ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 3, Number 3, Summer 1973

ON AN INTEGRAL INEQUALITY FOR DIVERGENCE-FREE FUNCTIONS

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1. **Introduction.** Let D be a bounded, two-dimensional domain with smooth boundary ∂D and $\phi_i(x_1, x_2) = \phi_i(x)$ $[i = 1, 2]$ any sufficiently smooth vector-valued function which is defined on D, vanishes on ∂D, and satisfies the divergence-free condition, $\phi_{j,j} = 0$ in D. Here the summation convention is used and a comma denotes differentiation; for example,

$$\phi_{j,j} = \frac{\partial \phi_1}{\partial x_1} + \frac{\partial \phi_2}{\partial x_2} .$$

Of interest in this work is the calculation of a positive constant λ such that

$$\int_D \phi_i \phi_i \, dx \leqq \frac{1}{\lambda} \int_D \phi_{i,j} \phi_{i,j} \, dx \tag{1}$$

when D can be enclosed in a wedge of angle π/α, $\alpha > \frac{1}{2}$. Ideally we would like to calculate an optimal value for λ; however, this does not seem possible and we shall, therefore, sharpen known results.

Inequality (1) has been employed in stability and uniqueness studies for the Navier-Stokes equations (see e.g. Serrin [11]) and in an examination of growth properties of solutions for a model of a dusty gas system (see Crooke [2]), among other applications.

It is generally possible to establish these types of inequalities by considering a corresponding variational problem. For inequality (1) we are interested in the following variational problem:

$$\hat{\lambda} = \inf_{\psi_i \in \Gamma(D)} \frac{\int_D \psi_{i,j} \psi_{i,j} \, dx}{\int_D \psi_i \psi_i \, dx} \tag{2}$$

where $\Gamma(D)$ denotes the class of Dirichlet integrable, vector-valued functions which are defined on D, vanish on ∂D and satisfy $\psi_{j,j} = 0$ in D. Hence, if λ is any lower bound for $\hat{\lambda}$ and ϕ_i any function belonging to $\Gamma(D)$, then

$$\lambda \leqq \inf_{\psi_i \in \Gamma(D)} \frac{\int_D \psi_{i,j} \psi_{i,j} \, dx}{\int_D \psi_i \psi_i \, dx} \leqq \frac{\int_D \phi_{i,j} \phi_{i,j} \, dx}{\int_D \phi_i \phi_i \, dx} ,$$

Received by the editors October 26, 1971.
AMS (MOS) subject classifications (1970). Primary 49G15, 49G05.

or

$$\int_D \phi_i \phi_i \, dx \leqq \frac{1}{\lambda} \int_D \phi_{i,j} \phi_{i,j} \, dx.$$

We note that if

$$\lambda' = \inf_{\psi_i \in \Gamma'(D)} \frac{\int_D \psi_{i,j} \psi_{i,j} \, dx}{\int_D \psi_i \psi_i \, dx}$$

where $\Gamma'(D)$ denotes the space of Dirichlet integrable, vector-valued functions which are defined on D and vanish on ∂D, then $\lambda' \leqq \hat{\lambda}$. Lower bounds for λ' abound in the literature. For example, the well-known Faber-Krahn inequality (see [3], [5]) states $\lambda' \geqq [j_0]^2/P^2$ where P is the radius of the circle having the same area as D and j_0 is the first zero of the Bessel function J_0. Payne and Weinberger [10] have obtained a sharper lower bound for λ' when D lies interior to a wedge of angle π/α, $\alpha \geqq 1$. The results of this paper will be applicable to similar domains; however, we shall be interested in lower bounds for $\hat{\lambda}$, not λ'. Serrin [11] and Velte [14] have presented lower bounds for $\hat{\lambda}$ which depend on the geometry of D.

2. **An eigenvalue inequality.** Suppose D is a two-dimensional, bounded domain which can be enclosed in a wedge of angle π/α, $\alpha > \frac{1}{2}$. That is, if r denotes the distance from the apex of the wedge, which is assumed to be at the origin, then

$$D \subset \{(r, \theta) : \theta \in (0, \pi/\alpha), r \in (0, R)\} = D_\alpha.$$

As can readily be seen by its definition, D_α is the sector of a circle of radius R. Noting that $\hat{\lambda}$ is a monotone function of domain, it is sufficient to consider computing a lower bound for the eigenvalue

$$\tilde{\lambda} = \inf_{\psi_i \in \Gamma(D_\alpha)} \frac{\int_{D_\alpha} \psi_{i,j} \psi_{i,j} \, dx}{\int_{D_\alpha} \psi_i \psi_i \, dx}, \tag{3}$$

since $\tilde{\lambda} \leqq \hat{\lambda}$.

To compute a lower bound for $\tilde{\lambda}$, we shall use Weinstein's (see [15], [16]) " method of intermediate problems". In order to employ this technique it is necessary to change the form of (3) by introducing a stream-function v such that $\psi_1 = v_{,2}$ and $\psi_2 = -v_{,1}$. It is easily demonstrated that with this definition of $v(x)$ variational problem (3) is transformed into

$$\tilde{\lambda} = \inf_{v \in \Omega(D_\alpha)} \frac{\int_{D_\alpha} (\Delta v)^2 \, dx}{\int_{D_\alpha} (\nabla v)^2 \, dx} \tag{4}$$

where $\Omega(D_\alpha)$ is the space of sufficiently smooth scalar functions which are defined on D_α and which with their normal derivatives vanish on ∂D_α. It might be noted that the positive constant $\tilde{\lambda}$ is the first eigenvalue of the buckling problem for an elastic plate occupying D_α. The corresponding problem for the square was one of the problems originally treated by Weinstein in [15]. Since the normal derivative of v, which we henceforth denote $\partial v/\partial n$, vanishes on ∂D_α, it follows that for any bounded function p defined on ∂D_α we have

$$\int_{\partial D_\alpha} p \frac{\partial v}{\partial n}\, dS = 0.$$

This leads us to our first intermediate problem:

$$\lambda = \inf_{v \in \Omega'(D_\alpha)} \frac{\int_{D_\alpha} (\Delta v)^2\, dx}{\int_{D_\alpha} (\nabla v)^2\, dx} \tag{5}$$

where $\Omega'(D_\alpha)$ is the space of sufficiently smooth scalar functions v defined on ∂D_α, satisfying the boundary conditions:

$$v = \partial v/\partial n = 0 \qquad \text{on } r = R, \tag{6.a}$$

$$v = 0 \qquad \text{on } \partial D_\alpha^*, \tag{6.b}$$

and

$$\int_{\partial D_\alpha^*} p \frac{\partial v}{\partial n}\, dS = 0 \tag{6.c}$$

where ∂D_α^* denotes that portion of ∂D_α for which $\theta = 0, \pi/\alpha$. Using the standard arguments of variational calculus, it follows that $\lambda \leqq \tilde{\lambda}$.

The Euler equation and associated boundary conditions for the variational problem (5) can be shown to be:

$$\Delta^2 v + \lambda \Delta v = 0 \qquad \text{in } D_\alpha, \tag{7.a}$$

$$v = \partial v/\partial n = 0 \qquad \text{on } r = R, \tag{7.b}$$

and

$$v = 0, \qquad \Delta v = ap \quad \text{on } \partial D_\alpha^* \tag{7.c}$$

where a is an undetermined constant and $\Delta^2 v = \Delta(\Delta v)$. Let v^j and λ^j $(j = 1, 2, \cdots)$ denote the eigenfunctions and corresponding eigenvalues of the base problem, i.e., problem (7) with $a = 0$. That is, v^j and λ^j are solutions of

$$\Delta^2 v^j + \lambda^j \Delta v^j = 0 \qquad \text{in } D_\alpha,$$

$$v^j = \partial v^j/\partial n = 0 \qquad \text{on } r = R,$$

and

$$v^j = \Delta(v^j) = 0 \qquad \text{on } \partial D_\alpha^*.$$

These eigenfunctions and eigenvalues can be computed explicitly, but as we shall see in the analysis that follows, it is only necessary to explicitly know the first three. They are

$$v^1(r, \theta) = r^\alpha \sin(\alpha\theta)[r^{-\alpha}J_\alpha(\sqrt{\lambda^1}r) - R^{-\alpha}J_\alpha(\sqrt{\lambda^1}R)]$$

and

$$\lambda^1 = \frac{[j_{\alpha+1}^{(1)}]^2}{R^2}$$

$$v^2(r, \theta) = \begin{cases} v_1{}^2 = r^\alpha \sin(2\alpha\theta)[r^{-2\alpha}J_{2\alpha}(\sqrt{\lambda_1{}^2}r) - R^{-2\alpha}J_{2\alpha}(\sqrt{\lambda_1{}^2}R)] \\ \text{or} \\ v_2{}^2 = r^\alpha \sin(\alpha\theta)[r^{-\alpha}J_\alpha(\sqrt{\lambda_2{}^2}r) - R^{-\alpha}J_\alpha(\sqrt{\lambda_2{}^2}R)] \end{cases}$$

and

$$\lambda^2 = \left\{\lambda_1{}^2 = \frac{[j_{2\alpha+1}^{(1)}]^2}{R^2} \text{ or } \lambda_2{}^2 = \frac{[j_{\alpha+1}^{(2)}]^2}{R}\right\},$$

$$v^3(r, \theta) = \begin{cases} v_1{}^3 = r^\alpha \sin(3\alpha\theta)[r^{-3\alpha}J_{3\alpha}(\sqrt{\lambda_1{}^3}) - R^{-3\alpha}J_{3\alpha}(\sqrt{\lambda_1{}^3}R)] \quad \text{or} \\ v_2{}^3 = r^\alpha \sin(2\alpha\theta)[r^{-2\alpha}J_{2\alpha}(\sqrt{\lambda_2{}^3}r) - R^{-2\alpha}J_{2\alpha}(\sqrt{\lambda_2{}^3}R)] \quad \text{or} \\ v_3{}^3 = r^\alpha \sin(\alpha\theta)[r^{-\alpha}J_\alpha(\sqrt{\lambda_3{}^3}r) - R^{-\alpha}J_\alpha(\sqrt{\lambda_3{}^3}R)] \quad \text{or} \\ v_4{}^3 = r^\alpha \sin(2\alpha\theta)[r^{-2\alpha}J_{2\alpha}(\sqrt{\lambda_4{}^3}r) - R^{-2\alpha}J_{2\alpha}(\sqrt{\lambda_4{}^3}R)] \quad \text{or} \\ v_5{}^3 = r^\alpha \sin(\alpha\theta)[r^{-\alpha}J_\alpha(\sqrt{\lambda_5{}^3}r) - R^{-\alpha}J_\alpha(\sqrt{\lambda_5{}^3}R)] \end{cases}$$

and

$$\lambda^3 = \left\{\lambda_1{}^3 = \frac{[j_{3\alpha+1}^{(1)}]^2}{R^2}, \lambda_2{}^3 = \frac{[j_{2\alpha+1}^{(2)}]^2}{R^2}, \lambda_3{}^3 = \frac{[j_{\alpha+1}^{(3)}]^2}{R^2}, \right.$$

$$\left. \lambda_4{}^3 = \frac{[j_{2\alpha+1}^{(1)}]^2}{R^2} \text{ or } \lambda_5{}^3 = \frac{[j_{\alpha+1}^{(2)}]^2}{R^2}\right\}.$$

In the above expressions $j_\nu^{(n)}$ denotes the nth zero of the Bessel function $J_\nu(\,\cdot\,)$. One should note that the superscripts in $v_j{}^i$ and $\lambda_j{}^i$ are to be interpreted as indices and not as powers.

Expanding $v(r, \theta)$ in a series of the eigenfunctions $v^j(r, \theta)$, we have

$$v(r, \theta) = \sum_{i=1}^{\infty} B(v, v^i) \frac{v^i}{k^i}$$

where we have set $B(v, v^i) = \int_{D_\alpha} v_{,j} v^i_{,j} \, dx$ and $k^i = B(v^i, v^i)$. We now develop an expression for $B(v, v^i)$ in terms of λ, λ^i and

$$\oint_{\partial D_\alpha^*} p \frac{\partial v^i}{\partial n} dS.$$

Integrating by parts and using the boundary conditions for v and v^i, one finds that

$$B(v, v^i) = - \int_{D_\alpha} v(\Delta v^i) \, dx = - \int_{D_\alpha} v^i(\Delta v) \, dx. \tag{8}$$

Using Green's first identity and the appropriate boundary conditions for v^i, we obtain

$$\int_{D_\alpha} v^i(\Delta^2 v) \, dx - \int_{D_\alpha} \Delta v(\Delta v^i) \, dx = - \oint_{\partial D_\alpha} \Delta v \frac{\partial v^i}{\partial n} \, dS. \tag{9}$$

Integrating the second term in (9) by parts twice, (9) becomes

$$\int_{D_\alpha} v^i(\Delta^2 v) \, dx - \int_{D_\alpha} v(\Delta^2 v^i) \, dx = - \oint_{\partial D_\alpha} \Delta v \frac{\partial v^i}{\partial n} dS.$$

Employing the differential equations satisfied by v and v^i, this identity transforms to

$$-\lambda \int_{D_\alpha} v^i(\Delta v) \, dx + \lambda^i \int_{D_\alpha} v(\Delta v) \, dx = - \oint_{\partial D_\alpha^*} \Delta v \frac{\partial v^i}{\partial n} \, dS.$$

With (8), we have then

$$(\lambda - \lambda^i) B(v, v^i) = - \oint_{\partial D_\alpha^*} \Delta v \frac{\partial v^i}{\partial n} \, dS = - \oint_{\partial D_\alpha^*} ap \frac{\partial v^i}{\partial n} \, dS,$$

or

$$B(v, v^i) = \frac{a \oint_{\partial D_\alpha^*} p(\partial v^i / \partial n) \, dS}{\lambda^i - \lambda} .$$

Our infinite series for $v(r, \theta)$ then becomes

$$v(r, \theta) = \sum_{i=1}^{\infty} \frac{\oint_{\partial D_\alpha^*} p(\partial v^i / \partial n) \, dS}{\lambda^i - \lambda} \frac{v^i}{k^i} .$$

Since $\oint_{\partial D_\alpha^*} p(\partial v/\partial n)\, dS$ must vanish, we find that λ must satisfy the expression

$$a \sum_{i=1}^{\infty} \frac{[\oint_{\partial D_\alpha^*} p(\partial v^i/\partial n)\, dS]^2}{k^i(\lambda^i - \lambda)} = 0. \tag{10}$$

At this point we choose our function $p(r)$ which has been, up to this time, arbitrary. Namely, we set $p(r) = \partial v^1/\partial n$ on ∂D_α^*. With respect to equation (10) we have two cases to consider.

Case I. If $a = 0$, then λ is one of the λ^i's. Since with the above choice of $p(r)$ we have that

$$\int_{\partial D_\alpha^*} p(r) \frac{\partial v^1}{\partial n}\, dS \neq 0 \quad \text{and} \quad \int_{\partial D_\alpha^*} p(r) \frac{\partial v_2{}^2}{\partial n}\, dS \neq 0,$$

then necessarily $\lambda = \lambda_1{}^2$.

Case II. Suppose $a \neq 0$. In this case we necessarily have that λ must satisfy

$$\sum_{i=1}^{\infty} \frac{[\oint_{\partial D_\alpha^*} p(\partial v^1/\partial n)\, dS]^2}{k^i(\lambda^i - \lambda)} = \sum_{i=1}^{\infty} \frac{(c^i)^2}{\lambda^i - \lambda} = 0 \tag{11}$$

where we have set

$$c^i = \frac{1}{\sqrt{k^i}} \int_{\partial D_\alpha^*} p \frac{\partial v^i}{\partial n}\, dS.$$

We now define the auxiliary function ψ such that

$$\Delta^2 \psi = 0 \qquad \text{in } D_\alpha, \tag{12.a}$$

$$\psi = \frac{\partial \psi}{\partial n} = 0 \qquad \text{on } r = R, \tag{12.b}$$

and

$$\psi = 0, \qquad \Delta\psi = p \quad \text{on } \partial D_\alpha^*. \tag{12.c}$$

Using the differential equations for v^i and ψ, together with Green's identity, we find that

$$\begin{aligned}\int_{\partial D_\alpha} p \frac{\partial v^i}{\partial n}\, dS &= \int_{\partial D_\alpha} \Delta\psi \frac{\partial v^i}{\partial n}\, dS = \int_{D_\alpha} \psi(\Delta^2 v^i)\, dx \\ &= -\lambda^i \int_{D_\alpha} \psi \Delta v^i\, dx = \lambda^i B(v^i, \psi).\end{aligned}$$

Hence, (11) becomes

$$\frac{(c^1)^2}{\lambda - \lambda^1} = \sum_{i=2}^{\infty} \frac{\lambda^i}{\lambda^i - \lambda} \left\{ \frac{\lambda^i}{k^i} [B(v^i, \psi)]^2 \right\}. \tag{13}$$

We now consider two subcases, depending on whether c_2 is zero or not.

Suppose $c_2 = 0$ (which is the case when $v^2 = v_1{}^2$); then (13) reduces to

$$\frac{(c^1)^2}{\lambda - \lambda^1} = \sum_{i=3}^{\infty} \frac{(\lambda^i)^2}{k^i(\lambda^i - \lambda)} [B(v^i, \psi)]^2.$$

Now either $\lambda \geqq \lambda^3$ or $\lambda < \lambda^3$. Since Case I leads to the best possible result, $\lambda = \lambda_1{}^2$, the only inequality of interest is when $\lambda < \lambda^3$. We then have, noting that for all $i > 3$

$$\lambda^i/(\lambda^i - \lambda) \leqq \lambda^3/(\lambda^3 - \lambda),$$

the result

$$\frac{(c^1)^2}{\lambda - \lambda^1} \leqq \frac{\lambda^3}{\lambda^3 - \lambda} \sum_{i=3}^{\infty} \frac{\lambda^i}{k^i} [B(v^i, \psi)]^2$$

or

$$\frac{(c^1)^2}{\lambda - \lambda^1} \leqq \frac{\lambda^3}{\lambda^3 - \lambda} \left\{ \sum_{i=1}^{\infty} \frac{\lambda^i}{k^i} [B(v^i, \psi)]^2 - \frac{\lambda^1}{k^1} [B(v^1, \psi)]^2 \right\}.$$

One can show by expanding $\psi(r, \theta)$ in a Fourier series,

$$\psi(r, \theta) = \sum_{i=1}^{\infty} B(v^i, \psi) \frac{v^i}{k^i},$$

that

$$\sum_{i=1}^{\infty} \frac{\lambda^i}{k^i} [B(v^i, \psi)]^2 = \int_{D_\alpha} (\Delta\psi)^2 \, dx.$$

This identity implies that

$$\frac{(c^1)^2}{\lambda - \lambda^1} \leqq \frac{\lambda^3}{\lambda^1(\lambda^3 - \lambda^1)} \left\{ \lambda^1 \int_{D_\alpha} (\Delta\psi)^2 \, dx - (c^1)^2 \right\}.$$

Finally, after considerable algebraic manipulation, one can show that

$$\lambda \geqq \lambda^1[1 + 1/(A - 1)]$$

where

$$A = \frac{\lambda^3 \lambda^1}{(c^1)^2(\lambda^3 - \lambda^1)} \int_{D_\alpha} (\Delta\psi)^2 \, dx \; .$$

This yields a lower bound for λ. However since $\int_{D_\alpha} (\Delta\psi)^2 \, dx$ is not easily computable, we present two lemmas which afford upper bounds for this quantity in terms of $\oint_{\partial D_\alpha^*} p^2 \, dS$.

LEMMA 1. *Let ψ satisfy* (12). *Then*

$$\int_{D_\alpha} (\Delta\psi)^2 \, dx \leqq \gamma^1 \oint_{\partial D_\alpha^*} p^2 \, dS$$

where γ^1 is a constant which depends on α and D_α and is given by

$$\gamma^1 = \frac{(3)^{3/8}}{(2)^{1/3}\sqrt{\Lambda} \sin(\pi/2\alpha)} \; .$$

Here Λ is any constant such that

$$\int_{D_\alpha} (\nabla u)^2 \, dx \leqq \frac{1}{\Lambda} \int_{D_\alpha} (\Delta u)^2 \, dx$$

where u is any sufficiently smooth function which with its normal derivative vanishes on ∂D_α.

PROOF. Integrating by parts twice and using boundary condition (12.c), we find that

$$\int_{D_\alpha} (\Delta\psi)^2 \, dx = \oint_{\partial D_\alpha^*} p \frac{\partial\psi}{\partial n} \, dS.$$

An application of the Schwarz inequality yields

$$(14) \quad \left[\int_{D_\alpha} (\Delta\psi)^2 \, dx \right]^2 \leqq \left[\oint_{\partial D_\alpha^*} (p)^2 \, ds \right] \left[\oint_{\partial D_\alpha^*} \left(\frac{\partial\psi}{\partial n} \right)^2 dS \right] .$$

Let $(\bar{x}_1, \bar{x}_2)$ be an arbitrary point in D and consider

$$(15) \quad \begin{aligned} &\int_{D_\alpha} (x_k - \bar{x}_k)\psi_{,k} \, \psi_{,jj} \, dx \\ &\quad = - \int_{D_\alpha} [(x_k - \bar{x}_k)\psi_{,k}]_{,j} \psi_{,j} \, dx \\ &\quad\quad + \oint_{\partial D_\alpha} (x_k - \bar{x}_k)\psi_{,k} \, \psi_{,j} n_j \, dS \end{aligned}$$

where n_j is the jth component of the outward normal vector on ∂D. It can be shown by simple manipulations that (15) collapses to

$$\int_{D_\alpha} (x_k - \bar{x}_k)\psi_{,k}\psi_{,jj}\,dx = \frac{1}{2}\int_{\partial D_\alpha^*} (x_k - \bar{x}_k)n_k \left(\frac{\partial\psi}{\partial n}\right)^2 dS. \tag{16}$$

We now suppose the point $(\bar{x}_1, \bar{x}_2)$ is chosen to lie on the line $\theta = \pi/2\alpha$. Furthermore, let d be a positive number such that $d \leqq 2(x_k - \bar{x}_k)n_k$ for all $x \in \partial D_\alpha^*$. If $q^2 = (x_k - \bar{x}_k)(x_k - \bar{x}_k)$ and $h = (x_k - \bar{x}_k)n_k$, then (16) becomes

$$\int_{\partial D_\alpha^*} h\left(\frac{\partial\psi}{\partial n}\right)^2 dS = 2\int_{D_\alpha} q\,\frac{\partial\psi}{\partial q}\,dx. \tag{17}$$

Employing Schwarz's inequality to (17), we have

$$\begin{aligned}&\left[\int_{\partial D_\alpha^*} h\left(\frac{\partial\psi}{\partial n}\right)^2 dS\right]^2 \\ &\quad\leqq 4\left[\int_{D_\alpha} \psi_{,j}\psi_{,j}\,dx\right]\left[\int_{D_\alpha} q^2(\psi_{,jj})^2\,dx\right].\end{aligned} \tag{18}$$

Integrating by parts and employing (12), one calculates:

$$\begin{aligned}\int_{D_\alpha} q^2(\psi_{,jj})^2\,dx = \int_{\partial D_\alpha^*} q^2 p\,\frac{\partial\psi}{\partial n}\,dS - 2\int_{\partial D_\alpha^*} h\left(\frac{\partial\psi}{\partial n}\right)^2 dS \\ + 4\int_{D_\alpha} \psi_{,j}\psi_{,j}\,dx.\end{aligned}$$

Returning to inequality (18), we have with the above identity, for all $\beta > 0$,

$$\begin{aligned}&d^2\left[\int_{\partial D_\alpha^*}\left(\frac{\partial\psi}{\partial n}\right)^2 dS\right]^2 \\ &\quad\leqq \left[4\int_{D_\alpha} \psi_{,j}\psi_{,j}\,dx\right]\left[\frac{1}{2\beta}\int_{\partial D_\alpha^*} \frac{q^4p^2}{h}\,dS\right. \\ &\qquad\qquad + \frac{(\beta-4)}{2}\int_{\partial D_\alpha^*} h\left(\frac{\partial\psi}{\partial n}\right)^2 dS \\ &\qquad\qquad \left.+ 4\int_{D_\alpha} \psi_{,}\psi_{,j}\,dx\right].\end{aligned}$$

Choosing $\beta = 4$, the above inequality simplifies to

$$d^2\left[\oint_{\partial D_\alpha^*}\left(\frac{\partial\psi}{\partial n}\right)^2 dS\right]$$
$$\leqq \frac{1}{2\Lambda d}\left[\int_{D_\alpha}(\Delta\psi)^2\,dx\right]\left[\oint_{\partial D_\alpha^*} q^4p^2\,dS\right]$$
$$+\frac{16}{\Lambda^2}\left[\int_{D_\alpha}(\Delta\psi)^2\,dx\right]^2$$

where we have made use of the inequality

$$\int_{D_\alpha}\psi_{,j}\,\psi_{,j}\,dx \leqq \frac{1}{\Lambda}\int_{D_\alpha}(\psi_{,jj})^2\,dx.$$

Using this upper bound for $\oint_{\partial D_\alpha^*}(\partial\psi/\partial n)^2\,dS$, inequality (14) becomes

$$\left[\int_{D_\alpha}(\Delta\psi)^2\,dx\right]^3$$
$$\leqq \left[\oint_{\partial D_\alpha^*} p^2\,dS\right]^2\left[\frac{1}{2\Lambda d^3}\oint_{\partial D_\alpha^*} q^4p^2\,dS\right.$$
$$\left.+\frac{16}{\Lambda^2 d^2}\int_{D_\alpha}(\Delta\psi)^2\,dx\right].$$

Using a form of the arithmetic-geometric mean inequality, we have for all $\sigma > 0$,

$$\left[\int_{D_\alpha}(\Delta\psi)^2\,dx\right]^3$$
$$\leqq \frac{3\sigma^2}{(3\sigma^2-1)}\left[\frac{d}{2\Lambda[\sin(\pi/2\alpha)]^4}+\frac{128\sigma}{3\Lambda^3 d^3}\right]$$
$$\cdot\left[\oint_{\partial D_\alpha^*} p^2\,dS\right]^3$$

where we have made use of the fact that

$$\sup_{D_\alpha}(q) = \frac{d}{\sin(\pi/2\alpha)}.$$

Recalling that the two constants d and σ are still at our disposal, we optimize them by the choices

1. $d = 4(\sigma)^{1/4}\sin(\pi/2\alpha)/\sqrt{\Lambda}$,
2. $\sigma = \sqrt{3}$.

This completes the proof of Lemma 1.

We remark that the upper bound for $\int_{D_\alpha}(\Delta\psi)^2\,dx$ presented in Lemma 1 deteriorates as α becomes large. For larger α we present the following lemma.

LEMMA 2. *If ψ is a sufficiently smooth function which satisfies* (12), *then for $\alpha > 1$ we have*

$$\int_{D_\alpha} (\Delta\psi)^2\,dx \leqq \gamma^2 \oint_{\partial D_\alpha^*} p^2\,dS$$

where $\gamma^2 = R\sin(\pi/\alpha)/2$.

PROOF. We define the auxiliary function ϕ such that

(19.a) $$\Delta^2\phi = 0 \qquad \text{in } D_\alpha,$$

(19.b) $$\phi = \Delta\phi = 0 \qquad \text{on } r = R,$$

and

(19.c) $$\phi = 0, \qquad \Delta\phi = p \quad \text{on } \partial D_\alpha^*.$$

With the above definition of ϕ it is not difficult to show that

$$\int_{D_\alpha} (\Delta\psi)^2\,dx \leqq \int_{D_\alpha} (\Delta\phi)^2\,dx.$$

It can be shown (see Payne [9]) that if $h(x) = h(x_1, x_2)$ is a harmonic function on a bounded domain D whose boundary ∂D has everywhere nonnegative average curvature and if ρ denotes the strip of minimum width that encloses D, then

$$\int_D |h|^q\,dx \leqq \frac{\rho}{2}\oint_{\partial D} |h|^q\,dS, \qquad q \geqq 1.$$

Since $\Delta\phi$ is a harmonic function in D_α, we have for $q = 2$

$$\int_{D_\alpha} (\Delta\phi)^2\,dx \leqq \frac{\rho}{2}\oint_{\partial D_\alpha} (\Delta\phi)^2\,dS.$$

However, $\Delta\phi = p$ on ∂D_α^* and zero on the rest of ∂D; therefore, since $\rho = R\sin(\pi/\alpha)$, $\alpha > 1$, we conclude that

$$\int_{D_\alpha} (\Delta\phi)^2\,dx \leqq \frac{R\sin(\pi/\alpha)}{2}\oint_{\partial D_\alpha^*} p^2\,dS.$$

This completes the proof of Lemma 2.

We remark that it is possible to improve Lemma 2 for specific values of α. In particular, Payne [9] has shown that

$$\int_D |h|^q\,dx \leqq \frac{\sigma_M}{2}\int_{\partial D} |h|^q\,dS$$

where σ_M is the maximum stress for the torsion problem on D. Saint-Venant (see Timoshenko [12]) has computed σ_M for the domain D_α when $\alpha = 1,\ 3/2,\ 3$. In particular, he showed that $\sigma_M = (0.849)R$ when $\alpha = 1$; $\sigma_M = (0.652)R$ when $\alpha = 3/2$; and $\sigma_M = (0.490)R$ when $\alpha = 3$. However, these numbers yield only marginal improvements over the results of Lemma 2.

In comparing the upper bounds for $\int_{D_\alpha}(\Delta\psi)^2\,dx$ afforded by Lemmas 1 and 2, we find that if $\gamma = \min[\gamma^1, \gamma^2]$, then $\gamma = \gamma^1$ for $\alpha = 1, 2, 3, 4$ and $\gamma = \gamma^2$ for $\alpha > 4$.

Finally, with the upper bounds for $\int_{D_\alpha}(\Delta\psi)^2\,dx$ we obtain the lower bound for λ in the case when $a \neq 0$ and $c_2 = 0$:

$$\lambda \geqq \lambda^1\{1 + 1/(A-1)\}$$

where

$$A = \frac{\lambda^3\lambda^1\gamma}{(c^1)^2(\lambda^3 - \lambda^1)}\int_{\partial D_\alpha^*} p^2\,dS$$

with the understanding that if $\alpha \leqq 1$, then $\gamma = \gamma^1$. With our choice of p, one can show (see Luke [6], Abramowitz and Stegun [1], and Tranter [13]) that

$$\int_{\partial D_\alpha^*} p(r)\frac{\partial v^1}{\partial n}\,dS = \int_{\partial D_\alpha^*} p^2\,dS = R^{-1}J_\alpha^{\,2}(j^{(1)}_{\alpha+1})$$

$$\cdot\left\{\frac{4\alpha^2}{4\alpha^2-1}\left[(j^{(1)}_{\alpha+1})^2 + \alpha + \tfrac{1}{2}\right]\right.$$

$$- \frac{4\alpha^2}{2\alpha-1}\left[1 + \frac{2^{\alpha-1}\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}{(j^{(1)}_{\alpha+1})^{\alpha-2}}\right.$$

$$\left.\left.\cdot\left(H_{\alpha+1}(j^{(1)}_{\alpha+1}) - \frac{2\alpha}{j^{(1)}_{\alpha+1}}H_\alpha(j^{(1)}_{\alpha+1})\right)\right] + \frac{2\alpha^2}{2\alpha-1}\right\}$$

and

$$k^1 = \frac{\pi[j^{(1)}_{\alpha+1}]^2}{4\alpha}J_\alpha(j^{(1)}_{\alpha+1}),$$

where here $\Gamma(\cdot)$ is the gamma function and $H_\nu(\cdot)$ denotes the Struve function of order ν. Lastly, since $\partial v_2{}^3/\partial n$ and $\partial v_4{}^3/\partial n$ are orthogonal to $p(r)$ on $\partial D_\alpha{}^*$, and $\lambda_5{}^3 \leqq \lambda_3{}^3$, we conclude that $\lambda^3 = \min[\lambda_1{}^3, \lambda_5{}^3]$.

The subcase when $c_2 \neq 0$ (which is the situation when $v^2 = v_2{}^2$ and $\lambda^2 = \lambda_2{}^2$) still remains. Using the same type of analysis as in the subcase when $c_2 = 0$, one can show that

$$\lambda \geqq \lambda^1[1 + 1/(B-1)]$$

where

$$B = \frac{\lambda^2\lambda^1\gamma}{(c^1)^2(\lambda^2 - \lambda^1)} \int_{\partial D_\alpha{}^*} p^2\, dS, \qquad \lambda^2 = \lambda_2{}^2.$$

Therefore, in any case or subcase we have

$$\lambda \geqq R^{-2}\min\{[j_{2\alpha+1}^{(1)}]^2, [j_{\alpha+1}^{(1)}]^2[1 + 1/(A-1)], [j_{\alpha+1}^{(1)}]^2[1 + 1/(B-1)]\}. \tag{20}$$

Putting this eigenvalue inequality in the context of our integral inequality, we have shown:

THEOREM 1. *Let* ϕ_i $(i = 1, 2)$ *be any sufficiently smooth, vector-valued function which is defined on a two-dimensional domain* D, *which can be enclosed in a wedge of angle* π/α, $\alpha > \frac{1}{2}$, *and side-length* R. *If in addition* ϕ_i *vanishes on the boundary of* D *and satisfies the divergence-free condition* $\phi_{j,j} = 0$ *in* D, *then*

$$\int_D \phi_i\phi_i\, dx \leqq \frac{1}{\lambda}\int_D \phi_{i,j}\phi_{i,j}\, dx$$

where

$$\lambda = R^{-2}\min\{[j_{2\alpha+1}^{(1)}]^2, [j_{\alpha+1}^{(1)}]^2[1 + 1/(A-1)], [j_{\alpha+1}^{(1)}]^2[1 + 1/(B-1)]\}.$$

3. **Conclusion.** Velte [14] has shown that $\lambda = \Lambda^1$ where Λ^1 is the first eigenvalue for the clamped plate problem on a two-dimensional, bounded domain D. Furthermore, Payne and Weinberger (see Payne [8]) have proven that $\Lambda^1 \geqq \bar{\lambda}^2$ (an inequality conjectured by Weinstein [15]) where $\bar{\lambda}^2$ is the second eigenvalue for the fixed membrane problem on D. Hence, applying these two results to our wedge domain D_α, the literature affords the lower bound for λ:

$$\lambda \geqq \bar{\lambda}^2 = R^{-2}\min\{[j_{2\alpha}^{(1)}]^2, [j_\alpha{}^{(2)}]^2\}. \tag{21}$$

For comparison purposes the lower bounds provided by the Weinstein-Velte result, (21), and our result, (20), have been computed for $\alpha = 1, 2, 3, 4$. This juxtaposition is presented in the following table where we have assumed for the sake of simplicity that $R = 1$.

Value of α	$\lambda_{WV} = \min\{[j_\alpha^{(2)}]^2, [j_{2\alpha}^{(1)}]^2\}$	$\lambda_C = \min\{[j_{2\alpha+1}^{(1)}]^2, [j_{\alpha+1}^{(1)}]^2[1 + 1/(A-1)], [j_{\alpha+1}^{(1)}]^2[1 + 1/(B-1)]\}$
1	$[j_2^{(1)}]^2 =$ 26.37	$1.3[j_2^{(1)}]^2 =$ 35.29
2	$[j_4^{(1)}]^2 =$ 57.58	$1.7[j_3^{(1)}]^2 =$ 70.77
3	$[j_3^{(2)}]^2 =$ 95.27	$2.1[j_4^{(1)}]^2 =$ 122.32
4	$[j_4^{(2)}]^2 =$ 122.42	$2.2[j_5^{(1)}]^2 =$ 168.68

In the above calculations we have let $\Lambda = \min\{[j_\alpha^{(2)}]^2, [j_{2\alpha}^{(1)}]^2\}$ and the values of the various Struve functions have been taken from [4]. Calculations of λ_C and λ_{WV} when $\alpha > 4$ seem to indicate that

$$\lambda_C > \frac{[j_{\alpha+1}^{(2)}]^2}{R^2} \geqq \frac{[j_\alpha^{(2)}]^2}{R^2} = \lambda_{WV}.$$

Finally, to show how our lower bound is sensitive to the positioning of the origin, let D be a right isosceles triangular region with equal sides of unit length. If the origin is taken at the midpoint of the hypotenuse, then we obtain from (20) with $\alpha = 1$,

$$\lambda \geqq 70.58.$$

Placing the origin at the right-angled corner, we have with $\alpha = 2$,

$$\lambda \geqq 70.77.$$

Finally, if we choose the origin at once of the acute-angled corners and letting $\alpha = 4$ in (20), we find

$$\lambda \geqq 84.34.$$

4. **Acknowledgement.** The author would like to express his gratitude to Professor L. E. Payne, with whom the author had many fruitful discussions.

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