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EXISTENCE OF A SOLUTION FOR THE PROBLEM WITH A CONCENTRATED SOURCE IN A SUBDIFFUSIVE MEDIUM

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ABSTRACT. By using Green's function, the problem is converted into an integral equation. It is shown that there exists a t_b such that, for $0 \leq t < t_b$, the integral equation has a unique nonnegative continuous solution u; if t_b is finite, then u is unbounded in $[0, t_b)$. Then, u is proved to be the solution of the original problem.

1. Introduction. Let a, b, α and T be positive real numbers with $0 < b < a$, $0 < \alpha < 1$. We consider the following fractional diffusion problem

$$(1.1) \qquad \begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2}{\partial x^2} D_t^{1-\alpha} u(x,t) & \\ \qquad\qquad +\delta(x-b) f(u(x,t)) & \text{in } (0,a) \times (0,T], \\ u(x,0) = \phi(x) & \text{on } [0,a], \\ u(0,t) = 0 = u(a,t) & \text{for } 0 < t \leq T, \end{cases}$$

where $D_t^{1-\alpha}$ denotes the Riemann-Liouville fractional derivative, $\delta(x-b)$ is the Dirac delta function and f and ϕ are given functions. We assume that $f(0) \geq 0$, $f'(u) > 0$, $f''(u) > 0$ for $u > 0$, and that ϕ is nontrivial and nonnegative on $[0, a]$ such that $\phi(0) = 0 = \phi(a)$.

When $\alpha = 1$, the problem (1.1) describes the heat diffusion problem involving a concentrated source at a particular position b. This local-

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ized input of thermal energy may lead to the blow-up of the solution. For references, we quote the papers by Chan [**1**], Chan and Boonklurb [**2**], Chan and Tian [**4, 5, 6**], Chan and Tragoonsirisak [**7, 8**], Chan and Treeyaprasert [**9**] and Olmstead and Roberts [**15**]. The effect of the moving source was studied by Kirk and Olmstead [**11**], and, more recently, by Chan, Sawangtong and Treeyaprasert [**3**].

For the case when $0 < \alpha < 1$, the fractional derivative problem is introduced to model diffusive behavior of mean square displacement of Brownian motion evolving on a slower than normal time scale. These problems were discussed by authors [**12, 13, 14, 16, 18**]. When applied to certain porous materials in which microscopic pores are filled with a substance that has a lower conductivity than that of the basic matrix material, the model can be formulated by the subdiffusive process.

Olmstead and Roberts [**14**] studied the subdiffusive problem with a concentrated source in the form

$$\begin{gathered}\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2}{\partial x^2} D_t^{1-\alpha} u(x,t) + \delta(x-b) f(u(x,t)),\\ (x,t) \in (0,l) \times (0,T),\end{gathered} \tag{1.2}$$

where $f > 0$, $f' > 0$, $f'' \geq 0$, and $\lim_{u\to\infty} f(u) = \infty$. By using Green's function $G_\alpha(x,t;\xi,\tau)$, they investigated the blow-up and asymptotic behavior of the solution at the singular point b, given by

$$u(b,t) = \int_0^t G_\alpha(b, t-\tau; b) f(u(b,\tau))\, d\tau.$$

In this paper, the properties of Green's function $G_\alpha(x,t;\xi,\tau)$ are investigated, and the existence and uniqueness of the solution of problem (1.1) is proved.

2. Green's function. The Riemann-Liouville fractional derivative is discussed in [**16**], and the limiting case of $\alpha = 1$ is associated with classical diffusion since D_t^0 is the identity operator. The Green function $G_\alpha(x,t;\xi,\tau)$ corresponding to the problem (1.1) satisfies: for x and ξ in $[0,a]$, and t and τ in $(-\infty,\infty)$,

$$\frac{\partial}{\partial t} G_\alpha(x,t;\xi,\tau) = \frac{\partial^2}{\partial x^2} D_t^{1-\alpha} G_\alpha(x,t;\xi,\tau) + \delta(x-\xi)\delta(t-\tau), \tag{2.1}$$

subject to $G_\alpha(x,t;\xi,\tau) = 0$ for $t < \tau$ and $G_\alpha(0,t;\xi,\tau) = 0 = G_\alpha(a,t;\xi,\tau)$.

It follows from Wyss and Wyss [**19,** Theorem 1] that Green's function $G_\alpha(x,t;\xi,\tau)$ can be expressed in terms of Green's function of the case $\alpha = 1$:

$$G_\alpha(x,t-\tau;\xi) = \int_0^\infty g_\alpha(z) G_1(x,(t-\tau)^\alpha z;\xi)\,dz,$$

where

$$G_1(x,t-\tau;\xi) = \frac{2H(t-\tau)}{a}\sum_{n=1}^\infty \sin\frac{n\pi\xi}{a}\sin\frac{n\pi x}{a} e^{-(n^2\pi^2/a^2)(t-\tau)}$$

with $H(t)$ being the Heaviside function, and

$$g_\alpha(z) = \sum_{k=0}^\infty \frac{(-1)^k z^k}{k!\Gamma(1-\alpha-\alpha k)} \quad \text{for } 0<\alpha<1, z>0,$$

known as Mainardi's function with $g_\alpha(z) \geq 0$ for $z \geq 0$,

$$\int_0^\infty g_\alpha(z)\,dz = 1,$$

and $g_\alpha(z)$ tends to 0 exponentially as $z \to \infty$. Thus,

$$G_\alpha(x,t-\tau;\xi) = \frac{2H(t-\tau)}{a}\sum_{n=1}^\infty \sin\frac{n\pi\xi}{a} \times \sin\frac{n\pi x}{a}\left(\int_0^\infty g_\alpha(z) e^{-(n^2\pi^2/a^2)(t-\tau)^\alpha z}\,dz\right),$$

which is positive for x and ξ in $(0,a)$, and $t > \tau$. In order to study the behavior of $G_\alpha(x,t-\tau;\xi)$, we consider the integral

$$\int_0^\infty g_\alpha(z) e^{-(n^2\pi^2/a^2)(t-\tau)^\alpha z}\,dz.$$

Let

$$I_k = \int_0^\infty z^k e^{-(n^2\pi^2/a^2)(t-\tau)^\alpha z}\,dz$$

for any $k = 0,1,2,\ldots$. Upon integration, we get for $k = 0$,

$$I_0 = \int_0^\infty e^{-(n^2\pi^2/a^2)(t-\tau)^\alpha z}\,dz = \frac{1}{(n^2\pi^2/a^2)(t-\tau)^\alpha};$$

for $k \geq 1$,

$$I_k = \frac{k!}{[(n^2\pi^2/a^2)(t-\tau)^\alpha]^{k+1}}.$$

For any fixed positive integer n, the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1-\alpha-\alpha k)[(n^2\pi^2/a^2)(t-\tau)^\alpha]^{k+1}}$$

converges uniformly for t in any compact subset of $(\tau, T]$. This gives

$$\begin{aligned} \text{(2.2)} \quad & \int_0^\infty g_\alpha(z) e^{-(n^2\pi^2/a^2)(t-\tau)^\alpha z}\, dz \\ & = \frac{1}{(n^2\pi^2/a^2)(t-\tau)^\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1-\alpha-\alpha k)[(n^2\pi^2/a^2)(t-\tau)^\alpha]^k}. \end{aligned}$$

For $\mu, \nu > 0$, the Mittag-Leffler function is defined as

$$E_{\mu,\nu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\nu + \mu k)},$$

which is entire for $z \in \mathbb{C}$ where $\mathbb{C}$ denotes the complex plane (cf., **[10]**). It follows from **[10,** page 10, Lemma 9.1] that the function has another representation form

$$E_{\mu,\nu}(z) = -\sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(\nu - \mu k)}$$

for any $z \in \mathbb{C}$. Based on the integral form of Mittag-Leffler $E_{\mu,\nu}(z)$, given by

$$E_{\mu,\nu}(z) = \frac{1}{2\pi i} \int_\Omega \frac{t^{\mu-\nu} e^t}{t^\mu - z}\, dt$$

for $\mathrm{Re}(\mu) > 0$, $\mathrm{Re}(\nu) > 0$, $z, \mu, \nu \in \mathbb{C}$, and Ω being a loop starting and ending at $-\infty$ and encircling the circular disk $|t| < |z|^{1/\alpha}$ in the positive sense, $|\arg t| < \pi$ on Ω, we obtain the asymptotic expansion (cf., **[16,** page 33])

$$\text{(2.3)} \qquad E_{\mu,\nu}(z) = -\sum_{r=1}^{N} \frac{1}{\Gamma(\nu - \mu r) z^r} + O\left(\frac{1}{z^{N+1}}\right),$$

as $|z| \to \infty$, $\beta \leq |\arg z| \leq \pi$ with $\beta \in (\pi\alpha/2, \alpha\pi)$.

From (2.2),

$$
\begin{aligned}
&\int_0^\infty g_\alpha(z) e^{-(n^2\pi^2/a^2)(t-\tau)^\alpha z}\,dz \\
&\quad = \frac{1}{(n^2\pi^2/a^2)(t-\tau)^\alpha} \\
&\quad \times \left\{ \frac{1}{\Gamma(1-\alpha)} + \sum_{k=1}^{\infty} \frac{(-1)^k}{\Gamma(1-\alpha-\alpha k)[(n^2\pi^2/a^2)(t-\tau)^\alpha]^k} \right\} \\
&\quad = \frac{1}{(n^2\pi^2/a^2)(t-\tau)^\alpha} \left\{ \frac{1}{\Gamma(1-\alpha)} - E_{\alpha,1-\alpha}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha \right) \right\}.
\end{aligned}
$$

By using the recurrence results $E_{\mu,\nu}(z) = zE_{\mu,\mu+\nu}(z) + (1/\Gamma(\nu))$ from [**10,** page 5, Theorem 5.1], we obtain

$$
E_{\alpha,1-\alpha}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha\right) = -\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha\right) + \frac{1}{\Gamma(1-\alpha)}.
$$

Then, (2.2) becomes

$$
\int_0^\infty g_\alpha(z) e^{-(n^2\pi^2/a^2)(t-\tau)^\alpha z}\,dz = E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha \right). \tag{2.4}
$$

It follows from (2.4) that

$$
G_\alpha(x,t-\tau;\xi) = \frac{2H(t-\tau)}{a} \sum_{n=1}^{\infty} \sin\frac{n\pi\xi}{a} \sin\frac{n\pi x}{a} E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha \right). \tag{2.5}
$$

Our next result gives the properties of $G_\alpha(x,t-\tau;\xi)$.

Lemma 2.1.

(a) *For* $\tau \in [0,T)$, $(x,t;\xi) \in ([0,a] \times (\tau,T]) \times [0,a]$, $G_\alpha(x,t-\tau;\xi)$ *is continuous.*
(b) *For each* $(\xi,\tau) \in [0,a] \times [0,T)$, $(G_\alpha)_t(x,t-\tau;\xi) \in C([0,a] \times (\tau,T])$.
(c) *For each* $(\xi,\tau) \in [0,a] \times [0,T)$, $D_t^{1-\alpha} G_\alpha(x,t-\tau;\xi)$ *is continuous for* $[0,a] \times (\tau,T]$.
(d) *For each* $(\xi,\tau) \in [0,a] \times [0,T)$, $(D_t^{1-\alpha} G_\alpha)_x(x,t-\tau;\xi)$ *and* $(D_t^{1-\alpha} G_\alpha)_{xx}(x,t-\tau;\xi)$ *are in* $C([0,a] \times (\tau,T])$.

(e) *If* $s \in C[0,T]$, *then*

$$\int_0^t G_\alpha(x, t-\tau; b) s(\tau)\, d\tau$$

is continuous for $x \in [0,a]$ *and* $t \in [0,T]$.

(f) *If* $s \in C[0,T]$, *then*

$$\int_0^t D_\tau^{1-\alpha}(G_\alpha(x,\tau; b)) s(\tau)\, d\tau$$

is continuous for $x \in [0,a]$ *and* $t \in [0,T]$.

Proof.

(a) From (2.5),

$$|G_\alpha(x, t-\tau; \xi)| \le \frac{2}{a} \sum_{n=1}^{\infty} \left| E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha \right) \right|. \tag{2.6}$$

For t in a compact subset of (τ, T), it follows from (2.3) that there exist some positive integer N_1^* and positive constant K_1 such that

$$\left| E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha \right) \right| \le \frac{K_1}{n^2(t-\tau)^\alpha} \tag{2.7}$$

for any $n \ge N_1^*$. Then, the series

$$\sum_{n=1}^{\infty} \left| E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha \right) \right|$$

converges uniformly for t in any compact subset of (τ, T). The result then follows.

(b) From [**10,** page 13, Theorem 11.1],

$$\frac{d}{dz}[z^{2\alpha} E_{\alpha,1+2\alpha}(Cz^\alpha)] = z^{2\alpha-1} E_{\alpha,2\alpha}(Cz^\alpha) \tag{2.8}$$

for $\alpha > 0$ and any constant C. Making use of the recurrence relation

$$E_{\alpha,1}(z) = z^2 E_{\alpha,1+2\alpha}(z) + \frac{1}{\Gamma(1)} + \frac{z}{\Gamma(1+\alpha)}$$

for $\alpha > 0$ and any $z \in \mathbb{C}$, we rewrite

$$E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha\right) = \left(\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha\right)^2 E_{\alpha,1+2\alpha}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha\right) + \frac{1}{\Gamma(1)} + \frac{-(n^2\pi^2/a^2)(t-\tau)^\alpha}{\Gamma(1+\alpha)}.$$

Differentiating both sides with respect to t, and making use of (2.8), we obtain

$$\frac{d}{dt}E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha\right) = \left(\frac{n^2\pi^2}{a^2}\right)^2(t-\tau)^{2\alpha-1}E_{\alpha,2\alpha}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha\right) - \frac{\alpha(n^2\pi^2/a^2)(t-\tau)^{\alpha-1}}{\Gamma(1+\alpha)}.$$

From (2.3), for large n,

$$E_{\alpha,2\alpha}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha\right) = -\left[\frac{1}{\Gamma(\alpha)(-(n^2\pi^2/a^2)(t-\tau)^\alpha)} + \sum_{r=3}^{N}\frac{1}{\Gamma(\alpha(2-r))(-(n^2\pi^2/a^2)(t-\tau)^\alpha)^r} + O\left(\frac{1}{(-(n^2\pi^2/a^2)(t-\tau)^\alpha)^{N+1}}\right)\right];$$

this gives, for some large positive number N_2^* with $n \geq N_2^*$, and some positive constant K_2,

$$\left|\frac{d}{dt}E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha\right)\right| = \frac{1}{(n^2\pi^2/a^2)(t-\tau)^{\alpha+1}} \times \left|\sum_{r=3}^{N}\frac{1}{\Gamma(\alpha(2-r))(n^2\pi^2/a^2)^{r-3}(t-\tau)^{(r-3)\alpha}}\right| + O\left(\frac{1}{n^{2(N-1)}(t-\tau)^{(N-1)\alpha+1}}\right) \leq \frac{K_2}{n^2(t-\tau)^{\alpha+1}}.$$

Therefore,

$$\left|\sum_{n=1}^{\infty}\frac{\partial}{\partial t}\sin\frac{n\pi\xi}{a}\sin\frac{n\pi x}{a}E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha\right)\right| \tag{2.9}$$

$$\leq \sum_{n=1}^{\infty} \left| \frac{d}{dt} E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha \right) \right| \leq \sum_{n=1}^{\infty} \frac{K_2}{n^2(t-\tau)^{\alpha+1}},$$

which converges uniformly with respect to $x \in [0,a]$ and t in any compact subset of $(\tau, T]$. This proves Lemma 2.1 (b).

(c) It follows from [**10,** page 14, Theorem 11.5] that

$$D_t^{1-\alpha} E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha \right) = (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha \right).$$

Using (2.3) and a similar argument as in the proof of (b), we obtain for some large positive number N_3^* with $n \geq N_3^*$, and some positive constant K_3,

$$\begin{aligned} &\left| (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha \right) \right| \\ &\leq (t-\tau)^{\alpha-1} \left| \sum_{r=2}^{N} \frac{1}{\Gamma(\alpha(1-r))((n^2\pi^2/a^2)(t-\tau)^\alpha)^r} \right| \\ &\quad + O\left(\frac{1}{n^{2(N+1)}(t-\tau)^{N\alpha+1}} \right) \\ &\leq \frac{K_3}{n^4(t-\tau)^{\alpha+1}}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\left| \sum_{n=1}^{\infty} D_t^{1-\alpha} \sin\frac{n\pi\xi}{a} \sin\frac{n\pi x}{a} E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha \right) \right| \\ &\qquad \leq \sum_{n=1}^{\infty} \left| (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha \right) \right| \\ &\qquad \leq \sum_{n=1}^{\infty} \frac{K_3}{n^4(t-\tau)^{\alpha+1}}, \end{aligned} \tag{2.10}$$

which converges uniformly with respect to $x \in [0,a]$ and t in any compact subset of $(\tau, T]$. This proves Lemma 2.1 (c).

(d) From Lemma 2.1 (c),

$$\left| \sum_{n=1}^{\infty} \frac{\partial}{\partial x} D_t^{1-\alpha} \sin\frac{n\pi\xi}{a} \sin\frac{n\pi x}{a} E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^\alpha \right) \right|$$

$$\le \sum_{n=1}^{\infty} \frac{n\pi}{a} \left| (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^{\alpha} \right) \right|$$
$$\le \sum_{n=1}^{\infty} \frac{\pi K_3}{a n^3 (t-\tau)^{\alpha+1}},$$

which converges uniformly with respect to $x \in [0,a]$ and t in any compact subset of $(\tau, T]$. Furthermore,

$$\left| \sum_{n=1}^{\infty} \frac{\partial^2}{\partial x^2} D_t^{1-\alpha} \sin\frac{n\pi\xi}{a} \sin\frac{n\pi x}{a} E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^{\alpha} \right) \right|$$
$$\le \sum_{n=1}^{\infty} \frac{n^2\pi^2}{a^2} \left| (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^{\alpha} \right) \right|$$
$$\le \sum_{n=1}^{\infty} \frac{\pi^2 K_3}{a^2 n^2 (t-\tau)^{\alpha+1}},$$

which converges uniformly with respect to $x \in [0,a]$ and t in any compact subset of $(\tau, T]$. Lemma 2.1 (d) is then proved.

(e) Let $S = \max_{t\in[0,T]} |s(t)|$, and let ϵ be any positive number such that $t - \epsilon > 0$. For any $x \in [0,a]$ and $\tau \in [0, t-\epsilon]$, it follows from (2.7) that

$$\sum_{n=1}^{\infty} \left| \frac{2}{a} \sin\frac{n\pi b}{a} \sin\frac{n\pi x}{a} E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^{\alpha} \right) s(\tau) \right| \le \frac{2S}{a} \sum_{n=1}^{\infty} \frac{a^2 K_1}{n^2\pi^2\epsilon^{\alpha}},$$

which converges uniformly. From the Weierstrass M-test,

$$\int_0^{t-\epsilon} G_\alpha(x, t-\tau; b) s(\tau)\, d\tau$$
$$= \sum_{n=1}^{\infty} \frac{2}{a} \int_0^{t-\epsilon} \sin\frac{n\pi b}{a} \sin\frac{n\pi x}{a} E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^{\alpha} \right) s(\tau)\, d\tau.$$

By using (2.7),

$$\sum_{n=1}^{\infty} \frac{2}{a} \int_0^{t-\epsilon} \sin\frac{n\pi b}{a} \sin\frac{n\pi x}{a} E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^{\alpha} \right) s(\tau)\, d\tau$$
$$\le \frac{2S}{a} \sum_{n=1}^{\infty} \int_0^{t-\epsilon} E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2}(t-\tau)^{\alpha} \right) d\tau$$

$$\leq \frac{2SK_1}{a} \sum_{n=1}^{\infty} \int_0^{t-\epsilon} \frac{1}{n^2(t-\tau)^\alpha}\, d\tau = \frac{2SK_1}{a} \sum_{n=1}^{\infty} \frac{t^{1-\alpha} - \epsilon^{1-\alpha}}{(1-\alpha)n^2},$$

which converges uniformly with respect to ϵ for $x \in [0, a]$ and $t \in [0, T]$. Since the above inequality holds for $\epsilon = 0$, it follows that, for $s \in C[0, T]$,

$$\sum_{n=1}^{\infty} \frac{2}{a} \int_0^{t-\epsilon} \sin\frac{n\pi b}{a} \sin\frac{n\pi x}{a} E_{\alpha,1}\bigg(- \frac{n^2\pi^2}{a^2}(t-\tau)^\alpha \bigg) s(\tau)\, d\tau$$

is a continuous function of x, t and ϵ. Therefore,

$$\begin{aligned}&\int_0^t G_\alpha(x, t-\tau; b) s(\tau)\, d\tau \\ &= \lim_{\epsilon\to 0} \sum_{n=1}^{\infty} \frac{2}{a} \int_0^{t-\epsilon} \sin\frac{n\pi b}{a} \sin\frac{n\pi x}{a} E_{\alpha,1}\bigg(- \frac{n^2\pi^2}{a^2}(t-\tau)^\alpha \bigg) s(\tau)\, d\tau\end{aligned}$$

is a continuous function for $x \in [0, a]$ and $t \in [0, T]$.

(f) Let $S = \max_{t\in[0,T]} |s(t)|$, and let ϵ be any positive number such that $t - \epsilon > 0$. For any $x \in [0, a]$ and $\tau \in [\epsilon, t]$, it follows from (2.10) that

$$\sum_{n=1}^{\infty} \left| \frac{2}{a} \sin\frac{n\pi b}{a} \sin\frac{n\pi x}{a} D_\tau^{1-\alpha}\bigg(E_{\alpha,1}\bigg(-\frac{n^2\pi^2}{a^2}(\tau)^\alpha\bigg)\bigg) s(\tau) \right| \leq \frac{2S}{a} \sum_{n=1}^{\infty} \frac{a^2K_3}{n^4\epsilon^{\alpha+1}},$$

which converges uniformly. From the Weierstrass M-test,

$$\begin{aligned}(2.11)\quad &\int_\epsilon^t D_\tau^{1-\alpha}(G_\alpha(x,\tau;b)) s(\tau)\, d\tau \\ &= \sum_{n=1}^{\infty} \frac{2}{a} \int_\epsilon^t \sin\frac{n\pi b}{a} \sin\frac{n\pi x}{a} D_\tau^{1-\alpha}\bigg(E_{\alpha,1}\bigg(- \frac{n^2\pi^2}{a^2}(\tau)^\alpha\bigg)\bigg) s(\tau)\, d\tau.\end{aligned}$$

From [**10,** pages 13, 14, Theorems 11.4, 11.5],

(2.12)

$$\begin{aligned}\int_0^t D_\tau^{1-\alpha} E_{\alpha,1}\bigg(- \frac{n^2\pi^2}{a^2}(\tau)^\alpha\bigg)\, d\tau &= \int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}\bigg(- \frac{n^2\pi^2}{a^2}\tau^\alpha\bigg)\, d\tau \\ &= t^\alpha E_{\alpha,1+\alpha}\bigg(- \frac{n^2\pi^2}{a^2} t^\alpha\bigg).\end{aligned}$$

From (2.4) and the monotone decreasing property of $E_{\alpha,1}(z)$ for $z \in [0,\infty)$, we have $|E_{\alpha,1}(z)| \leq 1$. It follows from the recurrence relation

$$E_{\alpha,1}(z) = zE_{\alpha,1+\alpha}(z) + \frac{1}{\Gamma(1)}$$

in [**10,** page 5, Theorem 5.1] that

$$\left| t^{\alpha} E_{\alpha,1+\alpha}\left(-\frac{n^2\pi^2}{a^2} t^{\alpha} \right) \right| \leq \frac{a^2}{n^2\pi^2} \left| 1 - E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2} t^{\alpha} \right) \right|.$$

By (2.11) and (2.12),

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{2}{a} \int_{\epsilon}^{t} \sin\frac{n\pi b}{a} \sin\frac{n\pi x}{a} D_{\tau}^{1-\alpha}\left(E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2}(\tau)^{\alpha} \right) \right) s(\tau)\, d\tau \\
&\quad \leq \frac{2S}{a} \sum_{n=1}^{\infty} \int_{\epsilon}^{t} \tau^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{n^2\pi^2}{a^2}\tau^{\alpha} \right) d\tau \\
&\quad \leq \frac{2SK_1}{a} \sum_{n=1}^{\infty} \frac{a^2}{n^2\pi^2} \left(\left| 1 - E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2} t^{\alpha} \right) \right| \right. \\
&\qquad\qquad\qquad\qquad \left. + \left| 1 - E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2} \epsilon^{\alpha} \right) \right| \right),
\end{aligned}$$

which converges uniformly with respect to ϵ for $x \in [0,a]$ and $t \in [0,T]$. Since the above inequality holds for $\epsilon = 0$, it follows that, for $s \in C[0,T]$,

$$\sum_{n=1}^{\infty} \frac{2}{a} \int_{\epsilon}^{t} \sin\frac{n\pi b}{a} \sin\frac{n\pi x}{a} D_{\tau}^{1-\alpha}\left(E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2}(\tau)^{\alpha} \right) \right) s(\tau)\, d\tau$$

is a continuous function of x, t and ϵ. Therefore,

$$\begin{aligned}
&\int_{0}^{t} D_{\tau}^{1-\alpha}(G_{\alpha}(x,\tau;b)) s(\tau)\, d\tau \\
&= \lim_{\epsilon\to 0} \sum_{n=1}^{\infty} \frac{2}{a} \int_{\epsilon}^{t} \sin\frac{n\pi b}{a} \sin\frac{n\pi x}{a} D_{\tau}^{1-\alpha}\left(E_{\alpha,1}\left(-\frac{n^2\pi^2}{a^2}(\tau)^{\alpha} \right) \right) s(\tau)\, d\tau
\end{aligned}$$

is a continuous function for $x \in [0,a]$ and $t \in [0,T]$. $\square$

3. Integral equation for problem (1.1). By using Green's function $G_\alpha(x, t-\tau; \xi)$, we obtain the integral equation

$$u(x,t) = \int_0^a G_\alpha(x,t;\xi)\phi(\xi)\,d\xi + \int_0^t G_\alpha(x,t-\tau;b)f(u(b,\tau))\,d\tau \tag{3.1}$$

for problem (1.1).

Theorem 3.1. *There exists a* t_b *such that, for* $0 \le t < t_b$, *the integral equation* (3.1) *has a unique nonnegative continuous solution* u. *If* t_b *is finite, then* u *is unbounded in* $[0, t_b)$.

Proof. For $(x,t) \in [0,a] \times [0,T]$, let us construct a sequence $\{u_i\}_{i=0}^{\infty}$ by

$$u_0(x,t) = \int_0^a G_\alpha(x,t;\xi)\phi(\xi)\,d\xi,$$

and, for $i = 0, 1, 2, \ldots$,

$$\frac{\partial u_{i+1}(x,t)}{\partial t} = \frac{\partial^2}{\partial x^2} D_t^{1-\alpha} u_{i+1}(x,t) + \delta(x-b)f(u_i(x,t)) \text{ in } (0,a)\times(0,T],$$
$$u_{i+1}(x,0) = \phi(x) \text{ on } [0,a],$$
$$u_{i+1}(0,t) = 0 = u_{i+1}(a,t) \text{ for } 0 < t \le T.$$

From (3.1),

$$u_{i+1}(x,t) = \int_0^a G_\alpha(x,t;\xi)\phi(\xi)\,d\xi + \int_0^t G_\alpha(x,t-\tau;b)f(u_i(b,\tau))\,d\tau.$$

We show that, for any $i = 0, 1, 2, \ldots$,

$$u_0 < u_1 < u_2 < \cdots < u_{i+1} \quad \text{in } (0,a)\times(0,T]. \tag{3.2}$$

Since

$$u_1(x,t) - u_0(x,t) = \int_0^t G_\alpha(x,t-\tau;b)f(u_0(b,\tau))\,d\tau,$$

it follows from G_α, u_0 and $f(u_0)$ being positive that $u_1(x,t) - u_0(x,t) > 0$ for $(x,t) \in (0,a)\times(0,T]$. Assume that, for some positive integer j, $u_0 < u_1 < u_2 < \cdots < u_j$ in $(0,a)\times(0,T]$. Since $u_{j-1} < u_j$ and $f' > 0$, we have

$$u_{j+1}(x,t)-u_j(x,t)=\int_0^t G_\alpha(x,t-\tau;b)(f(u_j(b,\tau))-f(u_{j-1}(b,\tau)))\,d\tau>0.$$

By the principle of mathematical induction, (3.2) holds.

Let $u=\lim_{i\to\infty}u_i$, and let M be a positive number such that $M>\sup_{x\in[0,a]}\phi(x)$. Since each u_i is nonnegative, we have that u is nonnegative. From Lemma 2.1 (a) and (3.1), u_i is continuous for $i=0,1,2,\ldots$. We would like to show that there exists a $t_1\le T$ such that $u(x,t)$ is continuous for $t\in[0,t_1]$. Note that

$$u_{i+1}(x,t)-u_i(x,t)=\int_0^t G_\alpha(x,t-\tau;b)(f(u_i(b,\tau))-f(u_{i-1}(b,\tau)))\,d\tau.$$

Let $S_i=\sup_{(x,t)\in[0,a]\times[0,t_1]}|u_i(x,t)-u_{i-1}(x,t)|$. It follows from the mean value theorem, (2.6) and (2.7) that

$$S_{i+1}\le f'(M)S_i\int_0^t G_\alpha(x,t-\tau;b)d\tau\le f'(M)\left(\sum_{n=1}^\infty \frac{2Kt^{1-\alpha}}{an^2(1-\alpha)}\right)S_i$$

for some positive constant K. By taking $\sigma_1\le t_1$ such that

$$f'(M)\left(\sum_{n=1}^\infty \frac{2Kt^{1-\alpha}}{an^2(1-\alpha)}\right)<1\quad\text{for } t\in[0,\sigma_1],$$

the sequence $\{u_i\}$ converges uniformly for $(x,t)\in[0,a]\times[0,\sigma_1]$, and hence, u is a nonnegative continuous solution of (3.1) for $(x,t)\in[0,a]\times[0,\sigma_1]$.

If $\sigma_1<t_1$, we replace the initial condition $u(x,0)=\phi(x)$ by $u(x,\sigma_1)$, which is known. Then, for $(x,t)\in[0,a]\times[\sigma_1,t_1]$,

$$\begin{aligned}u_{i+1}(x,t)=&\int_0^a G_\alpha(x,t-\sigma_1;\xi)u(\xi,\sigma_1)\,d\xi\\&+\int_{\sigma_1}^t G_\alpha(x,t-\tau;b)f(u_i(b,\tau))\,d\tau.\end{aligned}$$

From

$$\begin{aligned}u_{i+1}(x,t)-u_i(x,t)=&\int_{\sigma_1}^t G_\alpha(x,t-\tau;b)(f(u_i(b,\tau))\\&-f(u_{i-1}(b,\tau)))\,d\tau>0,\end{aligned}$$

we obtain

$$S_{i+1} \leq f'(M) S_i \int_{\sigma_1}^{t} G_\alpha(x, t-\tau; b)\, d\tau \leq f'(M) \left(\sum_{n=1}^{\infty} \frac{2K(t-\sigma_1)^{1-\alpha}}{an^2(1-\alpha)} \right) S_i.$$

From the previous choice of σ_1, we have $f'(M)(\sum_{n=1}^{\infty}(2K(t-\sigma_1)^{1-\alpha})/(an^2(1-\alpha))) < 1$ for $t \in [\sigma_1, \min\{2\sigma_1, t_1\}]$. Repeating the process, the sequence $\{u_i\}$ converges uniformly for any $(x,t) \in [0,a] \times [0,t_1]$, and hence, u is continuous.

In order to show the continuity of $D_t^{1-\alpha} u$, we have

$$\begin{aligned} D_t^{1-\alpha} u(x,t) = D_t^{1-\alpha} & \int_0^a G_\alpha(x,t;\xi)\phi(\xi)\, d\xi \\ & + D_t^{1-\alpha} \left(\int_0^t G_\alpha(x, t-\tau; b) f(u(b,\tau))\, d\tau \right). \end{aligned}$$

From Lemma 2.1 (c), $D_t^{1-\alpha} G_\alpha(x,t;\xi)$ is continuous. We have

$$D_t^{1-\alpha} \int_0^a G_\alpha(x,t;\xi)\phi(\xi)\, d\xi$$

is continuous for $(x,t) \in [0,a] \times (0,t_1]$. It follows from **[16,** page 99, (2.213)] that

$$\begin{aligned} D_t^{1-\alpha} & \left(\int_0^t G_\alpha(x, t-\tau; b) f(u(b,\tau))\, d\tau \right) \\ & = \int_0^t D_\tau^{1-\alpha}(G_\alpha(x,\tau;b)) f(u(b,t-\tau))\, d\tau \\ & \quad + \lim_{\tau \to 0^+} f(u(b,t-\tau)) D_\tau^{-\alpha} G_\alpha(x,\tau;b). \end{aligned}$$

From the continuity of f and Lemma 2.1 (c), we have

$$\int_0^t D_\tau^{1-\alpha} \left(G_\alpha(x,\tau;b) \right) f(u(b,t-\tau))\, d\tau$$

is continuous for $(x,t) \in [0,a] \times (0,t_1]$. Since $G_\alpha(x,\tau;b)$ is continuous for any $\tau \in (0,t_1]$ and $|G_\alpha(x,\tau;\xi)| \leq K\tau^{-\alpha}$ for some positive constant K, we have

$$D_\tau^{-\alpha} G_\alpha(x,\tau;b) = \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - s)^\alpha G_\alpha(x,s;b)\, ds$$

is continuous for any $\tau \in [0, t_1]$. Hence,

$$\lim_{\tau \to 0^+} f(u(b, t-\tau)) D_\tau^{-\alpha} G_\alpha(x, \tau; b) = 0.$$

Then, $D_t^{1-\alpha} u(x,t)$ is continuous for $(x,t) \in [0,a] \times (0, t_1]$, and

$$\begin{aligned} D_t^{1-\alpha} u(x,t) &= \int_0^a D_t^{1-\alpha} G_\alpha(x,t;\xi)\phi(\xi)\, d\xi \\ &\quad + \int_0^t D_\tau^{1-\alpha} \left(G_\alpha(x,\tau;b)\right) f(u(b,t-\tau))\, d\tau. \end{aligned} \tag{3.3}$$

In order to show uniqueness, we suppose that u and $\widetilde{u}$ are distinct solutions of the integral equation (3.1) on the interval $[0, t_1]$. Let $\Phi = \sup_{(x,t)\in[0,a]\times[0,t_1]} |u(x,t) - \widetilde{u}(x,t)| > 0$. From

$$|u(x,t) - \widetilde{u}(x,t)| = \left| \int_0^t G_\alpha(x, t-\tau; b)(f(u(b,\tau)) - f(\widetilde{u}(b,\tau)))\, d\tau \right|,$$

we obtain

$$\Phi \le f'(M)\Phi \int_0^t G_\alpha(x, t-\tau; b)\, d\tau < \Phi,$$

a contradiction when $t \in [0, \sigma_1]$. Thus, the solution u is unique.

For each $M > 0$, there exist t_1 such that the integral equation (3.1) has a unique nonnegative continuous solution u. Let t_b be the supremum of all t_1 such that the integral equation has a unique nonnegative continuous solution u. Suppose that $u(x,t)$ is bounded for any $(x,t) \in [0,a] \times [0, t_b]$. We consider the integral equation (3.1) for any $(x,t) \in [0,a] \times [t_b, \infty)$ with the initial condition $u(x,0) = \phi(x)$ replaced by $u(x, t_b)$, which is known:

$$u(x,t) = \int_0^a G_\alpha(x, t-t_b; \xi) u(\xi, t_b)\, d\xi + \int_{t_b}^t G_\alpha(x, t-\tau; b) f(u(b,\tau))\, d\tau.$$

For any given positive constant $M_1 > \sup_{x\in[0,a]} u(x, t_b)$, an argument as before shows that there exists some positive t_2 such that the equation has a unique continuous nonnegative solution u for any $(x,t) \in [0,a] \times [t_b, t_2]$. This contradicts the definition of t_b. Hence, u is unbounded in $[0, t_b)$ if t_b is finite. $\square$

4. Existence of the solution.

Theorem 4.1. *Problem* (1.1) *has a unique solution for* $0 \le t < t_b$.

Proof. From Lemma 2.1 (e),

$$\int_0^t G_\alpha(x, t-\tau; b) f(u(b,\tau))\, d\tau$$

exists for $x \in [0, a]$ and t in any compact subset $[t_3, t_4]$ of $[0, t_b)$. Thus, for any $x \in [0, a]$ and any $t_5 \in (0, t)$,

$$\begin{aligned}
&\int_0^t G_\alpha(x, t-\tau; b) f(u(b,\tau))\, d\tau \\
&\quad = \lim_{k\to\infty} \int_0^{t-1/k} G_\alpha(x, t-\tau; b) f(u(b,\tau))\, d\tau \\
&\quad = \lim_{k\to\infty} \bigg[\int_{t_5}^t \frac{\partial}{\partial \zeta} \bigg(\int_0^{\zeta - 1/k} G_\alpha(x, \zeta-\tau; b) f(u(b,\tau))\, d\tau \bigg) d\zeta \\
&\qquad\qquad + \int_0^{t_5 - 1/k} G_\alpha(x, t_5-\tau; b) f(u(b,\tau))\, d\tau \bigg].
\end{aligned}$$

For $\zeta - \tau \ge 1/k$, it follows from (2.9) that

$$|(G_\alpha)_\zeta(x, \zeta-\tau; b) f(u(b,\tau))| \le \sum_{n=1}^\infty \frac{K_2 k^{\alpha+1}}{n^2} f(u(b,\tau)),$$

which is integrable with respect to τ over $(0, \zeta - 1/k)$. It follows from the Leibnitz rule (cf., [**17,** page 380]) that

$$\begin{aligned}
&\frac{\partial}{\partial \zeta} \bigg(\int_0^{\zeta-1/k} G_\alpha(x, \zeta-\tau; b) f(u(b,\tau))\, d\tau \bigg) \\
&\quad = G_\alpha\Big(x, \frac{1}{k}; b\Big) f\Big(u\Big(b, \zeta - \frac{1}{k}\Big)\Big) \\
&\qquad + \int_0^{\zeta-1/k} (G_\alpha)_\zeta(x, \zeta-\tau; b) f(u(b,\tau))\, d\tau.
\end{aligned}$$

Consider the problem

$$\frac{\partial w(x, t-\tau; \xi)}{\partial t} = \frac{\partial^2}{\partial x^2} D_t^{1-\alpha} w(x, t-\tau; \xi) \quad \text{for } x \text{ and } \xi \text{ in } (0, a),$$

$$
\begin{aligned}
&0 \le \tau < t, \\
&w(0, t-\tau; \xi) = 0 = w(a, t-\tau; \xi) \quad \text{for } 0 \le \tau < t < T, \\
&\lim_{t \to \tau^+} w(x, t-\tau; \xi) = \delta(x - \xi).
\end{aligned}
$$

From the representation formula (3.1),

$$
\begin{aligned}
w(x, t-\tau; \xi) &= \int_0^a G_\alpha(x, t-\tau; \eta)\delta(\eta - \xi)\, d\eta \\
&= G_\alpha(x, t-\tau; \xi) \quad \text{for } t \ge \tau.
\end{aligned} \tag{4.1}
$$

It follows that $\lim_{t\to\tau^+} G_\alpha(x, t-\tau; \xi) = \delta(x-\xi)$. Therefore,

$$
\begin{aligned}
&\int_0^t G_\alpha(x, t-\tau; b) f(u(b,\tau))\, d\tau \\
&\quad = \int_{t_5}^t \lim_{k\to\infty} G_\alpha\left(x, \frac{1}{k}; b\right) f\left(u\left(b, \zeta - \frac{1}{k}\right)\right) d\zeta \\
&\qquad + \lim_{k\to\infty} \int_{t_5}^t \int_0^{\zeta - 1/k} (G_\alpha)_\zeta(x, \zeta-\tau; b) f(u(b,\tau))\, d\tau\, d\zeta \\
&\qquad + \int_0^{t_5} G_\alpha(x, t_5 - \tau; b) f(u(b,\tau))\, d\tau \\
&\quad = \delta(x-b) \int_{t_5}^t f(u(b,\zeta))\, d\zeta \\
&\qquad + \lim_{k\to\infty} \int_{t_5}^t \int_0^{\zeta - 1/k} (G_\alpha)_\zeta(x, \zeta-\tau; b) f(u(b,\tau))\, d\tau\, d\zeta \\
&\qquad + \int_0^{t_5} G_\alpha(x, t_5 - \tau; b) f(u(b,\tau))\, d\tau.
\end{aligned}
$$

Let

$$
h_k(x,\zeta) = \int_0^{\zeta - 1/k} (G_\alpha)_\zeta(x, \zeta-\tau; b) f(u(b,\tau))\, d\tau.
$$

For any $k > l$, we have

$$
h_k(x,\zeta) - h_l(x,\zeta) = \int_{\zeta - 1/l}^{\zeta - 1/k} (G_\alpha)_\zeta(x, \zeta-\tau; b) f(u(b,\tau))\, d\tau.
$$

Since $(G_\alpha)_t(x, t-\tau; b) \in C([0,a] \times (\tau, T])$ and $f(u(b,\tau))$ is a monotone function of τ, it follows from the second mean value theorem for integrals (cf., [**17,** page 328]) that, for any $x \neq b$ and any ζ in any compact subset $[t_6, t_7]$ of $(0, t_b)$, there exists some real number ν such that $\zeta - \nu \in (\zeta - 1/l, \zeta - 1/k)$ and

$$h_k(x,\zeta) - h_l(x,\zeta) = f\left(u\left(b, \zeta - \frac{1}{l}\right)\right) \int_{\zeta-1/l}^{\zeta-\nu} (G_\alpha)_\zeta(x, \zeta-\tau; b)\, d\tau + f\left(u\left(b, \zeta - \frac{1}{k}\right)\right) \int_{\zeta-\nu}^{\zeta-1/k} (G_\alpha)_\zeta(x, \zeta-\tau; b)\, d\tau.$$

From $(G_\alpha)_\zeta(x, \zeta-\tau; b) = -(G_\alpha)_\tau(x, \zeta-\tau; b)$, we have

$$\begin{aligned} h_k(x,\zeta) &- h_l(x,\zeta) \\ &= \left[f\left(u\left(b, \zeta - \frac{1}{k}\right)\right) - f\left(u\left(b, \zeta - \frac{1}{l}\right)\right)\right] G_\alpha(x, \nu; b) \\ &\quad + f\left(u\left(b, \zeta - \frac{1}{l}\right)\right) G_\alpha\left(x, \frac{1}{l}; b\right) - f\left(u\left(b, \zeta - \frac{1}{k}\right)\right) G_\alpha\left(x, \frac{1}{k}; b\right). \end{aligned}$$

Since, for $x \neq b$, $G_\alpha(x, \epsilon; b) \to 0$ as $\epsilon \to 0$ uniformly with respect to ζ, $\{h_k\}$ is a Cauchy sequence, and hence, $\{h_k\}$ converges uniformly with respect to ζ in any compact subset $[t_6, t_7]$ of $(0, t_b)$. Hence, for $x \neq b$,

$$\begin{aligned} \lim_{k\to\infty} \int_{t_5}^{t} \int_0^{\zeta-1/k} &(G_\alpha)_\zeta(x, \zeta-\tau; b) f(u(b,\tau))\, d\tau\, d\zeta \\ &= \int_{t_5}^{t} \lim_{k\to\infty} \int_0^{\zeta-1/k} (G_\alpha)_\zeta(x, \zeta-\tau; b) f(u(b,\tau))\, d\tau\, d\zeta \\ &= \int_{t_5}^{t} \int_0^{\zeta} (G_\alpha)_\zeta(x, \zeta-\tau; b) f(u(b,\tau))\, d\tau\, d\zeta. \end{aligned}$$

For $x = b$, it follows from $g_\alpha(z) > 0$ and $(G_1)_\zeta(b, (\zeta-\tau)^\alpha z; b) < 0$ that

$$\begin{aligned} &-(G_\alpha)_\zeta(b, \zeta-\tau; b) f(u(b,\tau)) \\ &\qquad = -\int_0^\infty g_\alpha(z) (G_1)_\zeta(b, (\zeta-\tau)^\alpha z; b)\, dz f(u(b,\tau)), \end{aligned}$$

which is positive. Thus, $\{-h_k(b,\zeta)\}$ is a nondecreasing sequence of nonnegative functions with respect to ζ. By the monotone convergence theorem,

$$\lim_{k\to\infty} \int_{t_5}^{t} \int_0^{\zeta-1/k} (G_\alpha)_\zeta(b, \zeta-\tau; b) f(u(b,\tau))\, d\tau\, d\zeta$$

$$= \int_{t_5}^{t} \lim_{k\to\infty} \int_0^{\zeta-1/k} (G_\alpha)_\zeta(b,\zeta-\tau;b) f(u(b,\tau))\,d\tau\,d\zeta$$
$$= \int_{t_5}^{t} \int_0^{\zeta} (G_\alpha)_\zeta(b,\zeta-\tau;b) f(u(b,\tau))\,d\tau\,d\zeta.$$

Thus,

$$\begin{aligned}
&\int_0^t G_\alpha(x,t-\tau;b) f(u(b,\tau))\,d\tau \\
&\quad = \delta(x-b)\int_{t_5}^{t} f(u(b,\zeta))\,d\zeta \\
&\qquad + \int_{t_5}^{t}\int_0^{\zeta} (G_\alpha)_\zeta(x,\zeta-\tau;b) f(u(b,\tau))\,d\tau\,d\zeta \\
&\qquad + \int_0^{t_5} G_\alpha(x,t_5-\tau;b) f(u(b,\tau))\,d\tau.
\end{aligned}$$

Differentiating the above equation with respect to t, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_0^t G_\alpha(x,t-\tau;b) f(u(b,\tau))\,d\tau \\
&\quad = \delta(x-b) f(u(b,t)) + \int_0^t (G_\alpha)_t(x,t-\tau;b) f(u(b,\tau))\,d\tau.
\end{aligned} \tag{4.2}$$

From Lemma 2.1 (d) and the Leibnitz rule, we have, for any x in any compact subset of $[0,a]$ and t in any compact subset in $(0,t_b)$,

$$\begin{aligned}
&\frac{\partial}{\partial x}\int_\epsilon^t D_\tau^{1-\alpha}(G_\alpha(x,\tau;b)) f(u(b,t-\tau))\,d\tau \\
&\qquad\qquad = \int_\epsilon^t (D_\tau^{1-\alpha}(G_\alpha)_x(x,\tau;b)) f(u(b,t-\tau))\,d\tau,
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial}{\partial x}\int_\epsilon^t (D_\tau^{1-\alpha} G_\alpha)_x(x,\tau;b) f(u(b,t-\tau))\,d\tau \\
&\qquad\qquad = \int_\epsilon^t (D_\tau^{1-\alpha} G_\alpha)_{xx}(x,\tau;b) f(u(b,t-\tau))\,d\tau.
\end{aligned}$$

For each x_1 in any compact subset of $[0, a]$,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_\epsilon^t D_\tau^{1-\alpha}(G_\alpha(x,\tau;b))f(u(b,t-\tau))\,d\tau \\
&\quad = \lim_{\epsilon\to 0}\int_{x_1}^x \left(\frac{\partial}{\partial\eta}\int_\epsilon^t D_\tau^{1-\alpha}(G_\alpha(\eta,\tau;b))f(u(b,t-\tau))\,d\tau\right)d\eta \\
&\qquad + \lim_{\epsilon\to 0}\int_\epsilon^t D_\tau^{1-\alpha}(G_\alpha(x_1,\tau;b))f(u(b,t-\tau))\,d\tau \\
&\quad = \lim_{\epsilon\to 0}\int_{x_1}^x\int_\epsilon^t (D_\tau^{1-\alpha}G_\alpha)_\eta(\eta,\tau;b)f(u(b,t-\tau))\,d\tau\,d\eta \\
&\qquad + \int_0^t D_\tau^{1-\alpha}(G_\alpha(x_1,\tau;b))f(u(b,t-\tau))\,d\tau.
\end{aligned} \tag{4.3}$$

We would like to show that

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{x_1}^x\int_\epsilon^t (D_\tau^{1-\alpha}G_\alpha)_\eta(\eta,\tau;b)f(u(b,t-\tau))\,d\tau\,d\eta \\
&\qquad = \int_{x_1}^x \lim_{\epsilon\to 0}\int_\epsilon^t (D_\tau^{1-\alpha}G_\alpha)_\eta(\eta,\tau;b)f(u(b,t-\tau))\,d\tau\,d\eta.
\end{aligned} \tag{4.4}$$

By the Fubini theorem (cf., [**17,** page 352]),

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{x_1}^x\int_\epsilon^t (D_\tau^{1-\alpha}G_\alpha)_\eta(\eta,\tau;b)f(u(b,t-\tau))\,d\tau\,d\eta \\
&\quad = \lim_{\epsilon\to 0}\int_\epsilon^t \left(f(u(b,t-\tau))\int_{x_1}^x (D_\tau^{1-\alpha}G_\alpha)_\eta(\eta,\tau;b)\,d\eta\right)d\tau \\
&\quad = \lim_{\epsilon\to 0}\int_\epsilon^t f(u(b,t-\tau))(D_\tau^{1-\alpha}G_\alpha(x,\tau;b) - D_\tau^{1-\alpha}G_\alpha(x_1,\tau;b))\,d\tau \\
&\quad = \int_0^t f(u(b,t-\tau))(D_\tau^{1-\alpha}G_\alpha(x,\tau;b) - D_\tau^{1-\alpha}G_\alpha(x_1,\tau;b))\,d\tau
\end{aligned}$$

which exists by Lemma 2.1 (f). Since

$$\begin{aligned}
&\int_0^t f(u(b,t-\tau))(D_\tau^{1-\alpha}G_\alpha(x,\tau;b) - D_\tau^{1-\alpha}G_\alpha(x_1,\tau;b))\,d\tau \\
&\qquad\qquad = \int_{x_1}^x\int_0^t (D_\tau^{1-\alpha}G_\alpha)_\eta(\eta,\tau;b)f(u(b,t-\tau))\,d\tau\,d\eta,
\end{aligned}$$

we have (4.4). From (4.3),

$$\frac{\partial}{\partial x}\int_0^t D_\tau^{1-\alpha}(G_\alpha(x,\tau;b))f(u(b,t-\tau))\,d\tau = \int_0^t (D_\tau^{1-\alpha}G_\alpha)_x(x,\tau;b))(u(b,t-\tau))\,d\tau. \tag{4.5}$$

For each x_2 in any compact subset of $[0,a]$,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_\epsilon^t (D_\tau^{1-\alpha}G_\alpha)_x(x,\tau;b)f(u(b,t-\tau))\,d\tau \\
&\quad= \lim_{\epsilon\to 0}\int_{x_2}^x \left(\frac{\partial}{\partial\eta}\int_\epsilon^t (D_\tau^{1-\alpha}G_\alpha)_\eta(\eta,\tau;b)f(u(b,t-\tau))\,d\tau\right)d\eta \\
&\qquad+ \lim_{\epsilon\to 0}\int_\epsilon^t (D_\tau^{1-\alpha}G_\alpha)_\eta(x_2,\tau;b)f(u(b,t-\tau))\,d\tau \\
&\quad= \lim_{\epsilon\to 0}\int_{x_2}^x\int_\epsilon^t (D_\tau^{1-\alpha}G_\alpha)_{\eta\eta}(\eta,\tau;b)f(u(b,t-\tau))\,d\tau\,d\eta \\
&\qquad+ \int_0^t (D_\tau^{1-\alpha}G_\alpha)_\eta(x_2,\tau;b)f(u(b,t-\tau))\,d\tau.
\end{aligned} \tag{4.6}$$

We would like to show that

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{x_2}^x\int_\epsilon^t (D_\tau^{1-\alpha}G_\alpha)_{\eta\eta}(\eta,\tau;b)f(u(b,t-\tau))\,d\tau\,d\eta \\
&\quad= \int_{x_2}^x \lim_{\epsilon\to 0}\int_\epsilon^t (D_\tau^{1-\alpha}G_\alpha)_{\eta\eta}(\eta,\tau;b)f(u(b,t-\tau))\,d\tau\,d\eta.
\end{aligned} \tag{4.7}$$

By the Fubini theorem,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{x_2}^x\int_\epsilon^t (D_\tau^{1-\alpha}G_\alpha)_{\eta\eta}(\eta,\tau;b)f(u(b,t-\tau))\,d\tau\,d\eta \\
&\quad= \lim_{\epsilon\to 0}\int_\epsilon^t \left(f(u(b,t-\tau))\int_{x_2}^x (D_\tau^{1-\alpha}G_\alpha)_{\eta\eta}(\eta,\tau;b)\,d\eta\right)d\tau \\
&\quad= \lim_{\epsilon\to 0}\int_\epsilon^t f(u(b,t-\tau))\big((D_\tau^{1-\alpha}G_\alpha)_\eta(x,\tau;b) - (D_\tau^{1-\alpha}G_\alpha)_\eta(x_2,\tau;b)\big)\,d\tau \\
&\quad= \int_0^t f(u(b,t-\tau))\left((D_\tau^{1-\alpha}G_\alpha)_\eta(x,\tau;b) - (D_\tau^{1-\alpha}G_\alpha)_\eta(x_2,\tau;b)\right)d\tau
\end{aligned}$$

which exists by (4.5). Therefore,

$$\int_0^t f(u(b,t-\tau))\left((D_\tau^{1-\alpha}G_\alpha)_\eta(x,\tau;b)-(D_\tau^{1-\alpha}G_\alpha)_\eta(x_2,\tau;b)\right)\,d\tau$$
$$=\int_{x_2}^x\int_0^t (D_\tau^{1-\alpha}G_\alpha)_{\eta\eta}(\eta,\tau;b)f(u(b,t-\tau))\,d\tau\,d\eta$$
$$=\int_{x_2}^x\lim_{\epsilon\to 0}\int_\epsilon^t (D_\tau^{1-\alpha}G_\alpha)_{\eta\eta}(\eta,\tau;b)f(u(b,t-\tau))\,d\tau\,d\eta,$$

where we have (4.7). From (4.6),

$$\int_0^t (D_\tau^{1-\alpha}G_\alpha)_x(x,\tau;b)f(u(b,t-\tau))\,d\tau$$
$$=\int_{x_2}^x\int_0^t (D_\tau^{1-\alpha}G_\alpha)_{\eta\eta}(\eta,\tau;b)f(u(b,t-\tau))\,d\tau\,d\eta$$
$$+\int_0^t (D_\tau^{1-\alpha}G_\alpha)_\eta(x_2,\tau;b)f(u(b,t-\tau))\,d\tau.$$

Thus,

$$\frac{\partial}{\partial x}\int_0^t (D_\tau^{1-\alpha}G_\alpha)_x(x,\tau;b)f(u(b,t-\tau))\,d\tau$$
$$=\int_0^t (D_\tau^{1-\alpha}G_\alpha)_{xx}(x,\tau;b)f(u(b,t-\tau))\,d\tau.$$

By (4.5),

$$\frac{\partial^2}{\partial x^2}\int_0^t (D_\tau^{1-\alpha}G_\alpha)(x,\tau;b)f(u(b,t-\tau))\,d\tau$$
$$=\int_0^t (D_\tau^{1-\alpha}G_\alpha)_{xx}(x,\tau;b)f(u(b,t-\tau))\,d\tau. \tag{4.8}$$

It follows from (3.1), (3.3), (4.2) and (4.8) that, for $x\in[0,a]$ and $0<t<t_b$,

(4.9)

$$\frac{\partial}{\partial t}u-\frac{\partial^2}{\partial x^2}D_t^{1-\alpha}u$$
$$=\frac{\partial}{\partial t}\left(\int_0^a G_\alpha(x,t;\xi)\phi(\xi)\,d\xi+\int_0^t G_\alpha(x,t-\tau;b)f(u(b,\tau))\,d\tau\right)$$

$$
\begin{aligned}
&- \frac{\partial^2}{\partial x^2}\bigg(\int_0^a D_t^{1-\alpha} G_\alpha(x,t;\xi)\phi(\xi)\, d\xi \\
&\qquad\qquad + \int_0^t \left(D_\tau^{1-\alpha} G_\alpha(x,\tau;b)\right) f(u(b,t-\tau))\, d\tau \bigg) \\
&= \int_0^a (G_\alpha)_t(x,t;\xi)\phi(\xi)\, d\xi + \delta(x-b) f(u(b,t)) \\
&\quad + \int_0^t (G_\alpha)_t(x,t-\tau;b) f(u(b,\tau))\, d\tau \\
&\quad - \int_0^a (D_t^{1-\alpha} G_\alpha)_{xx}(x,t;\xi)\phi(\xi)\, d\xi \\
&\quad - \int_0^t \left(D_\tau^{1-\alpha} G_\alpha\right)_{xx}(x,\tau;b) f(u(b,t-\tau))\, d\tau \\
&= \int_0^a \left((G_\alpha)_t - (D_t^{1-\alpha} G_\alpha)_{xx}\right)(x,t;\xi)\phi(\xi)\, d\xi + \delta(x-b) f(u(b,t)) \\
&\quad + \int_0^t (G_\alpha)_t(x,t-\tau;b) f(u(b,\tau))\, d\tau \\
&\quad - \int_0^t \left(D_\tau^{1-\alpha} G_\alpha\right)_{xx}(x,\tau;b) f(u(b,t-\tau))\, d\tau.
\end{aligned}
$$

From Lemma 2.1 (d) and the change of variable $s = t - \tau$, we obtain

$$
\begin{aligned}
&\int_0^t \left(D_\tau^{1-\alpha} G_\alpha\right)_{xx}(x,\tau;b) f(u(b,t-\tau))\, d\tau \\
&\qquad\qquad = -\int_t^0 \left(D_{t-s}^{1-\alpha} G_\alpha\right)_{xx}(x,t-s;b) f(u(b,s))\, ds.
\end{aligned}
$$

From $D_{t-s}^{1-\alpha} u(t-s) = D_t^{1-\alpha} u(t-s)$,

$$
\begin{aligned}
&\int_0^t \left(D_\tau^{1-\alpha} G_\alpha\right)_{xx}(x,\tau;b) f(u(b,t-\tau))\, d\tau \\
&\qquad\qquad = \int_0^t \left(D_t^{1-\alpha} G_\alpha\right)_{xx}(x,t-\tau;b) f(u(b,\tau))\, d\tau.
\end{aligned}
$$

From (4.9),

$$
\frac{\partial}{\partial t} u - \frac{\partial^2}{\partial x^2} D_t^{1-\alpha} u = \int_0^a \left((G_\alpha)_t - (D_t^{1-\alpha} G_\alpha)_{xx}\right)(x,t;\xi)\phi(\xi)\, d\xi
$$

$$+ \delta(x-b)f(u(b,t)) + \int_0^t \Big((G_\alpha)_t - \big(D_t^{1-\alpha}G_\alpha\big)_{xx}\Big)(x,t-\tau;b)f(u(b,\tau))d\tau.$$

Since

$$\int_0^a \left((G_\alpha)_t - (D_t^{1-\alpha}G_\alpha)_{xx}\right)(x,t;\xi)\phi(\xi)\,d\xi$$
$$= \delta(t)\int_0^a \delta(x-\xi)\phi(\xi)\,d\xi = 0 \quad \text{for } t = 0^+,$$

$$\int_0^t \left((G_\alpha)_t - \big(D_t^{1-\alpha}G_\alpha\big)_{xx}\right)(x,t-\tau;b)f(u(b,\tau))\,d\tau$$
$$= \delta(x-b)\lim_{\epsilon\to 0}\int_0^{t-\epsilon} \delta(t-\tau)f(u(b,\tau))\,d\tau = 0 \quad \text{for } t > \tau,$$

we obtain

$$\frac{\partial}{\partial t}u - \frac{\partial^2}{\partial x^2}D_t^{1-\alpha}u = \delta(x-b)f(u(b,t)).$$

From (3.1) and (4.1), we have, for $x \in [0,a]$,

$$\lim_{t\to 0} u(x,t) = \lim_{t\to 0}\int_0^a G_\alpha(x,t;\xi)\phi(\xi)\,d\xi = \phi(x).$$

Since $G_\alpha(0,t-\tau;\xi) = 0 = G_\alpha(a,t-\tau;\xi)$, we have $u(0,t) = 0 = u(a,t)$. Thus, the solution u of (3.1) is a solution of problem (1.1). Since a solution of the latter is a solution of the former, the theorem is proved. $\square$

REFERENCES

1. C.Y. Chan, *A quenching criterion for a multi-dimensional parabolic problem due to a concentrated nonlinear source*, J. Comp. Appl. Math. **235** (2011), 3724–3727.

2. C.Y. Chan and R. Boonklurb, *A blow-up criterion for a degenerate parabolic problem due to a concentrated nonlinear source*, Quart. Appl. Math. **65** (2007), 781–787.

3. C.Y. Chan, P. Sawangtong and T. Treeyaprasert, *Single blow-up point and critical speed for a parabolic problem with a moving nonlinear source on a semi-infinite interval*, Quart. Appl. Math. **73** (2015), 483–492.

4. C.Y. Chan and H.Y. Tian, *Single-point blow-up for a degenerate parabolic problem due to a concentrated nonlinear source*, Quart. Appl. Math. **61** (2003), 363–385.

5. ______, *Multi-dimensional explosion due to a concentrated nonlinear source*, J. Math. Anal. Appl. **295** (2004), 174–190.

6. ______, *A criterion for a multi-dimensional explosion due to a concentrated nonlinear source*, Appl. Math. Lett. **19** (2006), 298–302.

7. C.Y. Chan and P. Tragoonsirisak, *A multi-dimensional quenching problem due to a concentrated nonlinear source in R^N*, Nonlin. Anal. **69** (2008), 1494–1514.

8. ______, *A multi-dimensional blow up problem due to a concentrated nonlinear source in R^N*, Quart. Appl. Math. **69** (2011), 317–330.

9. C.Y. Chan and T. Treeyaprasert, *Blow-up criteria for a parabolic problem due to a concentrated nonlinear source on a semi-infinite interval*, Quart. Appl. Math. **70** (2012), 159–169.

10. H.J. Haubold, A.M. Mathai and R.K. Saxena, *Mittag-Leffler functions and their applications*, J. Appl. Math. (2011), 1–51, Art. ID 298628.

11. C.M. Kirk and W.E. Olmstead, *Blow-up in a reactive-diffusive medium with a moving heat source*, Z. Angew Math. Phys. **53** (2002), 147–159.

12. R. Metzler and J. Klafter, *The random walk's guide to anomalous diffusion, A fractional dynamics approach*, Phys. Rep. **339** (2000), 1–77.

13. ______, *The restaurant at the end of the random walk*: *Recent developments in the description of anomalous transport by fractional dynamics*, J. Phys. **37** (2004), 161–208.

14. W.E. Olmstead and C.A. Roberts, *Thermal blow-up in a subdiffusive medium*, SIAM J. Appl. Math. **69** (2008), 514–523.

15. ______, *Explosion in a diffusive strip due to a concentrated nonlinear source*, Meth. Appl. Anal. **1** (1994), 434–445.

16. I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, 1999.

17. K.R. Stromberg, *An introduction to classical real analysis*, Wadsworth International Group, Belmont, CA, 1981.

18. J. Trujillo, *Fractional models*: *Sub and super-diffusives, and undifferentiable solutions*, in *Innovation in engineering computational technology*, Sax-Coburg Publications, Stirling, Scotland, 2006.

19. M.M. Wyss and W. Wyss, *Evolution, its fractional extension and generalization*, Fract. Calc. Appl. Anal. **4** (2001), 273–284.

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