,\cdot\,,x)\|_p < |\xi|.$$

Hence, for small ρ, $\lim_{\mu\to 0}|\phi(\mu)| < |\xi|$. As discussed earlier, ϕ can have at most one ξ-value on $(0, \rho/2]$ and $[-\rho/2, 0)$. Therefore in this case there are no ξ-values on these intervals and so $\mu = 0$ is the unique value.

This completes the proof that T_ξ is well defined for all $(\eta, \xi) \in S_\sigma(\rho)$ and takes $S_\sigma(\rho)$ into $S_\sigma(\rho)$.

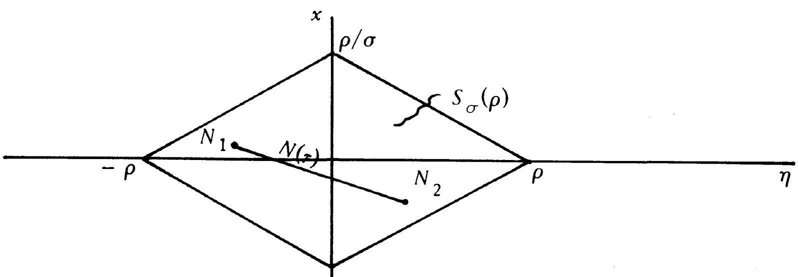
The following theorem guarantees the contraction property of T_ξ and hence our desired fixed point.

Theorem 3.2. *For ρ sufficiently small and γ as given in Theorem 3.1, the mapping T_ξ is a contraction on $S_\sigma(\rho)$ for each nonzero $\xi \in [-\gamma, \gamma]$.*

Proof. Let $\rho, \gamma, \xi, \sigma$ be chosen as in Theorem 3.1. Let $N_i \equiv (\eta_i, x_i)$ for $i = 1, 2$ be two points in $S_\sigma(\rho)$. We show that

$$\|T_\xi N_1 - T_\xi N_2\|_\sigma \leqq \tfrac{5}{8}\, \|N_1 - N_2\|_\sigma$$

where $\|N_i\|_\sigma = |\eta_i| + \sigma\|x_i\|$. Since $S_\sigma(\rho)$ is a convex set the "line segment" connecting N_1 and N_2 lies in $S_\sigma(\rho)$; namely $\{N(\tau) = N_1 + \tau(N_2 - N_1) : 0 \leqq \tau \leqq 1\}$.



We now think of ϕ as a function of the two variables τ and μ in writing

$$\phi(\tau, \mu) = r(\mu) \int_0^\omega v(s, \mu)(\lambda_0 + \eta(\tau)) f(s, x(s, \tau))\, ds \tag{3.9}$$

where

$$\begin{aligned} \eta(\tau) &= \eta_1 + \tau(\eta_2 - \eta_1), \\ x(s, \tau) &= x_1(s) + \tau(x_2(s) - x_1(s)). \end{aligned} \tag{3.10}$$

It follows from Theorem 3.1 that for each $0 \leqq \tau \leqq 1$ there is a unique $\mu = \mu(\tau)$ such that $\phi(\tau, \mu(\tau)) = \xi$ for $\xi \in [-\gamma, \gamma]$. If the implicit function theorem is applicable at each point $(\tau_0, \mu(\tau_0))$ then we would have, letting $\mu_0 = \mu(\tau_0)$,

$$|\mu'(\tau_0)| = \left| \frac{\partial\phi(\tau_0, \mu_0)/\partial\tau}{\partial\phi(\tau_0, \mu_0)/\partial\mu} \right|$$

and showing this value to be small would prove $|\mu(1) - \mu(0)| = |\mu_2 - \mu_1|$ is small which is our immediate goal.

For simplicity we first assume that $\mu(\tau) \neq 0$ for $0 \leqq \tau \leqq 1$, and proceed to use the implicit function theorem. Using (3.9) and (3.10) we can compute (the differentiation of f by τ is justified by (H4))

$$\frac{\partial\phi}{\partial\tau} = r(\mu) \int_0^\omega v(s, \mu) \Big[(\eta_2 - \eta_1) f(s, x) + (\lambda_0 + \eta(\tau)) \frac{\partial f}{\partial x}(s, x)(x_2 - x_1)(s) \Big]\, ds,$$

$$\frac{\partial\phi}{\partial\mu} = r'(\mu) \int_0^\omega v(s, \mu)(\lambda_0 + \eta(\tau)) f(s, x)\, ds + r(\mu) \int_0^\omega \frac{\partial v}{\partial\mu}(\lambda_0 + \eta(\tau)) f(s, x)\, ds.$$

Recall that $r(\mu) = [1 + o(1)]\, c/\mu$ for $c \neq 0$. Hence, in the first half of the expression for $\partial\phi/\partial\tau$

$$\begin{aligned} &\left| r(\mu) \int_0^\omega (\eta_2 - \eta_1) v f\, ds \right| \\ &\leqq r(\mu) |\eta_2 - \eta_1|\, \|v(\,\cdot\,, \mu)\|_q \|f(\,\cdot\,, x)\|_p \\ &\leqq \frac{K}{\mu} \|f(\,\cdot\,, x)\|_p |\eta_2 - \eta_1|, \end{aligned} \tag{3.11}$$

where $K \geqq 2c\|v(\cdot, \mu)\|_q$ for $|\mu| \leqq \rho/2$. Turning to the right half of our $\partial\phi/\partial\tau$ expression

$$\left| r(\mu) \int_0^\omega (\eta + \lambda_0) v \frac{\partial f}{\partial x} [x_2 - x_1] \, ds \right|$$

$$\leqq |r(\mu)| \, |\eta + \lambda_0| \, \|v(\cdot, \mu)\|_q \|(x_2 - x_1)\partial f/\partial x\|_p \tag{3.12}$$

$$\leqq \frac{K_1}{\mu} \|(x_2 - x_1)\partial f/\partial x\|_p.$$

Combining (3.11) and (3.12) we have, letting $K_2 = \max[K, K_1]$,

$$\left| \frac{\partial \phi}{\partial \tau} \right| \leqq \frac{K_2}{\mu} \left[\|f(\cdot, x)\|_p |\eta_2 - \eta_1| + \|(x_2 - x_1)\partial f/\partial x\|_p \right]. \tag{3.13}$$

It follows from Lemma 4.7 if ρ is sufficiently small and $\xi = \rho/(8C_0\sigma)$ that $\phi(\tau_0, \mu(\tau_0)) = \xi$ implies $\partial\phi(\tau_0, \mu(\tau_0))/\partial\mu \approx -\xi/\mu_0$. This fact coupled with the fact that $\|f(\cdot, x)\|_p = o(\|x\|)$, $\|(x_2 - x_1)\partial f/\partial x\|_p = o(\|x_2 - x_1\|)$ in (3.13) gives us, for small ρ and $\tilde{K}_0 \equiv \max[K_0, 1]$

$$|\mu'(\tau)| = \left| \frac{\partial \phi}{\partial \tau} \right| \Big/ \left| \frac{\partial \phi}{\partial \mu} \right| \leqq \frac{1}{4\tilde{K}_0} [|\eta_1 - \eta_2| + \|x_2 - x_1\|]$$

$$\leqq \frac{1}{4\tilde{K}_0} \|N_1 - N_2\|_\sigma \leqq \tfrac{1}{4} \|N_1 - N_2\|_\sigma. \tag{3.14}$$

From (3.14) it follows that if $\mu \neq 0$ for $0 \leqq \tau \leqq 1$ that $|\mu_2 - \mu_1| = |\mu(1) - \mu(0)| \leqq \frac{1}{4}\|N_1 - N_2\|_\sigma$. If $\mu = 0$ for some values of τ this result can be extended to those points by a simple continuity argument which will not be given.

It remains to show that $\sigma\|y_1 - y_2\| \leqq \frac{3}{8}\|N_1 - N_2\|_\sigma$. To this end consider

$$Ly_i = \mu_i y_i + (\lambda_0 + \eta_i) f(t, x_i(t)), \qquad i = 1, 2.$$

Subtracting and regrouping the right side gives

$$\begin{aligned} L[y_2 - y_1] &= \mu_2(y_2 - y_1) + (\mu_2 - \mu_1)y_1 \\ &\quad + (\lambda_0 + \eta_2)[f(t, x_2) - f(t, x_1)] + (\eta_2 - \eta_1) f(t, x_1) \\ &\equiv h(t). \end{aligned}$$

Once again applying Lemma 2.1 to $L[y_2 - y_1] = h(t)$ and using the fact that $(y_2 - y_1)(0) = \xi - \xi = 0$ and $\|y\| \geqq \|y\|_p$

$$\begin{aligned}\|y_2 - y_1\| &\leqq K_0\|h\|_p \\ &\leqq K_0[\,|\mu_2|\,\|y_2 - y_1\| + |\mu_2 - \mu_1|\,\|y_1\| \\ &\qquad + |\lambda_0 + \eta_2|\,\|f(\,\cdot\,, x_2) - f(\,\cdot\,, x_1)\|_p \\ &\qquad + |\eta_2 - \eta_1|\,\|f(\,\cdot\,, x_1)\|_p]\end{aligned}$$

Making use of $\|y_1\| \leqq \rho/2\sigma$, $\|f(\,\cdot\,, x_2) - f(\,\cdot\,, x_1)\|_p = o(\|x_2 - x_1\|)$, and $\|f(\,\cdot\,, x_1)\|_p = o(\|x_1\|)$ as $\|x_i\| \leqq \rho/\sigma \to 0$ it follows that

$$(1 - |\mu_2|K_0)\|y_2 - y_1\|$$

$$\leqq K_0\left[\frac{\rho}{2\sigma}\frac{1}{4K_0}\ \|N_2 - N_1\|_\sigma + \frac{1}{16K_0}\ \|x_2 - x_1\| + |\eta_2 - \eta_1|\frac{1}{16K_0}\frac{\rho}{\sigma}\right],$$

$$\|y_2 - y_1\|$$

$$\leqq 2K_0\left[\frac{\rho}{2\sigma}\frac{1}{4K_0}\ \|N_2 - N_1\|_\sigma + \frac{1}{16K_0}\|x_2 - x_1\| + |\eta_2 - \eta_1|\frac{1}{16K_0}\frac{\rho}{\sigma}\right]$$

$$= \frac{\rho}{4\sigma}\|N_2 - N_1\|_\sigma + \tfrac{1}{8}\left[\ \|x_2 - x_1\| + |\eta_2 - \eta_1|\rho/\sigma\right]$$

$$\leqq \frac{1}{4\sigma}\ \|N_2 - N_1\|_\sigma + \frac{1}{8\sigma}\cdot\|N_2 - N_1\|_\sigma = \frac{3}{8\sigma}\|N_2 - N_1\|_\sigma.$$

Hence

$$\begin{aligned}\|T_\xi N_2 - T_\xi N_1\|_\sigma &= \|\mu_2 - \mu_1\| + \sigma\|y_2 - y_1\| \\ &\leqq \tfrac{1}{4}\,\|N_2 - N_1\|_\sigma + \tfrac{3}{8}\|N_2 - N_1\|_\sigma = \tfrac{5}{8}\,\|N_2 - N_1\|_\sigma\end{aligned}$$

and the mappings T_ξ are contractions of $S_\sigma(\rho)$ into $S_\sigma(\rho)$. This proves the theorem.

The original goal is now an immediate corollary to Theorem 3.2.

THEOREM 3.3. *There is a $\gamma > 0$ such that for each $\xi \in [-\gamma, \gamma]$ there exists a unique $\lambda = \lambda(\xi)$ for which the solution to*

$$(3.15)\qquad L_0 y = \lambda(\xi)[y + f(t, y)], \qquad (y(0), y'(0)) = (\xi, m_0\xi),$$

lies in D.

PROOF. Apply the contraction mapping theorem to T_ξ; the resulting fixed point (for each $\xi \neq 0$) solves

$$Ly = \mu(\xi)y + (\lambda_0 + \mu(\xi))f(t, y),$$

$$(y(0), y'(0)) = (\xi, m_0\xi).$$

Then (3.15) follows by replacing $\lambda_0 + \mu(\xi)$ by $\lambda(\xi)$ and L by $L_0 + \lambda_0$ where L_0 is the original differential operator. Of course when $\xi = 0$, $y \equiv 0$.

4. **Some technical results.** Following are the lemmas referred to in §3.

LEMMA 4.1. *Let $\alpha(\mu)$ be defined by the relation*

$$v(0, \mu) = \cos \alpha(\mu), \qquad v'(0, \mu) = \sin \alpha(\mu), \tag{4.1}$$

where v is the D-solution to $(L - \mu)y = 0$ with normalized initial values. Then α is a differentiable function of μ (for small μ) and

$$\alpha'(\mu) = \left[\int_0^\omega v^2(s, \mu)\, ds \right] . \tag{4.2}$$

PROOF. This result follows from the classical limit point-limit circle analysis (e.g. see [6] or [7]) in the following manner (recall that in (H1) it was assumed that the eigenvalues of (2.1) correspond with the isolated spectrum of the classical analysis). We wish to show that (4.2) holds at an arbitrary μ_0. In the classical analysis one seeks the eigenfunctions in the form (in our case $\chi \in D$)

$$\chi(t; \mu, \beta) = y_1(t; \mu, \beta) + \tilde{m}(\mu, \beta) y_2(t; \mu, \beta), \tag{4.3}$$

where y_1 and y_2 solve $(L - \mu)y = 0$ with

$$y_1(0) = -\sin\beta, \qquad y_1'(0) = \cos\beta, \tag{4.4a}$$

$$y_2(0) = \cos\beta, \qquad y_2'(0) = \sin\beta. \tag{4.4b}$$

The parameter β is normally considered as fixed and the left boundary condition is $\chi(0)\sin\beta - \chi'(0)\cos\beta = 0$ (so that y_2 satisfied this). It turns out that, for fixed β, the function $\tilde{m}$ is a meromorphic function of μ with simple poles on the real axis. Moreover the poles of $\tilde{m}$ correspond to the eigenvalues (this last fact is easily seen from (4.3) and (4.4); at least for the limit point case where only one L^2 solution exists).

Hille [7] shows that the poles of $\tilde{m}$ have residues

$$\varphi(\mu_0) = -1 \Big/ \int_0^\omega |y_2|^2 < 0. \tag{4.5}$$

(Actually Hille shows this for the case $Lu = -u'' + qu$ and $\omega = \infty$, but the extension to our more general operator is straightforward.)

He also indicates the usefulness of considering β as a parameter and writing

$$\chi(t;\ \mu,\beta) = y_1(t;\ \mu,\beta) + \tilde{m}(\mu,\beta)y_2(t;\ \mu,\beta),$$

$$\chi(t;\ \mu,\gamma) = y_1(t;\ \mu,\gamma) + \tilde{m}(\mu,\gamma)y_2(t;\ \mu,\gamma).$$

Noting that $\chi(t;\ \mu,\beta)$ and $\chi(t;\ \mu,\gamma)$ are linearly dependent (so $\mathrm{Wron}[\chi(t;\ \mu,\beta),\chi(t;\ \mu,\gamma)] = 0$) it is easy to compute

$$\tan(\gamma - \beta) = (l(\mu,\gamma) - l(\mu,\beta))/(1 + l(\mu,\gamma)l(\mu,\beta)),$$

where $l = 1/\tilde{m}$ (hence we inspect the zeros of l which correspond to the poles of $\tilde{m}$). We now let $\beta = \alpha(\mu_0)$ and $\gamma = \alpha(\mu)$ and note that $l(\mu,\alpha(\mu)) = 0$ (see (4.3) and (4.4)). Hence

$$\tan[\alpha(\mu) - \alpha(\mu_0)] = -l(\mu,\alpha(\mu_0)).$$

Dividing both sides by $(\mu - \mu_0)$ and letting $\mu \to \mu_0$ we get

$$\lim_{\mu\to\mu_0} \frac{-l(\mu,\alpha(\mu_0))}{\mu - \mu_0} = -a_1 = \lim_{\mu\to\mu_0} \tan\frac{[\alpha(\mu) - \alpha(\mu_0)]}{\mu - \mu_0}$$

$$= \lim_{\mu\to\mu_0} \frac{\alpha(\mu) - \alpha(\mu_0)}{\mu - \mu_0} = \alpha'(\mu_0).$$

That is, $\alpha'(\mu_0) = -a_1$ where a_1 is the first Taylor's series coefficient of l at its zero μ_0. Since $a_1 = (\varphi(\mu_0))^{-1}$ we have

$$\alpha'(\mu_0) = -a_1 = \frac{-1}{\varphi(\mu_0)} = \int_0^\omega |y_2|^2 = \int_0^\omega |v(s,\mu_0)|^2$$

which proves the lemma.

LEMMA 4.2. $\|v(\cdot,\mu)\|_q$ *is continuous in* μ *for small* μ.

PROOF. Let μ_0 be fixed; we show that $|\mu - \mu_0| \to 0$ implies $\|v(\cdot,\mu) - v(\cdot,\mu_0)\|_q \to 0$. Denote $v(t,\mu_0)$ by v_0 and consider

$$L(v - v_0) = \mu v - \mu_0 v_0 = \mu(v - v_0) + (\mu - \mu_0)v_0 \equiv h(t).$$

We can apply Lemma 2.1 to $L(v - v_0) = h$ since we have assumed that (L^q, L^q) are admissible for $Ly = h$. For some constants C and K we have

$$\|v - v_0\|_q \leqq C|v - v_0|(0) + K[|\mu|\,\|v - v_0\|_q + |\mu - \mu_0|\,\|v_0\|_q].$$

Hence for μ small (so that $1 - K|\mu| > 0$)

$$\text{(4.6)} \quad \|v - v_0\|_q \leqq \frac{1}{1 - K|\mu|}[C|v - v_0|(0) + K|\mu - \mu_0| \, \|v_0\|_q],$$

$v(0, \mu) = \cos \alpha(\mu) \to \cos \alpha(\mu_0) = v(0, \mu_0)$ as $\mu \to \mu_0$ since Lemma 4.1 gives continuity of α. It follows that the right side of (4.6) $\to 0$ as $\mu \to \mu_0$.

LEMMA 4.3. *α is continuously differentiable in μ for small μ.*

PROOF. It remains to show that

$$\alpha'(\mu) = \int_0^\omega v^2(s, \mu)\, ds$$

is continuous. This is certainly true if $v(\,\cdot\,, \mu)$ is continuous in the L^2 norm and this fact is given by Lemma 4.2 if we replace q by 2 (recall that (L^2, L^2) is also admissible for $Ly = h$).

LEMMA 4.4. *$\partial v(t, \mu)/\partial \mu$ is in D and L^q and is a continuous function of μ in the L^q norm.*

PROOF. We know that

$$(L - \mu)v(t, \mu) = 0, \qquad (v(0, \mu), v'(0, \mu)) = (\cos \alpha(\mu), \sin \alpha(\mu)).$$

Formally differentiating the differential equation and initial values gives

$$\text{(4.7a)} \qquad (L - \mu)\frac{\partial v}{\partial \mu} = v,$$

$$\text{(4.7b)} \quad \left(\frac{\partial v}{\partial \mu}(0, \mu), \frac{\partial v'}{\partial \mu}(0, \mu) \right) = (-\sin \alpha(\mu)\alpha'(\mu), \cos \alpha(\mu)\alpha'(\mu)).$$

This can be justified using standard theorems (e.g. see [6]).

We now show $\partial v/\partial \mu$ is in D by the technique of §2; namely by using the Green's function. Note the initial values for $\partial v/\partial \mu$ are on $Y_\mu{}^\perp$, the orthogonal complement of Y_μ. According to Lemma 2.1 there exists a unique D-solution to $(L - \mu)y = v$ with initial conditions on $Y_\mu{}^\perp$. Referring again to (2.8) these initial conditions are given by

$$\mathrm{y}(0) = -\int_0^\omega \tilde{P}U^{-1}(s, \mu)\begin{pmatrix} 0 \\ v/a \end{pmatrix} ds,$$

where $\tilde{P}$ is the projection of R^2 onto $Y_\mu{}^\perp$ along Y_μ. Expanding out $\tilde{P}U^{-1}$ and taking the first component of $\mathrm{y}(0)$,

$$y(0) = \frac{(\cos \alpha(\mu))^{-1}}{m(\mu) + (m(\mu))^{-1}} \int_0^\omega -v^2(s, \mu)\, ds$$

$$= \frac{(\cos \alpha(\mu))^{-1}}{\tan \alpha(\mu) + \operatorname{ctn} \alpha(\mu)}(-\alpha'(\mu)) = -\sin \alpha(\mu)\alpha'(\mu)$$

which agrees with (4.7b). It follows that $\partial v/\partial \mu$ is *that* D-solution promised by Lemma 2.1.

To prove that $\partial v/\partial \mu$ is also in L^q exactly the same argument is repeated together with the fact that the pair (L^q, L^q) is admissible for $(L - \mu)y = h$.

Finally we must show that $\partial v(\,\cdot\,, \mu)/\partial \mu$ is continuous in μ in the L^q norm. Again we apply Lemma 2.1, this time with $B = D = L^q$ (in fact we will assume the constants C_0 and K_0 were chosen large enough to handle both cases (B, D) and (L^q, L^q)). Denoting $\partial v(t, \mu)/\partial \mu$ by v_μ and $\partial v(t, \mu_0)/\partial \mu$ by $v_{0\mu}$ we have

$$L(v_\mu - v_{0\mu}) = \mu v_\mu - \mu_0 v_{0\mu} + v - v_0$$
$$= (\mu - \mu_0)v_{0\mu} + (v_\mu - v_{0\mu})\mu + v - v_0.$$

So, by Lemma 2.1,

$$\|v_\mu - v_{0\mu}\|_q \leqq C_0|v_\mu(0) - v_{0\mu}(0)|$$
$$+ K_0[\,|\mu - \mu_0|\,\|v_{0\mu}\|_q + |\mu|\,\|v_\mu - v_{0\mu}\|_q + \|v - v_0\|_q].$$

Hence

$$\|v_\mu - v_{0\mu}\|_q \leqq \frac{1}{1 - K_0|\mu|}[C_0|v_\mu(0) - v_{0\mu}(0)| \tag{4.8}$$
$$+ K_0|\mu - \mu_0|\,\|v_{0\mu}\|_q + K_0\|v - v_0\|_q].$$

Since α, α', and $\|v(\,\cdot\,, \mu)\|_q$ are continuous it follows that the right side of (4.8) $\to 0$ as $\mu \to \mu_0$ and this established the lemma.

LEMMA 4.5. *For $g \in L^p$ and $F(\mu) \equiv \int_0^\omega v(s, \mu)g(s)\, ds$, F is continuously differentiable and*

$$F'(\mu) = \int_0^\omega \frac{\partial v}{\partial \mu} g\, ds. \tag{4.9}$$

PROOF. Let μ_0 be an arbitrary value; we show (4.9) for $\mu = \mu_0$ by showing that the integral in (4.9) is uniformly convergent for μ near μ_0. Let $v_\mu = \partial v(\,\cdot\,, \mu)/\partial \mu$ and $v_{0\mu} = \partial v(\,\cdot\,, \mu_0)/\partial \mu$. For $0 < a < b < \omega$

$$
\begin{aligned}
\left| \int_a^b v_\mu g \, ds \right| &= \left| \int_a^b (v_\mu - v_{0\mu})g + \int_a^b v_{0\mu} g \right| \\
&\leqq \left| \int_a^b (v_\mu - v_{0\mu})g \right| + \left| \int_a^b v_{0\mu} g \right| \\
&\leqq \left(\int_a^b |v_\mu - v_{0\mu}|^q \right)^{1/q} \left(\int_a^b |g|^p \right)^{1/p} \\
&\quad + \left(\int_a^b |v_{0\mu}|^q \right)^{1/q} \left(\int_a^b |g|^p \right)^{1/p} \\
&\leqq \left(\int_a^b |g|^p \right)^{1/p} (\|v_\mu - v_{0\mu}\|_q + \|v_{0\mu}\|_q).
\end{aligned} \tag{4.10}
$$

Since $\partial v(\cdot, \mu)/\partial\mu$ is continuous in μ in the L^q norm (by Lemma 4.4) then for $|\mu - \mu_0| \leqq \epsilon$, $\|v_\mu - v_{0\mu}\|_q \leqq 1$. It is now clear that the right side of (4.10) goes to zero as $a \to \omega$ uniformly for $|\mu - \mu_0| < \epsilon$. This justifies (4.9) for $\mu = \mu_0$. The uniform convergence also guarantees the continuity of F'.

Lemma 4.6. *The function* $r(\mu) = [\cos\alpha(\mu)(m(\mu) - m(0))]^{-1}$ *has the following behavior as* $\mu \to 0$

$$
r(\mu) = \frac{c}{\mu}[1 + o(1)], \qquad r'(\mu) = -\frac{c}{\mu^2}[1 + o(1)], \qquad c \neq 0.
$$

Proof. This is a simple computation using the facts: $m(\mu) = \tan\alpha(\mu)$ and $\alpha'(\mu) = \int_0^\omega v^2(s, \mu)\, ds \neq 0$.

Lemma 4.7. *Let* $(\eta, x) \in S_\sigma(\rho)$ *and*

$$
\phi(\mu) \equiv r(\mu) \int_0^\omega v(s, \mu)(\eta + \lambda_0) f(s, x(s))\, ds. \tag{4.11}
$$

Let $\mu_0 \neq 0$ *be in* $[-\rho/2, \rho/2]$ *such that* $\phi(\mu_0) = \xi = \rho/(8C_0\sigma)$. *Then, if* $\rho > 0$ *is sufficiently small and* $\sigma \geqq 1$,

$$
\phi'(\mu_0) \approx -\xi/\mu_0.
$$

In particular, the sign of $\phi'(\mu_0)$ *is* $\operatorname{sgn}(-\xi/\mu_0)$.

Proof. Differentiating (4.11) gives

$$
\begin{aligned}
\phi'(\mu) = r(\mu) &\int_0^\omega \frac{\partial v}{\partial \mu}(s, \mu)(\eta + \lambda_0) f(s, x)\, ds \\
&+ r'(\mu) \int_0^\omega v(s, \mu)(\eta + \lambda_0) f(s, x)\, ds.
\end{aligned}
$$

And at μ_0 we have

$$\text{(4.12)} \quad \phi'(\mu_0) = r(\mu_0) \int_0^\omega \frac{\partial v}{\partial \mu} (\eta + \lambda_0) f(s, x)\, ds + \frac{r'(\mu_0)}{r(\mu_0)} \xi.$$

By Lemma 4.6, $\xi r'(\mu_0)/r(\mu_0) \approx -\xi/\mu_0$ for small μ_0. We show that the integral in (4.12) is small compared to $-\xi/\mu_0$. Again using the Hölder inequality and Lemma 4.6,

$$\begin{aligned} &\left| r(\mu_0) \int_0^\omega \frac{\partial v}{\partial \mu} (\eta + \lambda_0) f(s, x)\, ds \right| \\ &\qquad \leqq (\rho + |\lambda_0|) |r(\mu_0)| \left\| \frac{\partial v}{\partial \mu}(\,\cdot\,, \mu) \right\|_q \|f(\,\cdot\,, x)\|_p \\ &\qquad \leqq \frac{K}{\mu_0} \|f(\,\cdot\,, x)\|_p. \end{aligned}$$

Since $\|f(\,\cdot\,, x)\|_p = o(\|x\|)$ and $\xi = \rho/(8C_0\sigma)$ it follows that for ρ sufficiently small and $\|x\| \leqq \rho/\sigma$ that the quantity $K\|f(\,\cdot\,, x)\|_p < \epsilon\xi$ for $0 < \epsilon \ll 1$ and the result follows.

5. **Conclusion.** In this paper the nonlinear perturbation term, $\lambda f(t, y)$, does not depend upon y'. The case $f = f(t, y, y')$ offers very little complication over the case treated; the major difference being the necessity to think of B and D as spaces of vector-valued functions. The restriction was made only to somewhat simplify the presentation.

An interesting question is whether this local bifurcation result can be extended to a global one in the spirit of Crandall and Rabinowitz [3]. The situation for the singular problems is complicated by the fact that the operator $\mathcal{G}$, $\mathcal{G}h = \int_0^\omega G(t, s)h(s)\, ds$, is not, in general, compact. Some singular problems of the type treated here will lead to compact operators; in which case the general theory would apply and provide some global behavior. It appears unlikely that global results will be forthcoming for the problem treated here. On the other hand, it is reasonable to expect the "bifurcation curve", $\{(\xi, \lambda(\xi)) : |\xi| \leqq \gamma\}$, to be smooth (probably C^1) and future investigation will likely establish this.

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