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DUAL AND BIDUAL LOCALLY CONVEX SPACES[1]

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ABSTRACT. A characterization of quasi-barreled spaces which are strong duals of quasi M-barreled spaces is given. Sufficient conditions are given for a quasi-barreled space E to be the strong bidual of a closed subspace F of E, when F has the inherited topology. These results are applied to spaces of continuous linear maps and to spaces of matrix maps between sequence spaces.

When is a locally convex space the strong dual (bidual) of a locally convex space? Many authors have addressed themselves to the first of these questions, see [6] for a partial bibliography. An answer to the second question, in the special case of Banach spaces, has been given by Shapiro, Shields, and Taylor in a paper yet unpublished. Their theorem reads:

THEOREM A. *Let F be a closed subspace of a Banach space E. The inclusion map of F into E can be extended to an isometry of F'' onto E if and only if there is a locally convex topology $\mathcal{T}$ on E satisfying*

(i) *the restriction of $\mathcal{T}$ to F is weaker than the norm topology on F,*
(ii) *the unit ball of F is $\mathcal{T}$-dense in the unit ball of E,*
(iii) *no proper closed subspace of F is $\mathcal{T}$-dense in E, and*
(iv) *the unit ball of E is $\mathcal{T}$-compact.*

In this paper we obtain results which generalize this theorem to general locally convex spaces. Our work also gives another answer to the first question listed above. In §3 we apply our results to spaces of matrix maps between sequence spaces.

We follow the terminology of Köthe [5] throughout. Hence if $E[\tau]$ is a locally convex space we will denote by $(E[\tau])'$ the topological dual of $E[\tau]$. Also, if a space is locally convex we mean it is also Hausdorff. We shall abbreviate locally convex space by l.c. space. We recall that $(E[\tau])'[\mathcal{T}_b(E)]$ is the strong dual of $E[\tau]$ and the strong bidual of $E[\tau]$ is the strong dual of the strong dual of $E[\tau]$. If F is a subspace of a l.c. space $E[\tau]$, the topology induced on F by τ will be denoted by τ_i. Also, if $\langle E_1, E_2\rangle$ is a dual system of vector spaces, then $\mathcal{T}_s(E_2, E_1)$, $\mathcal{T}_k(E_2, E_1)$, and $\mathcal{T}_b(E_2, E_1)$ denote the weak, Mackey, and strong topology on E_1 from E_2 respectively.

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1. **Dual spaces.** In this section we give a characterization of quasi-barreled spaces which are strong duals of quasi M-barreled spaces. The theorem will be applied to some examples at the end of this section.

We recall the definitions of quasi-barreled and quasi M-barreled from [5] and [6].

DEFINITIONS. A l.c. space $E[\tau]$ is said to be *quasi-barreled* (*quasi M-barreled*) if, in $(E[\tau])'$, each absolutely convex $\mathcal{T}_b(E)$-bounded set is τ-equicontinuous (relatively $\mathcal{T}_s(E)$-compact).

As remarked in [6] all quasi-barreled spaces are quasi M-barreled.

THEOREM 1. *Let $E[\tau]$ be a quasi-barreled space. Then $E[\tau]$ is the strong dual of a quasi M-barreled space if and only if there is an l.c. topology $\mathcal{T}$ on E satisfying*

(i) *$\mathcal{T}$ is weaker than τ,*

(ii) *each absolutely convex τ-bounded set B in E is $\mathcal{T}$-relatively compact, i.e. the $\mathcal{T}$-closure of B is $\mathcal{T}$-compact, and*

(iii) *there is a τ-neighborhood base of 0, $\mathcal{U}(0)$, consisting of $\mathcal{T}$-barrels.*

PROOF. If $E[\tau]$ is the strong dual of a quasi M-barreled space G, then the conclusion follows from the definition with $\mathcal{T} = \mathcal{T}_s(G, E)$.

Conversely, let $H = (E[\mathcal{T}])'$ and let $\mathcal{T}_0$ denote the topology on H of uniform convergence on the τ-bounded sets of E. Since $\mathcal{T}_s(H, E) \subset \mathcal{T}$, it follows from (ii) that $\mathcal{T}_0 \subset \mathcal{T}_k(E, H)$. Thus $(H[\mathcal{T}_0])' = E$.

Let $\mathcal{U}(0)$ be as in (iii) with $U \in \mathcal{U}(0)$. If $^\circ$ denotes the polar between E and H, then U° is $\mathcal{T}_s(E, H)$-bounded in H. $\mathcal{T}$ and $\mathcal{T}_s(H, E)$ are compatible topologies so from (iii) and the bipolar theorem $U = U^{\circ\circ}$. Thus $\tau \subset \mathcal{T}_b(H, E)$.

Let M be an absolutely convex $\mathcal{T}_s(E, H)$-bounded set. Then M° is an absolutely convex, absorbing, $\mathcal{T}_s(H, E)$-closed subset of E. Hence M° is a τ-barrel. From (ii) and the Banach-Mackey theorem [5, §20, 11], each τ-bounded set in E is $\mathcal{T}_b(H, E)$-bounded. M° is a $\mathcal{T}_b(H, E)$-neighborhood of 0, so M° absorbs each τ-bounded set in E. Using the hypothesis that $E[\tau]$ is quasi-barreled we have $\tau \supset \mathcal{T}_b(H, E)$.

If B is an absolutely convex $\mathcal{T}_b(H, E)$-bounded set, then B is τ-bounded. The $\mathcal{T}$-closure of B is $\mathcal{T}$-compact by (ii), and hence is $\mathcal{T}_s(H, E)$-compact. Therefore $H[\mathcal{T}_0]$ is quasi M-barreled.

REMARKS. If $E[\tau]$ is a normed space we can replace (ii) and (iii) of Theorem 1 by the condition that the unit ball of E is $\mathcal{T}$-compact.

Of course a Banach space is always the strong dual of a l.c. space, however the added conditions above characterize the Banach spaces which are duals of Banach spaces.

We give two examples using Theorem 1. The second example is well known from the theory of tensor products.

EXAMPLE 1. Let $E[\mathcal{T}]$ be a l.c. space and let B be an absolutely convex, compact subset of $E[\mathcal{T}]$. If $E(B)$ is the space generated by B with norm given by the gauge from B, then $E(B)$ is the dual of a normed space. In particular, let B be a closed bounded subset of a Montel space.

EXAMPLE 2. Let $F_1[\mathcal{T}_1]$ be a quasi-barreled space and $F_2[\mathcal{T}_2]$ a reflexive l.c. space. Let $\mathcal{L}_b(F_1, F_2)$ denote the space of continuous linear maps from $F_1[\mathcal{T}_1]$ into $F_2[\mathcal{T}_2]$ with the topology of uniform convergence on bounded sets of $F_1[\mathcal{T}_1]$ with respect to the topology $\mathcal{T}_2$ on F_2. If $\mathcal{L}_b(F_1, F_2)$ is a quasi-barreled space, then $\mathcal{L}_b(F_1, F_2)$ is the strong dual of a quasi M-barreled space.

PROOF. Since $F_1[\mathcal{T}_1]$ and $F_2[\mathcal{T}_2]$ are Mackey spaces we have $\mathcal{L}(F_1, F_{2s}) = \mathcal{L}(F_1, F_2)$, where $F_{2s} = F_2[\mathcal{T}_s(F_2[\mathcal{T}_2])']$ [8, p. 158]. Let $E[\tau]$ be $\mathcal{L}_b(F_1, F_2)$ and let $E[\mathcal{T}]$ be $\mathcal{L}_s(F_1, F_{2s})$, where the latter is the topology of uniform convergence on finite sets in F_1 with respect to the weak topology on F_2.

Condition (i) of Theorem 1 is clear. The theorem of Alaoglu-Bourbaki [1, p. 23] and the assumptions that $F_1[\mathcal{T}_1]$ is quasi-barreled and $F_2[\mathcal{T}_2]$ is reflexive prove that each bounded set in $E[\tau]$ is relatively $\mathcal{T}$-compact. Let $U = \{T \in E : T(B) \subset V\}$ where B is bounded in $F_1[\mathcal{T}_1]$ and V is a weakly closed neighborhood of 0 in $F_2[\mathcal{T}_2]$. If $\{T_\alpha\}$ is a net in U with $\{T_\alpha\}$ converging in $\mathcal{T}$ to an element T of E, then $T \in U$, since $\{T_\alpha(x)\}$ converges weakly for each x in B and V is weakly closed.

2. Strong biduals.

In this section we give sufficient conditions for a quasi-barreled space to be the strong bidual of a l.c. space.

THEOREM 2. *Let $E[\tau]$ be a quasi-barreled space with F a τ-closed subspace. Let F'' denote the bidual of $F[\tau_i]$. The inclusion map of F into E can be extended to an isomorphic homeomorphism of $F''[\mathcal{T}_b(F[\tau_i])']$ onto $E[\tau]$ if there is a l.c. topology $\mathcal{T}$ on E satisfying*

(i) *$\mathcal{T}$ is weaker than τ,*

(ii) *for each x in E there is a τ_i-bounded set B in F with x in the $\mathcal{T}$-closure of B in E,*

(iii) *no proper τ-closed subspace of F is $\mathcal{T}$-dense in E,*

(iv) *each absolutely convex τ-bounded set B in E is $\mathcal{T}$-relatively compact, and*

(v) *there is a τ-neighborhood base of 0 consisting of $\mathcal{T}$-barrels.*

PROOF. The conclusion will follow from Theorem 1 and its proof if we can show that $(F[\tau_i])' = H$, where $H = (E[\mathcal{T}])'$, and $\mathcal{T}_b(F, H)$ is compatible with the dual pair $\langle H, E\rangle$.

$(F[\mathcal{T}_i])' = H$, since (ii) implies that F is $\mathcal{T}$-dense in E. $\mathcal{T}$ is weaker than τ, so $H \subset (F[\tau_i])'$. Let $f \in (F[\tau_i])'$ with $f \neq 0$. Define G to be

$\{x \in F \mid f(x) = 0\}$. G is a τ-closed proper subset of F and hence by (iii) is not $\mathcal{T}_i$-dense in F. The Hahn-Banach theorem then guarantees the existence of a nonzero element g in H with $g(G) = 0$. It follows that $g = \alpha f$, for some scalar α [5, §9, 2]. Hence $H = (F[\tau_i])'$.

We have two dual pairs $\langle F, H\rangle$ and $\langle H, E\rangle$. We shall denote by $^{\square}$ and $^{\circ}$ the respective polar operations. A neighborhood base for $\mathcal{T}_b(F, H)$ is given by polars of τ_i-bounded sets in F. If $x \in E$, then (ii), the fact that $\mathcal{T}_s(H, E) \subset \mathcal{T}$, and the bipolar theorem imply the existence of an absolutely convex τ_i-bounded set B in F with $x \in B^{\square\circ}$. Hence $\mathcal{T}_s(E, H) \subset \mathcal{T}_b(F, H)$. Since the $\mathcal{T}$-closure in E of a τ_i-bounded set B is $\mathcal{T}$-compact it is also $\mathcal{T}_s(H, E)$-compact. Hence $B^{\square}$ contains a $\mathcal{T}_k(E, H)$-neighborhood. Thus $\mathcal{T}_b(F, H) \subset \mathcal{T}_k(E, H)$.

Remarks. A partial converse of Theorem 2 is given by Theorem 4 of [6]. Theorem 2 and this partial converse yield Theorem A of the introduction.

Condition (iii) of Theorem 2 holds if $(F[\tau_i])' = H$.

3. **Applications.** Our purpose in this section is to apply Theorem 2 to matrix maps between sequence spaces. We recall some definitions and terminology.

A *sequence space* is a vector space, λ, of sequences of scalars containing all finite sequences. In a sequence space, λ, algebraic and order structures will all be coordinatewise and e^i will indicate the ith coordinate vector. We denote by λ^* the *Köthe dual* of λ, as defined in [5, §30]. λ is said to be *perfect* if $\lambda = \lambda^{**}$. A set of sequences U is said to be normal if whenever $x \in U$ and $|y| = (|y_i|)$ is coordinatewise less than $|x|$ then $y \in U$. All perfect spaces are normal.

Since $\langle \lambda, \lambda^*\rangle$ forms a dual system we may speak of the usual dual system topologies on λ and λ^*. We are also interested in the *normal* topology on λ, which is the topology of uniform convergence on intervals $[-u, u]$, $u \in \lambda^*$, $u > 0$, where $[-u, u] = \{v \in \lambda^* \mid |v| \leqq u\}$.

If λ and μ are sequence spaces, then $\Sigma(\lambda, \mu)$ will denote the set of all infinite matrices which map λ into μ. (See [7] for further discussion.) As stated in [7], $\Sigma(\lambda, \mu)$ are those continuous linear maps from $\lambda[\mathcal{T}_s(\lambda^*)]$ into $\mu[\mathcal{T}_s(\mu^*)]$, if both λ and μ are normal. Hence $\Sigma(\lambda, \mu) \subset \mathcal{L}(\lambda, \mu)$, where $\mathcal{L}(\lambda, \mu)$ are those continuous linear maps from $\lambda[\mathcal{T}_b(\lambda^*)]$ into $\mu[\mathcal{T}_b(\mu^*)]$ [8, p. 158]. We may consider $\Sigma(\lambda, \mu)$ as a sequence space by ordering the coordinates of a matrix, so long as we preserve the same ordering for each matrix in the space.

We wish to consider two natural topologies on $\Sigma(\lambda, \mu)$ induced from the topologies $\mathcal{L}_b(\lambda[\mathcal{T}_b(\lambda^*)], \mu[\mathcal{T}_b(\mu^*)])$ and $\mathcal{L}_s(\lambda, \mu[\mathcal{T}_s(\mu^*)])$. Precisely, let τ denote the topology on $\Sigma(\lambda, \mu)$ of uniform convergence on $\mathcal{T}_b(\lambda^*, \lambda)$-bounded sets with respect to the $\mathcal{T}_b(\mu^*, \mu)$-topology, and let $\mathcal{T}_s$ denote the topology on $\Sigma(\lambda, \mu)$ of uniform convergence on finite sets in λ with respect to the $\mathcal{T}_s(\mu^*, \mu)$-topology.

THEOREM 3. *Let λ and μ be perfect sequence spaces with $\lambda[\mathcal{T}_b(\lambda^*)]$ a quasi-barreled space. Suppose $\Sigma(\lambda, \mu)$, as a sequence space, is normal and the intervals of $\Sigma(\lambda, \mu)$ form a fundamental system of τ-bounded sets. If $\Sigma(\lambda, \mu)$ is a bornological space under τ, then $\Sigma(\lambda, \mu)[\tau]$ is the strong bidual of the τ-closure of the finitely nonzero matrices.*

Before proving this theorem we shall set the notation for the remainder of the paper and prove some lemmas. For obvious reasons let E denote $\Sigma(\lambda, \mu)$ and let F denote the τ-closure in E of the finitely nonzero matrices.

In each of the lemmas we assume the hypotheses of Theorem 3.

LEMMA 1. *The τ-bounded sets in E are $\mathcal{T}_s$-relatively compact and there is a τ-neighborhood base of 0 consisting of $\mathcal{T}_s$-barrels.*

PROOF. If $\mathcal{B}$ is a τ-bounded set in E, then $\mathcal{B} \subset [-A, A]$ for some $A \in E$ by the hypothesis. $\lambda[\mathcal{T}_b(\lambda^*)]$ is quasi-barreled, so $\mathcal{B}$ is an equicontinuous set in $\mathcal{L}_b(\lambda, \mu)$. Coordinatewise convergence in E is weaker than $\mathcal{T}_s$, so $[-A, A]$ is $\mathcal{T}_s$ closed since $[-A, A]$ is coordinatewise compact. Now $\{B(x) : B \in \mathcal{B},\ x$ arbitrary but fixed in $\lambda\} \subset [-A|x|, A|x|]$, and the latter set is $\mathcal{T}_s(\mu^*, \mu)$-compact [5, §30, 6]. Thus by Alaoglu-Bourbaki [1, p. 23] $\mathcal{B}$ is relatively $\mathcal{T}_s$-compact.

For the second result let U be a τ-neighborhood of 0 in E of the form $\{T \in E \mid T(M) \subset V\}$ with M bounded in λ and V a $\mathcal{T}_b(\mu^*, \mu)$-neighborhood of 0 which is a $\mathcal{T}_s(\mu^*, \mu)$-barrel. Let $\{T_\alpha\} \subset U$ be a net which $\mathcal{T}_s$-converges to T in E. If $x \in M$, then $\{T_\alpha(x)\}$ converges weakly to $T(x)$ and $T(x) \in V$ since V is weakly closed. Thus $T \in U$.

LEMMA 2. *Given $A \in E$ there is a τ-bounded set $\mathcal{B} \subset F$ with A in the $\mathcal{T}_s$-closure of $\mathcal{B}$.*

PROOF. Let $A \in E$ and let $\mathcal{B} = [-|A|, |A|] \cap F$. Since E is normal $|A| \in E$ and $\mathcal{B}$ is τ-bounded by hypothesis. If $A = (a_{ij})$, define ${}^nA = ({}^na_{ij})$ by ${}^na_{ij} = a_{ij}$, for $i \leqq n, j \leqq n$, and ${}^na_{ij} = 0$, otherwise. Since λ, μ^*, and E are normal, $\sum_{i,j=1}^{\infty} |a_{ij}x_ju_i| < \infty$ for each $x \in \lambda$, $u \in \mu^*$. By the Fubini theorem $\sum_{i,j=1}^{n} a_{ij}x_ju_i$ converges to $\sum_{i,j=1}^{\infty} a_{ij}x_ju_i$, showing that $\{{}^nA\}$ $\mathcal{T}_s$-converges to A. Clearly $\{{}^nA\} \subset \mathcal{B}$.

If $a = (a_i)$ is any sequence of scalars we define λ_a to be all sequences $x = (x_i)$ for which $\sum_{i=1}^{\infty} |x_ia_i| < \infty$ [5, §30]. With this notation we can see that $E^* = \bigcap_{B \in E^+} \lambda_B$, where E^+ consists of those elements of E with positive coordinates. Clearly, $E^* \subset \bigcap_{B \in E^+} \lambda_B$. The other inclusion follows since E is normal and each λ_B is normal.

If g is in the algebraic dual of E and if for each x in E we have $g(x) = \sum_{n=1}^{\infty} x_n g(e^n)$, then we can represent g as a sequence. Namely, $g \sim u = (u_i)$ with $u_i = g(e^i)$, and $g(x) = \sum_{i=1}^{\infty} u_i x_i$. E^* consists precisely of those linear forms on E that can be represented as sequence since E is normal.

The finitely nonzero matrices are $\mathcal{T}_s$-dense in E by Lemma 2, so $(E[\mathcal{T}_s])'$ can be represented by a subspace of E^*. Each element of E^* is a bounded linear map on $E[\tau]$ so E^* consists of those τ-continuous linear functionals on E which can be represented as sequences since $E[\tau]$ is bornological.

Let $\mathcal{T}$ denote the topology $\mathcal{T}_s(E^*, E)$. If $A \in E^*$, $[-A, A]$ is $\mathcal{T}$-compact since $[-A, A]$ is $\mathcal{T}_s(E^*, E^{**})$-compact and is contained in E. $(E[\mathcal{T}_s])' \subset E^*$, so $\mathcal{T}_s((E[\mathcal{T}_s])', E) \subset \mathcal{T}$. We know $\mathcal{T}_s \supset \mathcal{T}_s((E[\mathcal{T}_s])', E)$.

Combining these facts $[-A, A]$ is compact in each topology and hence these topologies agree on τ-bounded sets of E. Thus each element of E is the $\mathcal{T}$-limit of a τ-bounded net in F; see Lemma 2.

Since there is a τ-neighborhood base of 0 consisting of $\mathcal{T}_s$-barrels each element in the base is also a $\mathcal{T}_s((E[\mathcal{T}_s])', E)$-barrel. Considering the comparison of topologies given above this neighborhood base consists of $\mathcal{T}$-barrels. Finally $\mathcal{T} \subset \tau$, since $E^* \subset (E[\tau])'$.

Remark. The above comparison of $\mathcal{T}_s$ and $\mathcal{T}$ can also be observed if we note that E^* is contained in the Grothendieck completion of $(E[\mathcal{T}_s])'$ under the $\mathcal{T}_0$ topology of the proof of Theorem 1.

Proof of Theorem 3. We will show that the E, F, τ, and $\mathcal{T}$ above satisfy the hypotheses for the corresponding symbols in Theorem 2. It is clear that (i), (ii), (iv), and (v) of Theorem 2 are satisfied.

The finitely nonzero sequences are dense in F, so $(F[\tau_i])'$ can be represented as sequences, i.e., $(F[\tau_i])' \subset E^*$. However, $(F[\tau_i])' \supset (F[\mathcal{T}_i])' = (E[\mathcal{T}])' = E^*$. Using the remark after Theorem 2 we have established condition (iii).

The following examples give applications of Theorem 3.

Example 1. $\Sigma(l^1, l^\infty)$ clearly satisfy the conditions of Theorem 3. In this case $\Sigma(l^1, l^\infty) = \mathcal{L}(l^1, l^\infty)$ and Theorem 3 shows that $\mathcal{L}_b(l^1, l^\infty)$ is the strong bidual of a Banach space.

F in this case does not consist of all compact maps since F does not contain all finite-dimensional maps.

We shall denote by $\mathfrak{B}$ the set of all infinite matrices of complex numbers with modulus less than or equal to 1. If λ is a collection of sequences of complex numbers, then $\lambda\mathfrak{B}$ will denote the set of all matrices obtained by forming the matrix product, AB, where $B \in \mathfrak{B}$ and A is a diagonal matrix whose diagonal is an element of λ. If μ is another set of sequences, we define $\mathfrak{B}\mu$ and $\lambda\mathfrak{B}\mu$ similarly.

EXAMPLE 2. Let μ be a perfect sequence space which is bornological under the $\mathcal{T}_b(\mu^*, \mu)$-topology. If the intervals form a fundamental system of $\mathcal{T}_b(\mu^*, \mu)$-bounded sets, then $\Sigma(l^1, \mu)$ satisfies the conditions of Theorem 3.

PROOF. It is essentially shown in [7, Proposition 3.3] that $\Sigma(l^1, \mu)$ is normal. Each element of $\Sigma(l^1, \mu)$ is continuous from the norm of l^1 into $\mu[\mathcal{T}_b(\mu^*)]$. Hence each element of $\Sigma(l^1, \mu)$ maps the unit ball of l^1 into an interval in μ. From [4], $\Sigma(l^1, \mu) = \mu\mathfrak{B}$. If $\mathcal{B}$ is a τ-bounded set in $\Sigma(l^1, \mu)$, then $\mathcal{B}$ is equicontinuous and thus maps the unit ball of l^1 into an interval in μ, i.e., $\{B(x) \mid B \in \mathcal{B}, \|x\|_1 \leqq 1\} \subset [-y, y]$ for some $y \in \mu$. Let $A = (a_{ij})$ be defined by $a_{ij} = y_i$. A is in $\Sigma(l^1, \mu)$ and $\mathcal{B} \subset [-A, A]$. It is easy to see that the intervals in $\Sigma(l^1, \mu)$ are τ-bounded. Hence the intervals of $\Sigma(l^1, \mu)$ form a fundamental system of τ-bounded sets.

It remains to show that $\Sigma(l^1, \mu)$ is bornological under the τ-topology. Let W be an absolutely convex set in $\Sigma(l^1, \mu)$ which absorbs all τ-bounded sets. If D_y denotes the diagonal matrix with diagonal y, then define $\mathring{U} = \{y \in \mu : D_yB \in W$, for each $B \in \mathfrak{B}\}$. If x and y are in $\mathring{U}$, α and β are scalars with $|\alpha| + |\beta| \leqq 1$, then $(\alpha D_x + \beta D_y)B = \alpha D_xB + \beta D_yB \in \alpha W + \beta W \subset W$, so $\mathring{U}$ is absolutely convex. If M is a bounded set in μ, then by the hypothesis $M \subset [-x, x]$ for some x in μ. Since W absorbs all bounded sets of $\Sigma(l^1, \mu)$ there is an $\alpha > 0$ with $[-D_xB_1, D_xB_1] \subset \alpha W$, where B_1 is the matrix with all 1's. If $y \in M$ and $B \in \mathfrak{B}$, $D_yB \in [-D_xB_1, D_xB_1]$, so $y \in \alpha\mathring{U}$. Since $\mu[\mathcal{T}_b(\mu^*)]$ is bornological $\mathring{U}$ is a neighborhood of zero.

Let U be a normal neighborhood contained in $\mathring{U}$ and let $V = U\mathfrak{B}$. Since $V \subset W$ we need only show that V is a τ-neighborhood. If $A \in \Sigma(l^1, \mu)$ and A maps the unit ball of l^1 into U, then from the proof of 2.5 [4], A is in V. (The reader can easily construct a straightforward proof that A is in V.) Hence V is a τ-neighborhood and $\Sigma(l^1, \mu)$ is bornological in the τ-topology.

An *echelon* space is a sequence space $\lambda = \bigcap_k (1/a^k)\lambda_k$ where (a^k) is a sequence of positive sequences, (λ_k) is a sequence of perfect spaces which are Banach spaces in their strong topologies from their Köthe duals such that

(i) $l^1 \subset \lambda_k \subset l^\infty$,

(ii) $a^k\lambda_k{}^* \subset a^{k+1}\lambda^*_{k+1}$.

For details about echelon spaces, see [3]. An echelon space is of order 1 (order ∞) if each λ_k is l^1 (l^∞). An echelon space is nuclear if and only if it is an echelon space of order 1 with each a^k/a^{k+1} an element of l^1. A nuclear echelon space is also an echelon space of order ∞ with the same sequence, (a^k) [2].

EXAMPLE 3. Let λ and μ be echelon spaces of order ∞ with $\lambda^*[\mathcal{T}_b(\lambda)]$ a quasi-barreled space. Then $\Sigma(\lambda^*, \mu)$ satisfies the conditions of Theorem 3.

PROOF. $\mathcal{L}_b(\lambda^*, \mu)$ is a metrizable space since λ^* has a fundamental sequence of bounded sets and μ is metrizable [3]. Thus $\Sigma(\lambda^*, \mu)$ is bornological in the τ-topology. $\lambda^* = \bigcup_{k=1}^{\infty} a_k l^1$, for some sequence (a^k), see [3], and it is easy to see that $\Sigma(\lambda^*, \mu) = \bigcap_{k=1}^{\infty}[\Sigma(a^k l^1, \mu)]$. Combining this with Example 2 we have that $\Sigma(\lambda^*, \mu)$ is normal.

If $\mathcal{B}$ is τ-bounded in $\Sigma(\lambda^*, \mu)$, then for each bounded set M in λ^*, $\{B(x) \mid x \in M, B \in \mathcal{B}\}$ is bounded in μ. If M is bounded in $a^k l^1$, for some k, then M is bounded in λ^*, see [3]. Hence $\mathcal{B}$ is τ-bounded in $\Sigma(a^k l^1, \mu)$ for each k, and from Example 2 the coordinatewise sup of $\mathcal{B}$ is a matrix in each $\Sigma(a^k l^1, \mu)$ and hence also in $\Sigma(\lambda^*, \mu)$. From the definition of the τ-topology it is easy to see that intervals in $\Sigma(\lambda^*, \mu)$ are τ-bounded.

REMARK. Assume the context of Example 3 and further hypothesis that $\lambda[\mathcal{T}_b(\lambda^*)]$ is nuclear with $\mu[\mathcal{T}_k(\mu^*)]$ reflexive. From [4, 2.6] $\mathcal{L}(\lambda^*, \mu) = \mu \mathfrak{B} \lambda$. As in Example 2 if U is a normal neighborhood of 0 in $\lambda[\mathcal{T}_b(\lambda^*)]$ and V is a normal neighborhood of 0 in $\mu[\mathcal{T}_b(\mu^*)]$, then $V \mathfrak{B} U$ is a neighborhood of 0 in $\mathcal{L}_b(\lambda^*, \mu)$. From the hypothesis on μ the finitely nonzero sequences are dense in $\mu[\mathcal{T}_b(\mu^*)]$. Combining these facts we have $F = E$ and $\mathcal{L}_b(\lambda^*, \mu)$ is reflexive. This is a special case of Corollary 2, p. 176 of [8].

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