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A CHAIN CONDITION FOR GROUPS

THOMAS S. SHORES

ABSTRACT. In this article a chain condition for groups, called the bounded chain condition, is studied; this chain condition includes the ascending and descending chain conditions as special cases. It is shown that every locally radical group which satisfies the bounded chain condition on subgroups must satisfy the ascending or descending chain condition on subgroups, i.e., such groups are Artinian or Noetherian. The same conclusion holds for nilpotent groups which satisfy the bounded chain condition on abelian subgroups.

The ascending and descending chain conditions (ACC and DCC, resp.) for subgroups have been studied extensively as essentially independent phenomena. Our intent in this note is to examine a new chain condition which is a common consequence of both of these chain conditions. By a *chain* of subgroups we mean a double sequence of subgroups $\cdots, C_{-1}, C_0, C_1, \cdots$ of a given group such that $C_i \subseteq C_{i+1}$ for all integers i. Such a chain is *bounded* in case there is a positive integer n such that all terms C_i are equal for $i > n$ or all terms C_i are equal for $i < -n$. If every chain of subgroups of the group G is bounded, we say G has the *bounded chain condition* (BCC) on subgroups. The BCC on subgroups (or for that matter on any lattice) can be thought of as a minimal generalization of the ACC and DCC.

Two questions are of concern to us: First of all, it is only natural to ask how far the BCC extends the ACC and DCC. Clearly the BCC on subgroups is implied by both the ACC and the DCC on subgroups. Whether or not every group with the BCC on subgroups in fact has the ACC or DCC (i.e., is Artinian or Noetherian) appears to be a difficult question. We shall show in Theorem 3 that every locally radical group with the BCC on subgroups is Artinian or Noetherian.

The second question we consider arises from the following fact: It is known that for groups that are "close" to abelian, such as solvable groups, the ACC or DCC on *abelian* subgroups implies the same condition for all subgroups. Does the BCC exhibit this type of behavior? The answer is no in most cases. In fact we show in the next section that there exists a hypercentral metabelian group which has the BCC

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on abelian subgroups and is neither Artinian nor Noetherian. Such a group is locally radical and does not have the BCC on subgroups by Theorem 3. On the other hand, nilpotent groups with the BCC on abelian subgroups do satisfy the BCC for subgroups (and are therefore Artinian or Noetherian).

Let us recall here that a group G is hypercentral if some term of the upper central series of G is itself G. If a finite term of the upper central series is G, we say G is nilpotent. Also, G is a solvable group if G has a finite normal series with abelian factors and a radical group if G has an ascending normal series of subgroups whose factors are locally nilpotent. Some more notation: Let G be a group and $\mathfrak{p}$ a group theoretic class.

$Z(G)$ = center of G $(= Z_1(G))$.

$Z_\alpha(G)$ = αth term of the upper central series of G.

G is a $\mathfrak{p}$-group = G belongs to the class $\mathfrak{p}$.

G is a locally $\mathfrak{p}$ = finitely generated subgroups of G are $\mathfrak{p}$-groups.

G is hyper-$\mathfrak{p}$ = nontrivial epimorphic images of G have nontrivial normal $\mathfrak{p}$-subgroups.

G is almost $\mathfrak{p}$ = G has a normal $\mathfrak{p}$-subgroup whose index in G is finite.

$\mathfrak{c}$ = the class of groups which are Artinian or Noetherian.

If S is a subset of G, $\langle S\rangle$ is the subgroup generated by S and $C(S)$ is the centralizer of S in G.

1. **Nilpotent groups and the BCC.** We begin by examining abelian groups with the BCC.

LEMMA 1. *Every abelian group G with the BCC on subgroups is an $\mathfrak{c}$-group; if G has an element of infinite order, then G is Noetherian.*

PROOF. Let G be abelian with the BCC. If G has an element x of infinite order, set $G_n = \langle x^{2^n}\rangle$ and obtain a strictly decreasing sequence of subgroups G_n, $n = 0, 1, \cdots$. Thus the factor group G/G_0 has no strict infinite ascending sequences of subgroups; for the inverse image in G of such a sequence together with the sequence of subgroups G_n, $n = 0, 1, \ldots$, would yield a chain of subgroups of G which is not bounded. This is a contradiction. Hence G/G_0 is a Noetherian group. But G_0 is Noetherian and extensions of Noetherian groups by Noetherian groups are themselves Noetherian. Hence G is an $\mathfrak{c}$-group.

Next suppose G is a periodic group. Then G contains no nontrivial infinite direct sums of subgroups. For if this were so, say G had subgroups A_n, $n = 0, \pm 1, \pm 2, \ldots$, we could let B_n be the subgroup of G generated by the groups A_k, $k \leqq n$, and obtain an unbounded chain of subgroups

$$\cdots \subset B_{-1} \subset B_0 \subset B_1 \subset \cdots.$$

Now every periodic abelian group has a direct summand which is either cyclic or quasi-cyclic (i.e., of Prüfer type) by a theorem of Kulikov (see item 5.2.10 of [4, p. 98]). It follows readily that G must be a finite direct sum of cyclic and quasi-cyclic subgroups. Since these subgroups are Artinian, so is G. Hence G is an $\mathfrak{c}$-group; in any case the lemma follows.

There are several results about automorphism groups which will be needed in the sequel:

1. Every solvable group of automorphisms of an abelian Noetherian group is itself Noetherian (see Satz 2 of [**2**, p. 171] for a proof).

2. Every periodic group of automorphisms of an abelian Artinian group is finite (see Lemma 1.2(b) of [4, p. 405]). The main result on nilpotent groups is as follows:

THEOREM 1. *The following are equivalent conditions on the nilpotent group G:*

(I) *G is an $\mathfrak{c}$-group.*

(II) *G has the BCC on subgroups.*

(III) *G has the BCC on abelian subgroups.*

PROOF. That (I) implies (II) and (II) implies (III) is obvious. Suppose that the nilpotent group G satisfies (III). Choose a maximal normal abelian subgroup A of G. Since G is nilpotent, $C(A) = A$. Also A is an $\mathfrak{c}$-group by Lemma 1. The group $G/A = G/C(A)$ is essentially a group of automorphisms of A. If A is a Noetherian group, then so is G/A by item 1 preceding this theorem. It follows that G itself is Noetherian.

Now suppose that A is an Artinian group. If G/A is periodic, then this group is finite by item 2 so that G must be Artinian. So it remains to consider the case in which G/A is not periodic. If A is finite, then G is Noetherian and we are done. Suppose that A is infinite and choose an element $x \in G - A$ whose image in G/A has infinite order. We shall complete the proof by deriving a contradiction. The group $H = A\langle x\rangle$ is nilpotent and has the BCC on abelian subgroups. It follows that there exists an integer n such that, for all $a \in A$,

$$[a, x, \cdots, x] = 1,$$

where x is iterated n times in the commutator expression above. (Here $[u, v] = u^{-1}v^{-1}uv$.) If we represent the identity endomorphism of A as 1_A, the endomorphism of A induced by conjugation by x as ϕ and write the group operation of A additively, then the commutator identity above becomes

$$a(1_A - \phi)^n = 0.$$

Since a was arbitrary, $A(1_A - \phi)^n = 0$. Therefore there is an integer $k \leqq n$ such that the kernel K of the restriction of endomorphism $(1_A - \phi)$ to the group $A(1_A - \phi)^k$ is infinite. K is an infinite periodic abelian group with the BCC on subgroups. It follows from the proof of Lemma 1 that K has a quasi-cyclic subgroup L. Since ϕ acts as the identity on L, x centralizes L. Hence $L\langle x\rangle$ is an abelian subgroup of G. But $L\langle x\rangle$ contains the non-Artinian subgroup $\langle x\rangle$ and non-Noetherian subgroup L. Hence $L\langle x\rangle$ is not an $\mathfrak{c}$-group. By Lemma 1 this is a contradiction and thus Theorem 1 is proved.

EXAMPLE. Let p be a prime and A a quasi-cyclic p-group, i.e., A is generated by elements $a_1, a_2, \cdots$ where $a_1{}^p = 1$ and $a_{n+1}^p = a_n$, $n = 1, 2, \cdots$. Next define an automorphism ϕ of A by the equation

$$a_{n+1}\phi = (a_{n+1})^{1+p+\cdots+p^n}, \qquad n = 1, 2, \ldots.$$

It is readily seen that ϕ is well-defined, since the order of a_n is p^n. Now let G be the split extension of A by $\langle\phi\rangle$. Then $Z(G) = \langle a_1\rangle$. A simple induction shows that $Z_n(G) = \langle a_n\rangle$ for all positive n. Hence $Z_{\omega+1}(G) = G$ and G is a metabelian hypercentral group. Let C be an abelian subgroup of G. If $A \subseteq C$, then $C = A(C \cap \langle\phi\rangle)$. Since no positive power of ϕ is in $Z(G)$, $C \cap \langle\phi\rangle = 1$ and $A = C$. On the other hand if $A \cap C \subset A$, then $A \cap C$ is a finite group. Also $A \cap C \lhd C$, so that

$$C/(A \cap C) = AC/A$$

which is cyclic and hence Noetherian. Thus C is Noetherian and every abelian subgroup of G is an $\mathfrak{c}$-group. Therefore G has the BCC on abelian subgroups. But G does not have the BCC on subgroups, for G has an unbounded chain of subgroups, namely,

$$\cdots \subset \langle a_1, \phi^4\rangle \subset \langle a_1, \phi^2\rangle \subset \langle a_1, \phi\rangle \subset \langle a_1, a_2, \phi\rangle \subset \cdots.$$

REMARK 1. It is readily seen that any solvable group with the ACC or DCC on subnormal subgroups enjoys the same chain condition on all subgroups. Even in the simplest cases the BCC does not exhibit this type of behavior. For let G be as in the previous example and K a normal subgroup of G. If $K \cap A \subset A$, then $K \cap A$ is finite and it is easily checked that $G/(K \cap A) \cong G$. Also the image of A in $G/(K \cap A)$ is the maximum divisible subgroup, so this image is self-centralizing. But $AK/(K \cap A)$ is the direct sum of $A/(K \cap A)$ and

$K/(K \cap A)$, so $K \subseteq A$. Hence the only normal subgroups of G are the subgroups of A and the subgroups containing A. It follows that the only subnormal subgroups of G are the normal subgroups. Therefore G has the BCC on subnormal subgroups but not on all subgroups.

REMARK 2. While the BCC on abelian subgroups does not ensure the BCC on all subgroups for the class of solvable groups, it does at least ensure the solvability of radical groups. Call a group a minimax group if it has a Noetherian normal subgroup with Artinian factor group. Then the following is true: Every hyper-(almost radical) group with the BCC on abelian subgroups is an almost solvable polyminimax group. This fact is an immediate consequence of our Lemma 1 and Remark 2.3 in a recent paper of B. Amberg [1].

2. Solvable groups and the BCC.

LEMMA 2. *Every solvable group with the BCC on subgroups is an* $\mathfrak{C}$*-group.*

PROOF. Let G be a solvable group and denote the nth derived term of G by $G^{(n)}$. (Thus $G^{(1)}$ is the commutator subgroup of G and $G^{(n+1)} = (G^{(n)})^{(1)}$.) The first integer n such that $G^{(n)} = 1$ is the derived length of G and the proof is by induction on n. For $n = 1$, G is abelian and the result follows from Lemma 1. Suppose that $n > 1$ and the result is true for groups of shorter derived length. Let $K = G^{(1)}$ and $C = C(K)$. Note that $C^{(1)} \subseteq Z(C)$, so that C is a nilpotent group. By Theorem 1, C is an $\mathfrak{C}$-group.

The group K is also an $\mathfrak{C}$-group by the induction hypothesis. If K has an element of infinite order, say x, then the factor group G/K has no infinite ascending sequences of subgroups. For the inverse image in G of such a sequence, together with the decreasing sequence of subgroups $\langle x^{2^i} \rangle$, $i = 1, 2, \cdots$, yields an unbounded chain in G. Therefore both K and G/K are Noetherian groups, so that G is Noetherian. Thus we may assume that K is periodic. If K is also Noetherian, then it must be finite since periodic solvable groups are locally finite. In this case G/C is isomorphic to a group of automorphisms of K and therefore finite. Since C is an $\mathfrak{C}$-group, it follows that G is an $\mathfrak{C}$-group. Finally consider the case in which K is infinite Artinian. Then K has a characteristic abelian subgroup A of finite index in K (see, e.g., [4, p. 403]). Such a subgroup is Artinian and hence a direct sum of cyclic and quasi-cyclic subgroups. Thus for some prime p and p-primary component of A it is infinite. Let A_i be the subgroup of elements of A whose order is a divisor of p^i, $i = 1, 2, \cdots$. The A_i are characteristic in A, hence also characteristic subgroups of G. Also we obtain an increasing sequence of subgroups

$$A_1 \subset A_2 \subset \cdots.$$

If G/K is not periodic, then there is an element $x \in G$ whose image in G/K has infinite order. Thus we can form the unbounded chain of subgroups

$$\cdots \subset A_1\langle x^4\rangle \subset A_1\langle x^2\rangle \subset A_1\langle x\rangle \subset A_2\langle x\rangle \subset \cdots.$$

Since G has the BCC on subgroups, this is impossible and G/K is periodic. Now G/K has the BCC on subgroups since it is an epimorphic image of G. By Lemma 1 G/K is an $\mathfrak{c}$-group. Hence G/K is Artinian in any case. It follows that G is also Artinian and Lemma 2 is proved.

THEOREM 2. *If G is a hyper-(almost radical) group, then the following are equivalent conditions on G:*

(I) *G has the BCC on subgroups.*

(II) *G is an almost solvable $\mathfrak{c}$-group.*

PROOF. Clearly (II) implies (I). Conversely, if G satisfies (I), then G is almost solvable by Remark 2. Any solvable subgroup of G also has the BCC on subgroups and is therefore an $\mathfrak{c}$-group by Lemma 2. Thus G is a finite extension of an $\mathfrak{c}$-group and hence G is itself an $\mathfrak{c}$-group.

THEOREM 3. *If G is a locally radical group, then the following are equivalent conditions on G:*

(I) *G has the BCC on subgroups.*

(II) *G is a solvable $\mathfrak{c}$-group.*

PROOF. Clearly (II) implies (I). Conversely, suppose G is locally radical and satisfies (I). Since subgroups of G also have the BCC on subgroups, G is locally a solvable $\mathfrak{c}$-group by Theorem 2. Suppose that G has an element x of infinite order. If G were not finitely generated, we could find elements $x = x_1, x_2, x_3, \ldots$ such that if $S_i = \langle x_1, \cdots, x_i\rangle$, then $S_1 \subset S_2 \subset \cdots$. Thus we have the unbounded chain

$$\cdots \subset \langle x^2\rangle \subset \langle x\rangle \subset S_1 \subset S_2 \subset \cdots.$$

This is impossible, so G is finitely generated and solvable. Hence G is an $\mathfrak{c}$-group by Lemma 2. On the other hand if G has no elements of infinite order, then every abelian subgroup of G has the BCC and is periodic. This implies that every abelian subgroup of G is Artinian, so G has the DCC on abelian subgroups. We now apply a result of S. N. Černikov which says that every locally solvable group whose abelian subgroups satisfy the minimum condition is solvable (see [3]); it follows that G is solvable and therefore an $\mathfrak{C}$-group by Lemma 2. Hence (I) implies (II).

REMARK 3. One might expect to improve Theorem 3 by replacing "radical" by "almost radical." Such an extension would involve some difficult unsolved problems in the theory of finiteness conditions in groups. In this connection we note that V. Sunkov has recently shown in [6] that locally finite groups (and therefore also locally almost radical groups) whose abelian subgroups are Artinian are likewise Artinian. However, it still seems to be unknown whether or not locally solvable groups whose abelian subgroups are Noetherian are likewise Noetherian.

REMARK 4. The question of whether or not every group with the BCC is an $\mathfrak{C}$-group remains open. A related problem is that of finding a group with the BCC which is not almost solvable. In this connection we point out that it is still not known whether or not there exists an $\mathfrak{C}$-group which is not almost solvable.

REFERENCES

1. B. Amberg, *Groups with maximum conditions*, Pacific J. Math. **32** (1970), 9–19. MR **40** #7362.

2. R. Baer, *Auflösbare Gruppen mit Maximalbedingung*, Math. Ann. **129** (1955), 139–173. MR **16**, 994.

3. S. N. Černikov, *On locally solvable groups satisfying the minimality condition for subgroups*, Mat. Sb. **28** (**70**) (1951), 119–129. (Russian) MR **12**, 477.

4. Annemarie Schlette, *Artinian, almost abelian groups and their groups of automorphisms*, Pacific J. Math. **29** (1969), 403–425. MR **40** #241.

5. W. R. Scott, *Group theory*, Prentice-Hall, Englewood Cliffs, N. J., 1964. MR **29** #4785.

6. V. Sunkov, *On locally finite groups with minimum condition on abelian subgroups*, Algebra i Logika **9** (1970), 220–248. (Russian).

UNIVERSITY OF NEBRASKA, LINCOLN, NEBRASKA 68508