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SURFACE INTEGRAL FORMULATION OF THE INTERIOR TRANSMISSION PROBLEM

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ABSTRACT. We consider a surface integral formulation of the so-called interior transmission problem that appears in the study of inverse scattering problems from dielectric inclusions. In the case where the magnetic permeability contrast is zero, the main originality of our approach consists in still using classical potentials for the Helmholtz equation but in weaker trace space solutions. One major outcome of this study is to establish Fredholm properties of the problem for relaxed assumptions on the material coefficients. For instance, we allow the contrast to change sign inside the medium. We also show how one can retrieve discreteness results for transmission eigenvalues in some particular situations.

1. Introduction. The theory of inverse scattering for acoustic and electromagnetic waves, is an active area of research with significant developments in the past few years and, more specifically, the so-called interior transmission problem (ITP) and its transmission eigenvalues [**3, 4, 6–9, 11, 12, 15, 17, 19, 21, 22**]. Although simply stated, the interior transmission problem is not covered by the standard theory of elliptic partial differential equations since, as it stands, it is neither elliptic nor self-adjoint. We provide here a new approach to study the interior transmission problem using a surface integral equation formulation. We shall consider here the scalar case that corresponds, for instance, with two-dimensional transverse polarizations of electromagnetic waves. We treat separately the case where the magnetic permeability contrast does not vanish and has a fixed sign at the boundary of the domain (case 1) and the case where this contrast is zero (case 2).

The main original motivation behind this study was the design of a numerical method to solve ITP in the case of piece-wise constant

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index of refraction and compute transmission eigenvalues for general geometries [**10, 13**]. However, we found out that the surface integral formulation of the problem also presents some theoretical interests. For instance, establishing the equivalence between this formulation and the volumetric formulation of the problem in case 2 requires the introduction of non standard results on potentials. This is due to the fact that the space of (variational) solutions is $L^2(D)$ with Laplacian in $L^2(D)$, where D is the domain of the inclusion. Hence, the natural spaces for solutions to the integral equation would be $H^{-1/2}(\partial D) \times H^{-3/2}(\partial D)$, since the unknowns correspond with the traces and conormal traces of the (variational) solutions. Regularity, continuity and coercivity properties of the used potentials in those trace spaces are the main novel ingredients of our study. We relied in particular on the theory of pseudo-differential operators to derive regularity properties. Then, by using appropriate denseness arguments, classical trace formulae are generalized to potentials with densities having weaker regularities. Coercivity properties of the potentials are analyzed in the cases of purely imaginary wavenumbers. Let us emphasize here that an alternative (theoretical) approach to treating this case would have been to consider potentials with kernels related to the fundamental solution of the biharmonic operator. However, this approach would have been less intuitive (in the case of ITP) and less appropriate for numerical considerations.

The second and probably more important interest of this integral equation formulation is related to the study of ITP with relaxed assumptions on the sign of the contrasts. More specifically, we allow the difference between the index of refraction of the inclusion and of the background to change sign inside D. The variational method, employed for instance in [**5, 14**] to treat the case of inclusions with cavities, fails to establish the Fredholm nature of the ITP in those situations. Using the surface integral approach we are able to prove that the ITP is of Fredholm type if the contrast is constant and positive (or negative) only in the neighborhood of the boundary. We deduce, in particular, that the set of transmission eigenvalues is still discrete in some specific cases where uniqueness can be shown for a particular wavenumber. The main drawback of this method is that it can only treat the question of discreteness of the set of transmission eigenvalues and not existence. This type of result is similar to the one recently established in [**25**] for case 2 using the notion of upper triangular compact operators (see also [**18**]). For case 1, similar results have been derived in [**2**] using the variational approach and the notion of T-coercivity and, in the case of regular coefficients, we refer to [**20**] where even weaker assumptions on the contrast sign are needed to obtain discreteness of the spectrum. The approach in [**20**] can also be exploited to establish existence of transmission eigenvalues [**21**].

The interior transmission problem and outline of the paper. Consider a simply connected and bounded region $D \subset \mathbf{R}^d$, ($d = 2$ or $d = 3$) where the boundary $\Gamma := \partial D$ is of class C^2. The general form of the scalar isotropic interior transmission problem can be written as:

$$
\begin{cases}
\nabla \cdot \frac{1}{\mu(x)} \nabla w + k^2 n(x) w = 0 & \text{in } D, \\
\Delta v + k^2 v = 0 & \text{in } D, \\
w = v & \text{on } \Gamma, \\
\frac{1}{\mu} \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} & \text{on } \Gamma,
\end{cases}
\tag{1}
$$

where $v, w \in H^1(D)$ if $\mu \neq 1$ and $v, w \in L^2(D)$ such that $u := w - v \in H^2(D)$ if $\mu = 1$.

Definition 1.1. A transmission eigenvalue is a value of k for which the interior transmission problem (1) has a non trivial solution.

In the first part, in order to introduce the surface integral equation method, we shall treat the simple case where n and μ are constant. We distinguish the case $\mu \neq 1$, for which the basic tools are the same as for classical transmission problems, from the case $\mu = 1$ and $n \neq 1$, where new ingredients have to be used. We then consider the more general cases where the latter assumptions hold only on a neighborhood of the boundary.

The outline of this article is the following. In Section 2, we first recall some classical results from potential theory associated with the Helmholtz operator. These results are then used to derive and analyze a surface integral formulation of ITP in the the case where $\mu \neq 1$ is constant and n is constant. After studying regularity properties of

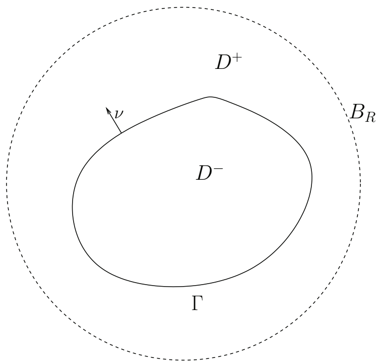

FIGURE 1. Domains and notation.

the potentials for densities in $H^{-3/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, we treat in Section 3 the case $\mu = 1$ and $n \neq 1$. Finally, in Section 4, we show how one can extend the results of previous sections to the cases where the assumptions on the coefficients hold only in a neighborhood of the boundary.

2. Surface integral equation formulation of the ITP in the case $\mu \neq 1$.

2.1. Some classical results from potential theory. We denote by $D^+ := \{\mathbf{R}^d \setminus \overline{D}\} \cap B_R$ a bounded exterior domain, where B_R denotes a ball containing D and denote by $D^- := D$ the interior domain. Let ν be the unit normal to Γ directed to the exterior of D (see Figure 1).

If u is a regular function defined in $D^+ \cup D^-$, we denote by

$$u^\pm(x_\Gamma) := \lim_{h \downarrow 0^\pm} u(x_\Gamma + h\nu(x_\Gamma)),$$
$$\frac{\partial u^\pm}{\partial \nu}(x_\Gamma) := \lim_{h \downarrow 0^\pm} \nabla u(x_\Gamma + h\nu(x_\Gamma)) \cdot \nu(x_\Gamma) \quad \text{for } x_\Gamma \in \Gamma.$$

Moreover, we define the jumps on Γ:

$$[u]_\Gamma(x_\Gamma) := u^+(x_\Gamma) - u^-(x_\Gamma)$$
$$\left[\frac{\partial u}{\partial \nu}\right]_\Gamma(x_\Gamma) := \frac{\partial u^+}{\partial \nu}(x_\Gamma) - \frac{\partial u^-}{\partial \nu}(x_\Gamma) \quad \text{for } x_\Gamma \in \Gamma.$$

We shall keep this notation for non regular functions if these trace operators can be continuously extended (in an appropriate function space) to these functions.

Let Φ_k be the outgoing Green function associated with the Helmholtz operator with wavenumber $k \in \mathbf{C}$ with non negative real and imaginary parts. We recall that

$$\Phi_k(x,y) = \frac{e^{\mathrm{i}k|x-y|}}{4\pi|x-y|} \quad \text{for } d = 3$$

and

$$\Phi_k(x,y) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|) \quad \text{for } d = 2,$$

where $H_0^{(1)}$ denotes the Hankel function of the first kind of order 0. We then define the single and double layer potentials for regular densities φ, respectively, by:

$$\begin{aligned}
(\mathrm{SL}_k^\Gamma \varphi)(x) &:= \int_\Gamma \Phi_k(x,y)\varphi(y)\, ds(y),\\
(\mathrm{DL}_k^\Gamma \varphi)(x) &:= \int_\Gamma \frac{\partial \Phi_k}{\partial \nu(y)}(x,y)\varphi(y)\, ds(y) \quad \text{for } x \in \mathbf{R}^d \setminus \Gamma.
\end{aligned}$$

The following theorem summarizes some classical results from potential theory that can be found for instance in [**16, 23, 24**].

Theorem 2.1. *The single-layer potential* $\mathrm{SL}_k^\Gamma : H^{-1/2}(\Gamma) \to H^1(D^\pm)$ *and the double layer potential* $\mathrm{DL}_k^\Gamma : H^{1/2}(\Gamma) \to H^1(D^\pm)$ *are bounded and give rise to bounded linear operators*

$$\begin{aligned}
S_k^\Gamma &: H^{-1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma), &\qquad K_k^\Gamma &: H^{1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma),\\
K_k'^\Gamma &: H^{-1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma), &\qquad T_k^\Gamma &: H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma),
\end{aligned}$$

such that, for all $\varphi \in H^{-1/2}(\Gamma)$ *and* $\psi \in H^{1/2}(\Gamma)$,

$$\begin{cases}
(\mathrm{SL}_k^\Gamma \varphi)^\pm = S_k^\Gamma \varphi & \textit{and } (\mathrm{DL}_k^\Gamma \psi)^\pm = K_k^\Gamma \psi \pm \dfrac{1}{2}\psi \quad \textit{in} \quad H^{1/2}(\Gamma),\\
\frac{\partial(\mathrm{SL}_k^\Gamma \varphi)}{\partial \nu}^\pm = K_k'^\Gamma \varphi \mp \frac{1}{2}\varphi & \textit{and } \frac{\partial(\mathrm{DL}_k^\Gamma \psi)}{\partial \nu}^\pm = T_k^\Gamma \psi \ \textit{in} \quad H^{-1/2}(\Gamma).
\end{cases}$$

We recall that, for regular densities φ and ψ, the surface potentials S_k^Γ, K_k^Γ, $K_k'^\Gamma$ and T_k^Γ can be expressed as

$$
\begin{aligned}
(S_k^\Gamma\varphi)(x) &= \int_\Gamma \Phi_k(x,y)\varphi(y)\,ds(y),\\
(K_k^\Gamma\psi)(x) &= \int_\Gamma \frac{\partial\Phi_k}{\partial\nu(y)}(x,y)\psi(y)\,ds(y),\\
(K_k'^\Gamma\varphi)(x) &= \int_\Gamma \frac{\partial\Phi_k}{\partial\nu(x)}(x,y)\varphi(y)\,ds(y),\\
(T_k^\Gamma\psi)(x) &= \lim_{\varepsilon\to 0}\int_{\Gamma,|y-x|>\varepsilon} \frac{\partial^2\Phi_k}{\partial\nu(y)\partial\nu(x)}(x,y)\psi(y)\,ds(y)
\end{aligned}
\tag{2}
$$

for all $x \in \Gamma$.

In the following, when there is no confusion on the notation, we shall simplify the notation and only use SL_k, DL_k, S_k, K_k, K_k' and T_k.

2.2. Derivation of the surface integral equation. We consider in this section the simpler case where μ and n are constant in D and $\mu \neq 1$. We first write an equivalent formulation of the problem in terms of surface integral equations then prove that the operator associated with this formulation is Fredholm of index 0.

Let k be the wavenumber appearing in system (1); we shall denote

$$k_0 := k \quad \text{and} \quad k_1 := \sqrt{\mu n}k.$$

Consider $(v,w) \in H^1(D)\times H^1(D)$ a solution to (1), and set

$$\alpha := \frac{\partial v}{\partial\nu}\bigg|_\Gamma = \frac{1}{\mu}\frac{\partial w}{\partial\nu}\bigg|_\Gamma \in H^{-1/2}(\Gamma) \quad \text{and} \quad \beta := v|_\Gamma = w|_\Gamma \in H^{1/2}(\Gamma).$$

Since v and w satisfy $\Delta v + k_0^2 v = 0$ and $\Delta w + k_1^2 w = 0$ in D, then it is well known [**11**] that these solutions can be expressed using the following integral representation

$$v = \mathrm{SL}_{k_0}\alpha - \mathrm{DL}_{k_0}\beta, \qquad w = \mu\mathrm{SL}_{k_1}\alpha - \mathrm{DL}_{k_1}\beta \quad \text{in } D. \tag{3}$$

From the boundary conditions of (1), $w = v$ and $\partial v/\partial \nu|_\Gamma = (1/\mu)(\partial w/\partial \nu)|_\Gamma$ and the jump properties of potentials (see Theorem 2.1), one easily verifies that α and β satisfy

$$Z(k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0, \tag{4}$$

where

$$Z(k) := \begin{pmatrix} \mu S_{k_1} - S_{k_0} & -K_{k_1} + K_{k_0} \\ -K'_{k_1} + K'_{k_0} & 1/\mu T_{k_1} - T_{k_0} \end{pmatrix}.$$

Equation (4) forms the surface integral equation formulation of (1). The equivalence between the two formulations is ensured after guaranteeing that non trivial solutions of (4) define, through (3), non trivial solutions to (1). Using Green's integral theorem, it is easily seen that solutions to (1) correspond with non radiating solutions. More precisely, if we define the far field operators $P_i^\infty : H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \to L^2(\Omega)$ for $i = 0, 1$, where $\Omega := \{x \in \mathbf{R}^d / ||x|| = 1\}$, by (the integrals are understood as duality pairing)

$$P_0^\infty(\alpha, \beta)(\widehat{x}) = \frac{1}{4\pi}\int_\Gamma \left(\beta(y)\frac{\partial e^{-\mathrm{i}k\widehat{x}\cdot y}}{\partial \nu(y)} - \alpha(y)e^{-\mathrm{i}k\widehat{x}\cdot y}\right) ds(y),$$

$$P_1^\infty(\alpha, \beta)(\widehat{x}) = \frac{1}{4\pi}\int_\Gamma \left(\beta(y)\frac{\partial e^{-\mathrm{i}k_1\widehat{x}\cdot y}}{\partial \nu(y)} - \frac{1}{\mu}\alpha(y)e^{-\mathrm{i}k_1\widehat{x}\cdot y}\right) ds(y),$$

then the far field pattern of v given by $P_0^\infty(\alpha, \beta)$ and the far field pattern of w given by $P_1^\infty(\alpha, \beta)$ vanish.

The following theorem indicates that one of the latter conditions is sufficient to ensure the equivalence between the two formulations of ITP (see also Remark 2.2).

Theorem 2.2. *Assume that the wavenumber k is real and positive. The three following assertions are equivalent*:

(i) *There exists $(v, w) \in H^1(D) \times H^1(D)$ a non trivial solution to* (1).

(ii) *There exists $(\alpha, \beta) \neq (0, 0)$ in $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ such that*

$$Z(k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0 \quad \textit{and} \quad P_0^\infty(\alpha, \beta) = 0.$$

(iii) *There exists* $(\alpha, \beta) \neq (0,0)$ *in* $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ *such that*

$$Z(k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0 \quad \textit{and} \quad P_1^{\infty}(\alpha,\beta) = 0.$$

Proof. It only remains to show that (ii) implies (i) and that (iii) implies (i). Assume that there exist $\alpha \in H^{-1/2}(\Gamma)$ and $\beta \in H^{1/2}(\Gamma)$ satisfying $Z(k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0$. We define $v := \mathrm{SL}_{k_0}\alpha - \mathrm{DL}_{k_0}\beta$ and $w := \mu\mathrm{SL}_{k_1}\alpha - \mathrm{DL}_{k_1}\beta$ in $\mathbf{R}^d \setminus \Gamma$. The regularity of the single and double layer potentials shows that v and w are in $H^1(D)$ and they satisfy $\Delta v + k^2 v = 0$ and $\nabla \cdot \dfrac{1}{\mu}\nabla w + k^2 n w = 0$ in D.

First assume that $P_0^{\infty}(\alpha,\beta) = 0$. We shall show that $v \neq 0$. From Rellich's lemma, we deduce that $v = 0$ in $\mathbf{R}^d \setminus D$. Assume that $v = 0$ also in D. We have in particular $[v]_\Gamma = [\partial v/\partial\nu]_\Gamma = 0$ and from the jump properties of the single and double layer potentials we also have $[v]_\Gamma = -\beta$ and $[\partial v/\partial\nu]_\Gamma = -\alpha$. This contradicts the fact that $(\alpha,\beta) \neq (0,0)$. Then $v \neq 0$ in D.

If we assume now that $P_1^{\infty}(\alpha,\beta) = 0$, we can similarly show that $w \neq 0$. Indeed, from Rellich's lemma, we deduce that $w = 0$ in $\mathbf{R}^d \setminus \overline{D}$. Now if we assume that $w = 0$ also in D, we have in particular that $[w]_\Gamma = [\partial w/\partial\nu]_\Gamma = 0$. From the expression of w and the jump properties of the single and double layer potentials, we also have $[w]_\Gamma = -\beta$ and $[\partial w/\partial\nu]_\Gamma = -\mu\alpha$ which contradicts the fact that $(\alpha,\beta) \neq (0,0)$. Then $w \neq 0$ in D. ◻

Remark 2.1. Since v and w have the same Cauchy data on Γ, if either v or w is different from zero, then the other one is necessarily different from zero too.

Remark 2.2. Another possibility to ensure equivalence between the surface integral and volumetric formulations of ITP would have been to use the so called Calderòn projectors. More precisely, its is well known (see for instance [**22**]) that the pairs $(\alpha,\beta) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ that coincide with normal traces and traces of solutions $u \in H^1(D)$ to the

Helmholtz equation $\Delta u + k^2 u = 0$ in D can be characterized as elements of the kernel of the Calderòn projector (for the exterior problem)

$$P(k) = \begin{pmatrix} S_k & -K_k - I/2 \\ K'_k - I/2 & -T_k \end{pmatrix}.$$

We preferred to use the farfield operator since it is easier to handle in numerical applications and is also more convenient to use in the case of $\mu = 1$ (studied in Section 3).

2.3. Analysis of the operator $Z(k)$. We first decompose $Z(k)$ as the sum of a coercive operator and a compact operator showing that this operator is Fredholm of index 0. We then show the discreteness of the set of transmission eigenvalues using analytic Fredholm theory.

Theorem 2.3. *The operator $Z(k) : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is Fredholm of index zero, and is analytic on $k \in \mathbf{C} \setminus \mathbf{R}^-$.*

Proof. We use the following decomposition

$$\begin{aligned} Z(k) = &\begin{pmatrix} (\mu-1)S_{\mathrm{i}|k_0|} & 0 \\ 0 & (1/\mu - 1)T_{\mathrm{i}|k_0|} \end{pmatrix} \\ &+ \begin{pmatrix} \mu(S_{k_1} - S_{\mathrm{i}|k_0|}) & 0 \\ 0 & 1/\mu(T_{k_1} - T_{\mathrm{i}|k_0|}) \end{pmatrix} \\ &+ \begin{pmatrix} S_{\mathrm{i}|k_0|} - S_{k_0} & 0 \\ 0 & T_{\mathrm{i}|k_0|} - T_{k_0} \end{pmatrix} \\ &+ \begin{pmatrix} 0 & K_{k_1} - K_{k_0} \\ K'_{k_1} - K'_{k_0} & 0 \end{pmatrix}. \end{aligned} \tag{5}$$

It is well known that, for any purely imaginary $k := \mathrm{i}\kappa$, with $\kappa \in \mathbf{R}$, the trace of the single layer potential and the normal derivative of the double layer potential are coercive on their corresponding spaces (see, for instance, [**23**, Section 33] for the case $\kappa = 0$). Then the first operator of the right-hand side is invertible from $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ into $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$. Moreover, the three last operators in the decomposition of $Z(k)$ are compact from $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$

into $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ [**16**]. Consequently, the operator $Z(k) : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is Fredholm of index zero.

The analyticity of the operator $Z(k)$ is a consequence of the analyticity of the kernels of the potentials and the fact that the derivative with respect to k does not increase the singularity of the surface potentials. □

To apply the analytic Fredholm theorem and conclude on the discreteness of the set of transmission eigenvalues, we need one more result that ensures the injectivity of $Z(k)$ for at least one k. The main tool of the following proof is the T-coercivity [**2**].

Lemma 2.4. *Assume that $\mu - 1$ and $1 - n$ are both either positive or negative, and let k be a positive real. Then the operator $Z(\mathrm{i}k) : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is injective.*

Proof. Assume that $Z(\mathrm{i}k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0$. Let us define

$$v := \mathrm{SL}_{\mathrm{i}k_0}\alpha - \mathrm{DL}_{\mathrm{i}k_0}\beta \quad \text{in } \mathbf{R}^d \setminus \Gamma \text{ and } w := \mu \mathrm{SL}_{\mathrm{i}k_1}\alpha - \mathrm{DL}_{\mathrm{i}k_1}\beta \text{ in } \mathbf{R}^d \setminus \Gamma.$$

The relation $Z(\mathrm{i}k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0$ implies that, on Γ, we have $w^\pm = v^\pm$ and $(1/\mu)(\partial w^\pm/\partial \nu) = (\partial v^\pm/\partial \nu)$. Consequently, the pair $(w, v) \in H^1(\mathbf{R}^d \setminus \Gamma)^2$ are a solution to

$$\begin{cases} \Delta v - k^2 v = 0 & \text{in } \mathbf{R}^d \setminus \Gamma \\ \nabla \cdot (1/\mu)\nabla w - k^2 n w = 0 & \text{in } \mathbf{R}^d \setminus \Gamma \\ w^\pm = v^\pm & \text{on } \Gamma \\ (1/\mu)(\partial w^\pm/\partial \nu) = (\partial v^\pm/\partial \nu) & \text{on } \Gamma. \end{cases}$$

Let us define the Hilbert space $\mathbf{H} := \{(w, v) \in H^1(\mathbf{R}^d \setminus \Gamma)^2,\ w^\pm = v^\pm \text{ on } \Gamma\}$. First, we observe that $(w, v) \in \mathbf{H}$. Now, let $(\varphi, \psi) \in \mathbf{H}$. Multiplying the first equation by ψ and the second by φ, integrating by parts on both side of Γ and using the jump relations on the boundary, we get the following variational formulation: find (w, v) in $\mathbf{H}$ such that $a_k((w, v), (\varphi, \psi)) = 0$ for all (φ, ψ) in $\mathbf{H}$ where

$$a((w, v), (\varphi, \psi)) := \int_{\mathbf{R}^d} \frac{1}{\mu}\nabla w \cdot \nabla\overline{\varphi} + k^2 n w \overline{\varphi} - \int_{\mathbf{R}^d} \nabla v \cdot \nabla\overline{\psi} + k^2 v \overline{\psi}.$$

1. Case where $\mu < 1$ and $n > 1$. Let $\mathcal{O}$ be a neighborhood of Γ. Let us define the cutoff function χ with compact support in $\mathcal{O}$ such that $\chi = 1$ on Γ and

$$\begin{aligned} T : \mathbf{H} \times \mathbf{H} &\longrightarrow \mathbf{H} \times \mathbf{H} \\ (w, v) &\longmapsto (w, -v + 2\chi w). \end{aligned}$$

Assume that there exists a $C > 0$ such that $||\nabla\chi||_\infty \leq C$.

$$\begin{aligned} a((w,v), T(w,v)) &= \int_{\mathbf{R}^d \setminus \Gamma} \left(\frac{1}{\mu} |\nabla w|^2 + k^2 n |w|^2 + |\nabla v|^2 + k^2 |v|^2 \right) dx \\ &\quad - 2 \int_{\mathcal{O}} \chi \nabla v \cdot \nabla \overline{w} - 2k^2 \int_{\mathcal{O}} \chi v \overline{w} - 2 \int_{\mathcal{O}} \overline{w} \nabla v \cdot \nabla \chi \\ &\geq \frac{1}{\mu} ||\nabla w||^2_{L^2(\mathbf{R}^d)} + k^2 n ||w||^2_{L^2(\mathbf{R}^d)} \\ &\quad + ||\nabla v||^2_{L^2(\mathbf{R}^d)} + k^2 ||v||^2_{L^2(\mathbf{R}^d)} \\ &\quad - \gamma ||\nabla v||^2_{L^2(\mathbf{R}^d)} - \frac{1}{\gamma} ||\nabla w||^2_{L^2(\mathbf{R}^d)} \\ &\quad - C\eta ||\nabla v||^2_{L^2(\mathbf{R}^d)} - \frac{C}{\eta} ||w||^2_{L^2(\mathbf{R}^d)} \\ &\quad - k^2 \delta ||v||^2_{L^2(\mathbf{R}^d)} - \frac{k^2}{\delta} ||w||^2_{L^2(\mathbf{R}^d)} \\ &= \left(\frac{1}{\mu} - \frac{1}{\gamma} \right) ||\nabla w||^2_{L^2(\mathbf{R}^d)} \\ &\quad + \left(k^2 \left(n - \frac{1}{\delta} \right) - \frac{C}{\eta} \right) ||w||^2_{L^2(\mathbf{R}^d)} \\ &\quad + (1 - \gamma - C\eta) ||\nabla v||^2_{L^2(\mathbf{R}^d)} \\ &\quad + k^2 (1 - \delta) ||v||^2_{L^2(\mathbf{R}^d)}. \end{aligned}$$

Let $\mu < \gamma < 1$, $1/n < \delta < 1$ and η be such that $1 - \gamma - C\eta > 0$ is fixed. Then, if k is large enough to have $k^2(n - (1/\delta)) - (C/\eta) > 0$, we deduce that a is T-coercive. As a consequence, $w = 0$ and $v = 0$ are the only solutions. From the equality $[v]_\Gamma = -\beta$ and $[\partial v/\partial \nu]_\Gamma = -\alpha$, we get that $\alpha = \beta = 0$ and finally $Z(\mathrm{i}k) : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is injective.

2. Case where $\mu > 1$ and $n < 1$. Let $\mathcal{O}$ be a neighborhood of Γ. Let us define the cutoff function χ with compact support in $\mathcal{O}$ such that

$\chi = 1$ on Γ and

$$\begin{aligned} T : \mathbf{H} \times \mathbf{H} &\longrightarrow \mathbf{H} \times \mathbf{H} \\ (w, v) &\longmapsto (w - 2\chi v, -v). \end{aligned}$$

Assume that there exists a $C > 0$ such that $||\nabla \chi||_\infty \leq C$.

$$\begin{aligned} a((w,v), T(w,v)) = &\int_{\mathbf{R}^d \setminus \Gamma} \left(\frac{1}{\mu} |\nabla w|^2 + k^2 n |w|^2 + |\nabla v|^2 + k^2 |v|^2 \right) dx \\ &- \frac{2}{\mu} \int_{\mathcal{O}} \chi \nabla w \cdot \nabla \overline{v} - \frac{2}{\mu} \int_{\mathcal{O}} \overline{v} \nabla w \cdot \nabla \chi - 2k^2 n \int_{\mathcal{O}} \chi w \overline{v} \\ \geq &\frac{1}{\mu} ||\nabla w||^2_{L^2(\mathbf{R}^d)} + k^2 n ||w||^2_{L^2(\mathbf{R}^d)} \\ &+ ||\nabla v||^2_{L^2(\mathbf{R}^d)} + k^2 ||v||^2_{L^2(\mathbf{R}^d)} \\ &- \frac{\eta}{\mu} ||\nabla w||^2_{L^2(\mathbf{R}^d)} - \frac{1}{\mu\eta} ||\nabla v||^2_{L^2(\mathbf{R}^d)} \\ &- \frac{C\delta}{\mu} ||\nabla w||^2_{L^2(\mathbf{R}^d)} - \frac{C}{\mu\delta} ||v||^2_{L^2(\mathbf{R}^d)} \\ &- \frac{k^2 n}{\gamma} ||w||^2_{L^2(\mathbf{R}^d)} - k^2 n \gamma ||v||^2_{L^2(\mathbf{R}^d)} \\ = &\frac{1}{\mu} (1 - \eta - C\delta) ||\nabla w||^2_{L^2(\mathbf{R}^d)} \\ &+ k^2 n \left(1 - \frac{1}{\gamma} \right) ||w||^2_{L^2(\mathbf{R}^d)} \\ &+ \left(1 - \frac{1}{\mu\eta} \right) ||\nabla v||^2_{L^2(\mathbf{R}^d)} \\ &+ \left(k^2 \left(n - \frac{1}{\beta} \right) - \frac{C}{\eta} \right) ||v||^2_{L^2(\mathbf{R}^d)}. \end{aligned}$$

Let $1/\mu < \eta < 1$, $1 < \gamma < 1/n$ and δ be such that $1 - \eta - C\delta > 0$ is fixed. Then, if k is large enough to have $k^2(1 - n\gamma) - (C/\mu\delta) > 0$, we deduce that a is T-coercive. As a consequence, $w = 0$ and $v = 0$ are the only solutions. From the equality $[v]_\Gamma = -\beta$ and $[\partial v / \partial \nu]_\Gamma = -\alpha$ we get that $\alpha = \beta = 0$ and finally $Z(\mathrm{i}k) : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is injective. $\square$

We now can state a concluding theorem for this section, which is a classical result on ITP [**9**] related to the discreteness of transmission eigenvalues for contrasts that does not change sign and is now a straightforward corollary of Theorem 2.3, Lemma 2.4 and the fact that the set of transmission eigenvalues is contained in the set of points where $Z(k)$ is not invertible.

Theorem 2.5. *Assume that $\mu - 1$ and $1 - n$ are either positive or negative. Then the set of transmission eigenvalues is discrete.*

3. The case $\mu = 1$. In this section, we assume that $\mu = 1$ and treat the case where n is constant and $n \neq 1$.

We first observe that the analysis of the previous section cannot be carried out in the current case since the operator $Z(k) : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is compact for $\mu = 1$ (see decomposition (5)). This is somehow predictable since the natural spaces for the solutions is $v \in L^2(D)$ and $\Delta v \in L^2(D)$; therefore, the boundary values $\alpha := \partial v/\partial\nu|_\Gamma$ and $\beta := v|_\Gamma$ now live respectively in $H^{-3/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ and not in the classical spaces $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. Consequently, one needs to analyze the operator $Z(k)$ as acting on $H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$.

The first step is then to analyze/generalize the properties of the single and the double layer potentials in these spaces.

3.1. Single and double layer potentials for densities in $H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$.

3.1.1. Trace properties. Since the volume potential

$$(\widehat{\mathrm{SL}}_k\varphi)(x) := \int_D \Phi_k(x,y)\varphi(y)\,dy, \quad x \in \mathbf{R}^d,$$

defines a pseudo-differential operator of order -2 (see [**16**]), this implies in particular that (see [**16**, Theorem 8.5.8]) $\mathrm{SL}_k : H^{-3/2}(\Gamma) \to L^2(D^\pm)$ and $\mathrm{DL}_k : H^{-1/2}(\Gamma) \to L^2(D^\pm)$ are continuous. Moreover, by obvious denseness arguments, for any densities $\varphi \in H^{-3/2}(\Gamma)$ and $\psi \in H^{-1/2}(\Gamma)$, $\mathrm{SL}_k\varphi$ and $\mathrm{DL}_k\psi$ satisfy the Helmholtz equation in the distributional sense in $\mathbf{R}^d \setminus \Gamma$. Therefore, if one defines

$$L^2_\Delta(D^\pm) := \left\{u \in L^2(D^\pm),\ \Delta u \in L^2(D^\pm)\right\}$$

equipped with the graph norm, then one easily deduces that $\mathrm{SL}_k : H^{-3/2}(\Gamma) \to L^2_\Delta(D^\pm)$ and $\mathrm{DL}_k : H^{-1/2}(\Gamma) \to L^2_\Delta(D^\pm)$ are continuous. More importantly, one can generalize the results of Theorem 2.1 in the following sense.

Theorem 3.1. *The single-layer potential* $\mathrm{SL}_k : H^{-3/2}(\Gamma) \to L^2_\Delta(D^\pm)$ *and the double layer potential* $\mathrm{DL}_k : H^{-1/2}(\Gamma) \to L^2_\Delta(D^\pm)$ *are bounded and give rise to bounded linear operators*

$$S_k : H^{-3/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma), \qquad K_k : H^{-1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma),$$
$$K'_k : H^{-3/2}(\Gamma) \longrightarrow H^{-3/2}(\Gamma), \qquad T_k : H^{-1/2}(\Gamma) \longrightarrow H^{-3/2}(\Gamma),$$

such that for all $\varphi \in H^{-3/2}(\Gamma)$ *and* $\psi \in H^{-1/2}(\Gamma)$,

$$\begin{cases} (\mathrm{SL}_k\varphi)^\pm = S_k\varphi & \text{and } (\mathrm{DL}_k\psi)^\pm = K_k\psi \pm \psi/2 \text{ in } H^{-1/2}(\Gamma), \\ \frac{\partial(\mathrm{SL}_k\varphi)}{\partial\nu}^\pm = K'_k\varphi \mp \varphi/2 & \text{and } \frac{\partial(\mathrm{DL}_k\psi)}{\partial\nu}^\pm = T_k\psi \text{ in } H^{-3/2}(\Gamma). \end{cases}$$

Proof. The first part of the theorem is already proven. The jump and trace properties will be deduced from Theorem 2.1 using a denseness argument. More specifically, let $\varphi \in H^{-3/2}(\Gamma)$ and $\varphi_n \in H^{-1/2}(\Gamma)$ be such that $\varphi_n \underset{n\to\infty}{\to} \varphi$ in $H^{-3/2}(\Gamma)$. Using the continuity of SL_k from $H^{-3/2}(\Gamma)$ into $L^2(D^\pm)$, we have that $\mathrm{SL}_k\varphi_n$ converges to $\mathrm{SL}_k\varphi$ in $L^2(D^\pm)$ and $\Delta\mathrm{SL}_k\varphi_n$ converges to $\Delta\mathrm{SL}_k\varphi$ in $L^2(D^\pm)$. For all $v \in L^2_\Delta(D^\pm)$, we define the trace $v^\pm \in H^{-1/2}(\Gamma)$ of v on Γ by using the identity

$$\langle v^\pm, \varphi\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = \mp \int_{D^\pm} v\Delta w \pm \int_{D^\pm} w\Delta v,$$

where $w \in H^2(D^\pm)$ such that $w = 0$ and $\partial w/\partial\nu = \varphi$ on Γ. Furthermore,

$$||v^\pm||_{H^{-1/2}(\Gamma)} := \sup_{||\varphi||_{H^{1/2}(\Gamma)}=1} \langle v^\pm, \varphi\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \le C||v||_{L^2_\Delta(D^\pm)}.$$

We deduce that

$$||(\mathrm{SL}_k\varphi_n)^\pm - (\mathrm{SL}_k\varphi)^\pm||_{H^{-1/2}(\Gamma)} \le C||\mathrm{SL}_k\varphi_n - \mathrm{SL}_k\varphi||_{L^2_\Delta(D^\pm)},$$

and consequently $(\mathrm{SL}_k\varphi_n)^\pm \underset{n\to\infty}{\to} (\mathrm{SL}_k\varphi)^\pm$ in $H^{-1/2}(\Gamma)$. We then get the jump property

$$0 = [\mathrm{SL}_k\varphi_n]_\Gamma \underset{n\to\infty}{\longrightarrow} [\mathrm{SL}_k\varphi]_\Gamma \text{ in } H^{-1/2}(\Gamma).$$

For all $v \in L^2_\Delta(D^\pm)$, we define the normal derivative $\partial v/\partial\nu^\pm \in H^{-3/2}(\Gamma)$ of v on Γ through

$$\left\langle \frac{\partial v}{\partial \nu}^\pm, \varphi \right\rangle_{H^{-3/2}(\Gamma), H^{3/2}(\Gamma)} = \pm \int_{D^\pm} v\Delta w \mp \int_{D^\pm} w\Delta v,$$

where $w \in H^2(D^\pm)$ such that $w = \varphi$ and $\partial w/\partial\nu = 0$ on Γ. Then,

$$\left\| \frac{\partial v}{\partial \nu}^\pm \right\|_{H^{-3/2}(\Gamma)} := \sup_{||\varphi||_{H^{3/2}}=1} \left\langle \frac{\partial v}{\partial \nu}^\pm, \varphi \right\rangle_{H^{-3/2}(\Gamma), H^{3/2}(\Gamma)} \leq C||v||_{L^2_\Delta(D^\pm)}.$$

We deduce that

$$\left| \frac{\partial(\mathrm{SL}_k\varphi_n)}{\partial\nu}^\pm - \frac{\partial(\mathrm{SL}_k\varphi)}{\partial\nu}^\pm \right\|_{H^{-3/2}(\Gamma)} \leq C||\mathrm{SL}_k\varphi_n - \mathrm{SL}_k\varphi||_{L^2_\Delta(D^\pm)}.$$

Since $[\partial(\mathrm{SL}_k\varphi_n)/(\partial\nu)]_\Gamma = -\varphi_n$, by taking the limit as $n \to \infty$ one obtains the jump property $[\partial(\mathrm{SL}_k\varphi)/(\partial\nu)]_\Gamma = -\varphi$.

The proof of the jump properties for the double-layer potential follows a very similar procedure and is left to the reader. □

3.1.2. Regularity of the single and double-layer potentials. In this subsection, we generalize some regularity results for densities in $H^{-3/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$. For two given wavenumbers k and k' (with non negative real parts) we define

$$\widehat{\mathcal{SL}}_{k,k'} := \widehat{\mathrm{SL}}_k - \widehat{\mathrm{SL}}_{k'}, \qquad \mathcal{SL}_{k,k'} := \mathrm{SL}_k - \mathrm{SL}_{k'}.$$

Theorem 3.2. *The pseudo-differential operator $\widehat{\mathcal{SL}}_{k,k'}$ is of order -4.*

Proof. First, we consider the case $d = 3$. We use the power series of the exponential

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

Let $k \neq k'$ and denote $z = x - y$. Then, the kernel of $\widehat{\mathcal{SL}}_{k,k'}$ has the expansion

$$\begin{aligned} a(x,z) &:= \frac{e^{\mathrm{i}k|z|} - e^{\mathrm{i}k'|z|}}{4\pi|z|} \\ &= \frac{\mathrm{i}}{4\pi}(k-k') - \frac{1}{4\pi}\sum_{j=0}^{\infty} \frac{\mathrm{i}^j}{(j+2)!}(k^{j+2} - {k'}^{j+2})|z|^{j+1} \\ &= \frac{\mathrm{i}}{4\pi}(k-k') + \sum_{j=0}^{\infty} a_{j+1}(x,z), \end{aligned}$$

where

$$a_{j+1}(x,z) := \frac{-\mathrm{i}^j}{4\pi(j+2)!}(k^{j+2} - {k'}^{j+2})|z|^{j+1}, \quad \text{for all } j \geq 0,$$

which satisfies

$$a_p(x, tz) = t^p a_p(x,z).$$

From [**16**, Theorem 7.1.1], we deduce that

$$\widehat{\mathrm{SL}}_{k,k'}\varphi(x) = \int_D a(x, x-y)\varphi(y)\,dy,$$

where a is a pseudo homogeneous kernel of degree 1 is a pseudo-differential operator of order -4.

Now, for $d = 2$, the kernel of $\widehat{\mathcal{SL}}_{k,k'}$ is

$$a(x,z) := \frac{\mathrm{i}}{4}\left(H_0^{(1)}(k|z|) - H_0^{(1)}(k'|z|)\right).$$

From [**1**], for all $t \in \mathbf{C}$, we have

$$H_0^{(1)}(t) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!^2}\left(\frac{t}{2}\right)^{2p}\theta(p) + \frac{2\mathrm{i}}{\pi}\ln(t)\sum_{p=0}^{\infty}\frac{(-1)^p}{p!^2}\left(\frac{t}{2}\right)^{2p},$$

where

$$\theta(p) := 1 + \frac{2\mathrm{i}}{\pi}(C - \ln 2) - \frac{2\mathrm{i}}{\pi}\sum_{m=1}^{p}\frac{1}{m}$$

and

$$C := \lim_{p\to+\infty}\left\{\sum_{m=1}^{p}\frac{1}{m} - \ln p\right\}$$

denotes the Euler's constant. Then,

$$\begin{aligned}
a(x,z) = {} & \frac{\mathrm{i}}{4}\sum_{p=0}^{\infty}\frac{(-1)^{p+1}}{(p+1)!^2}\left(k^{2p+2} - {k'}^{2p+2}\right)\left(\frac{|z|}{2}\right)^{2p+2}\theta(p) \\
& + \frac{1}{2\pi}\sum_{p=0}^{\infty}\frac{(-1)^{p}}{(p+1)!^2}\left(k^{2p+2}\ln(k) - {k'}^{2p+2}\ln(k')\right)\left(\frac{|z|}{2}\right)^{2p+2} \\
& + \frac{1}{2\pi}(\ln k - \ln k') \\
& + \frac{1}{2\pi}\ln|z|\sum_{p=0}^{\infty}\frac{(-1)^{p}}{(p+1)!^2}\left(k^{2p+2} - {k'}^{2p+2}\right)\left(\frac{|z|}{2}\right)^{2p+2} \\
= {} & f(x,z) + \sum_{j=0}^{\infty} p_{j+2}(x,z)\ln|z|,
\end{aligned}$$

where $f \in C^\infty(D\times\mathbf{R}^d)$ and

$$p_{j+2}(x,z) = \begin{cases} 0 & \text{if } j \text{ is odd,} \\ \frac{1}{2\pi}\frac{(-1)^{p+1}}{(p+1)!^2}\left(k^{j+2} - {k'}^{j+2}\right)\left(\frac{|z|}{2}\right)^{j+2} & \text{if } j = 2p. \end{cases}$$

The function p_q satisfies $p_q(x,tz) = t^q p_q(x,z)$ and, consequently, the kernel of $\widehat{\mathcal{SL}}_{k,k'}$ is a pseudo-homogeneous kernel of degree 2. From [**16**, Theorem 7.1.1], we deduce that $\widehat{\mathcal{SL}}_{k,k'}$ is a pseudo-differential operator of order -4. □

Now, one can remark that the operator defined by $\mathcal{DL}_{k,k'} := \mathrm{DL}_k - \mathrm{DL}_{k'}$ is such that

$$\mathcal{DL}_{k,k'}(\varphi) = -\nabla\mathcal{SL}_{k,k'}(\varphi\nu)$$

since $(\partial \Phi_k)/(\partial \nu(y)) = \nabla_y \Phi_k \cdot \nu(y) = -\nabla_x \Phi_k \cdot \nu(y)$. Therefore, the regularity of the operator $\mathcal{DL}_{k,k'}$ can be deduced from the regularity of $\mathcal{SL}_{k,k'}$.

Using Theorem 3.2 and [**16**, Theorem 8.5.8], we can generalize the regularity results on $\mathcal{SL}_{k,k'}$ and $\mathcal{DL}_{k,k'}$ for densities in $H^{-3/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, respectively.

Corollary 3.3. *$\mathcal{SL}_{k,k'} : H^{-3/2}(\Gamma) \to H^2(D)$ and $\mathcal{DL}_{k,k'} : H^{-1/2}(\Gamma) \to H^2(D)$ are continuous for all $k \neq k'$.*

In the case where $\mu = 1$, we need to find more regular operators for the compact part in the Fredholm decomposition of the operator corresponding to the interior transmission problem. To this end, we eliminate the principal part of the asymptotic developments of the kernels of the potentials and consider the operators $\widehat{\mathcal{SL}}_{k,k'} + \gamma(k,k')\widehat{\mathcal{SL}}_{\mathrm{i}|k|,\mathrm{i}|k'|}$ where

$$\gamma(k,k') := \frac{k^2 - k'^2}{|k|^2 - |k'|^2}.$$

Theorem 3.4. *$\widehat{\mathcal{SL}}_{k,k'} + \gamma(k,k')\widehat{\mathcal{SL}}_{\mathrm{i}|k|,\mathrm{i}|k'|}$ is a pseudo-differential operator of order -5 for all $k \neq k'$, $k, k' \in \mathbf{C}\backslash\mathbf{R}^-$.*

Proof. The proof follows the same idea as in the proof of Theorem 3.2. First, let us consider the case $d = 3$.

We consider the kernel of $\widehat{\mathcal{SL}}_{k,k'} + \gamma(k,k')\widehat{\mathcal{SL}}_{\mathrm{i}|k|,\mathrm{i}|k'|}$. It can be written in the form

$$\begin{aligned}\widetilde{a}(x,z) &:= \frac{e^{\mathrm{i}k|z|} - e^{\mathrm{i}k'|z|}}{4\pi|z|} + \frac{e^{-|kz|} - e^{-|k'z|}}{4\pi|z|} \\ &= \frac{1}{4\pi}\left[\mathrm{i}(k-k') - \frac{k^2 - k'^2}{|k| + |k'|}\right] \\ &\quad - \frac{1}{4\pi}\sum_{j=0}^{\infty}\frac{1}{(j+3)!}\Big[\mathrm{i}^{j+1}(k^{j+3} - k'^{\,j+3})\end{aligned}$$

$$+ (-1)^j \left(|k|^{j+3} - |k'|^{j+3}\right) \gamma(k,k')\Big] |z|^{j+2}$$
$$= \frac{1}{4\pi}\left[\mathrm{i}(k-k') - \frac{k^2-k'^2}{|k|+|k'|}\right] + \sum_{j=0}^{\infty} \widetilde{a}_{j+2}(x,z),$$

where

$$\widetilde{a}_{j+2}(x,z) := -\frac{1}{4\pi(j+3)!} \times \left[\mathrm{i}^{j+1}(k^{j+3}-k'^{j+3}) + (-1)^j \left(|k|^{j+3}-|k'|^{j+3}\right)\gamma(k,k')\right] |z|^{j+2},$$

for all $j \geq 0$, which satisfies

$$\widetilde{a}_p(x,tz) = t^p \widetilde{a}_p(x,z).$$

From [**16**, Theorem 7.1.1], we deduce that

$$\left(\widehat{\mathcal{SL}}_{k,k'} + \gamma(k,k')\widehat{\mathcal{SL}}_{\mathrm{i}|k|,\mathrm{i}|k'|}\right)\varphi(x) = \int_D \widetilde{a}(x,x-y)\varphi(y)\,dy,$$

where $\widetilde{a}$ is a pseudo-homogeneous kernel of degree 2, is a pseudo-differential operator of order -5.

Let us now consider the case $d = 2$. The kernel of $\widehat{\mathcal{SL}}_{k,k'} + \gamma(k,k')\widehat{\mathcal{SL}}_{\mathrm{i}|k|,\mathrm{i}|k'|}$ is

$$\widetilde{a}(x,z) = \frac{\mathrm{i}}{4}\left(H_0^{(1)}(k|z|) - H_0^{(1)}(k'|z|) + H_0^{(1)}(\mathrm{i}|kz|) - H_0^{(1)}(\mathrm{i}|k'z|)\right)$$
$$= \frac{\mathrm{i}}{4}\sum_{p=1}^{\infty}\frac{1}{(p+1)!^2}\left[(-1)^{p+1}\left(k^{2p+2}-k'^{2p+2}\right)\right.$$
$$\left. + \left(|k|^{2p+2}-|k'|^{2p+2}\right)\gamma(k,k')\right]\left(\frac{|z|}{2}\right)^{2p+2}\theta(p)$$
$$+ \frac{1}{2\pi}\sum_{p=0}^{\infty}\frac{1}{(p+1)!^2}\left[(-1)^p\left(k^{2p+2}\ln(k) - k'^{2p+2}\ln(k')\right)\right.$$
$$\left. + \left(|k|^{2p+2}\ln(\mathrm{i}|k|) - |k'|^{2p+2}\ln(\mathrm{i}|k'|)\right)\gamma(k,k')\right]\left(\frac{|z|}{2}\right)^{2p+2}$$
$$+ \frac{1}{2\pi}(\ln k - \ln k' + \ln|k| - \ln|k'|)$$

$$
\begin{aligned}
&+\frac{1}{2\pi}\ln|z|\sum_{p=0}^{\infty}\frac{1}{(p+2)!^2}\left[(-1)^{p+1}\left(k^{2p+4}-k'^{2p+4}\right)\right.\\
&\left.-\left(|k|^{2p+4}-|k'|^{2p+4}\right)\gamma(k,k')\right]\left(\frac{|z|}{2}\right)^{2p+4}\\
=\;&\widetilde{f}(x,z)+\sum_{j=0}^{\infty}\widetilde{p}_{j+4}(x,z)\ln|z|,
\end{aligned}
$$

where $\widetilde{f}\in C^\infty(D\times\mathbf{R}^d)$ and $\widetilde{p}_{j+4}(x,z)=0$ if j is odd and

$$
\begin{aligned}
\widetilde{p}_{j+4}(x,z)=\frac{1}{2\pi(p+2)!^2}&\left[(-1)^{p+1}\left(k^{j+4}-k'^{j+4}\right)\right.\\
&\left.-\left(|k|^{j+4}-|k'|^{j+4}\right)\gamma(k,k')\right]\left(\frac{|z|}{2}\right)^{j+4}\ln|z|\quad\text{if } j=2p.
\end{aligned}
$$

The function $\widetilde{p}_q$ satisfies $\widetilde{p}_q(x,tz)=t^q\widetilde{p}_q(x,z)$ and, consequently, the kernel of $\widehat{\mathcal{SL}}_{k,k'}+\gamma(k,k')\widehat{\mathcal{SL}}_{\mathrm{i}|k|,\mathrm{i}|k'|}$ is a pseudo-homogeneous kernel of degree 4. We deduce that $\widehat{\mathcal{SL}}_{k,k'}+\gamma(k,k')\widehat{\mathcal{SL}}_{\mathrm{i}|k|,\mathrm{i}|k'|}$ is a pseudo-differential operator of order -6 (then also of order -5). $\square$

We can immediately deduce the following corollary for the regularities of $\mathcal{SL}_{k,k'}+\gamma(k,k')\mathcal{SL}_{\mathrm{i}|k|,\mathrm{i}|k'|}$ and $\mathcal{DL}_{k,k'}+\gamma(k,k')\mathcal{DL}_{\mathrm{i}|k|,\mathrm{i}|k'|}$.

Corollary 3.5. *$\mathcal{SL}_{k,k'}+\gamma(k,k')\mathcal{SL}_{\mathrm{i}|k|,\mathrm{i}|k'|}:H^{-3/2}(\Gamma)\to H^3(D)$ and $\mathcal{DL}_{k,k'}+\gamma(k,k')\mathcal{DL}_{\mathrm{i}|k|,\mathrm{i}|k'|}:H^{-1/2}(\Gamma)\to H^3(D)$ are continuous for all $k\neq k'$, $k,k'\in\mathbf{C}\backslash\mathbf{R}^-$.*

Now that we have generalized the properties of the potentials in the weaker spaces $H^{-3/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, we can treat the interior transmission problem and study the Fredholm property of the corresponding operator.

3.2. Surface integral formulation. The procedure to derive a surface integral formulation of problem (1) is similar to the case $\mu\neq 1$. Consider $(v,w)\in L^2(D)\times L^2(D)$ a solution to (1), and set

$$
\alpha:=\left.\frac{\partial v}{\partial\nu}\right|_\Gamma=\left.\frac{\partial w}{\partial\nu}\right|_\Gamma\in H^{-3/2}(\Gamma)\quad\text{and}\quad\beta:=v|_\Gamma=w|_\Gamma\in H^{-1/2}(\Gamma).
$$

Since v and w satisfy $\Delta v + k_0^2 v = 0$ and $\Delta w + k_1^2 w = 0$ in D, similarly to the case $\mu \neq 1$, these solutions can be written

$$v = \mathrm{SL}_{k_0}\alpha - \mathrm{DL}_{k_0}\beta, \qquad w = \mathrm{SL}_{k_1}\alpha - \mathrm{DL}_{k_1}\beta \quad \text{in } D. \tag{6}$$

Then $u := w - v$ can be written in the form $u = \mathcal{SL}_{k_1,k_0}\alpha - \mathcal{DL}_{k_1,k_0}\beta$. From the boundary conditions of (1), $u|_\Gamma = 0$ and $\partial u/\partial\nu|_\Gamma = 0$, α and β satisfy $Z(k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0$, where

$$Z(k) := \begin{pmatrix} S_{k_1} - S_{k_0} & -K_{k_1} + K_{k_0} \\ -K'_{k_1} + K'_{k_0} & T_{k_1} - T_{k_0} \end{pmatrix}.$$

Again, the far fields generated by v and w are equal to zero, and we have $P_0^\infty(\alpha,\beta) = 0$ and $P_1^\infty(\alpha,\beta) = 0$ where P_0^∞ and P_1^∞ are defined in Section 2 with $\mu = 1$.

Theorem 3.6. *The three following assertions are equivalent.*

(i) *There exist* $v, w \in L^2(D)$ *such that* $w - v \in H^2(D)$, *a nontrivial solution to* (1).

(ii) *There exist* $\alpha \neq 0$ *in* $H^{-3/2}(\Gamma)$ *and* $\beta \neq 0$ *in* $H^{-1/2}(\Gamma)$ *such that*

$$Z(k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0 \quad \textit{and} \quad P_0^\infty(\alpha,\beta) = 0.$$

(iii) *There exist* $\alpha \neq 0$ *in* $H^{-3/2}(\Gamma)$ *and* $\beta \neq 0$ *in* $H^{-1/2}(\Gamma)$ *such that*

$$Z(k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0 \quad \textit{and} \quad P_1^\infty(\alpha,\beta) = 0.$$

Proof. The proof is similar to the proof of Theorem 2.2 with $v, w \in L^2(D)$. The regularity of the difference $u := w - v$ results from Corollary 3.3. ◻

3.3. Fredholm property of the surface integral operator $Z(k)$. Similarly to the case where $\mu \neq 1$, we want to show that $Z(k)$ is of Fredholm type. We shall make the following decomposition:

$$Z(k) = -\gamma(k_1,k_0)Z(\mathrm{i}|k|) + (Z(k) + \gamma(k_1,k_0)Z(\mathrm{i}|k|)).$$

Indeed, from Corollary 3.5, the operator $Z(k) + \gamma(k_1, k_0)Z(\mathrm{i}|k|) : H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ is compact. Thus, it only remains to show that $Z(\mathrm{i}|k|) : H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ is coercive.

Remark 3.1. In the following, an operator $A : H \to H'$ is said to be coercive if

$$|\langle Ax, x\rangle_{H,H'}| \geq C||x||_H^2$$

for all $x \in H$ where $\langle \cdot, \cdot \rangle_{H,H'}$ denotes the duality pairing between H and its dual H'.

Lemma 3.7. $Z(\mathrm{i}|k|) : H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ *is coercive.*

Proof. For the sake of presentation, we denote by $\kappa_0 := |k_0|$ and $\kappa_1 := |k_1|$.

Let α be in $H^{-3/2}(\Gamma)$ and $\beta \in H^{-1/2}(\Gamma)$. Let us consider the following problem:

$$(7) \qquad \begin{cases} (\Delta - \kappa_0^2)(\Delta - \kappa_1^2)u = 0 & \text{in } \mathbf{R}^d \setminus \Gamma \\ [\Delta u]_\Gamma = \beta(\kappa_1^2 - \kappa_0^2) & \text{on } \Gamma \\ \left[\frac{\partial(\Delta u)}{\partial \nu}\right]_\Gamma = \alpha(\kappa_1^2 - \kappa_0^2) & \text{on } \Gamma. \end{cases}$$

Multiplying the first equation by $\varphi \in H^2(\mathbf{R}^d)$, integrating by parts on both sides of Γ and using the jump conditions on Γ, we can formulate the variational formulation as follows : find $u \in H^2(\mathbf{R}^d)$ such that

$$(8) \quad \int_{\mathbf{R}^d\setminus\Gamma} (\Delta u - \kappa_0^2 u)(\Delta\overline{\varphi} - \kappa_1^2\overline{\varphi})\, dx \\ = -\int_\Gamma (\kappa_1^2 - \kappa_0^2)\left(\alpha\overline{\varphi} - \beta\frac{\partial\overline{\varphi}}{\partial\nu}\right) ds(x), \quad \text{for all } \varphi \in H^2(\mathbf{R}^d).$$

One can remark that $u = \mathcal{SL}_{\mathrm{i}\kappa_1,\mathrm{i}\kappa_0}\alpha - \mathcal{DL}_{\mathrm{i}\kappa_1,\mathrm{i}\kappa_0}\beta$ is a solution to (8). Using the Lax-Milgram theorem, the existence and uniqueness of a solution $u \in H^2(\mathbf{R}^d)$ to (8) can be established. Thus, the only solution to (8) is $u = \mathcal{SL}_{\mathrm{i}\kappa_1,\mathrm{i}\kappa_0}\alpha - \mathcal{DL}_{\mathrm{i}\kappa_1,\mathrm{i}\kappa_0}\beta$. In particular,

$$u|_\Gamma = (S_{\mathrm{i}\kappa_1} - S_{\mathrm{i}\kappa_0})\alpha - (K_{\mathrm{i}\kappa_1} - K_{\mathrm{i}\kappa_0})\beta$$

and

$$\left.\frac{\partial u}{\partial \nu}\right|_{\Gamma} = (K'_{\mathrm{i}\kappa_1} - K'_{\mathrm{i}\kappa_0})\alpha - (T_{\mathrm{i}\kappa_1} - T_{\mathrm{i}\kappa_0})\beta.$$

For $\varphi = u$ in (8), we get

$$\begin{aligned}(9)\quad \int_{\mathbf{R}^d\setminus\Gamma} &(\Delta u - \kappa_0^2 u)(\Delta\overline{u} - \kappa_1^2\overline{u})\,dx \\ &= -\int_{\Gamma}(\kappa_1^2 - \kappa_0^2)\left(\alpha\overline{u} - \beta\frac{\partial\overline{u}}{\partial\nu}\right)ds(x).\end{aligned}$$

From the inequality

$$\int_{\mathbf{R}^d\setminus\Gamma}(\Delta u - \kappa_0^2 u)(\Delta\overline{u} - \kappa_1^2\overline{u})\,dx \geq C||u||^2_{H^2(\mathbf{R}^d)}$$

and (9), we obtain

$$(10)\quad \left|\langle\alpha, u\rangle_{H^{-3/2}(\Gamma),H^{3/2}(\Gamma)} - \left\langle\beta, \frac{\partial u}{\partial\nu}\right\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}\right| \geq C'||u||^2_{H^2(\mathbf{R}^d)}.$$

Now let us show that there exists a $C_1 > 0$ such that $||\alpha||_{H^{-3/2}(\Gamma)} \leq C_1||u||_{H^2(\mathbf{R}^d)}$. Let $\varphi \in H^{3/2}(\Gamma)$ be such that $||\varphi||_{H^{3/2}(\Gamma)} = 1$. Then, there exists $\widetilde{\varphi} \in H^2(\mathbf{R}^d)$ such that $\widetilde{\varphi}|_\Gamma = \varphi$ and $\partial\widetilde{\varphi}/\partial\nu|_\Gamma = 0$. From (8), we have that

$$\begin{aligned}|\langle\alpha, \varphi\rangle_{H^{-3/2}(\Gamma),H^{3/2}(\Gamma)}| &= \frac{1}{|\kappa_1^2 - \kappa_0^2|}\left|\int_{\mathbf{R}^d\setminus\Gamma}(\Delta u - \kappa_0^2 u)(\Delta\overline{\widetilde{\varphi}} - \kappa_1^2\overline{\widetilde{\varphi}})\,dx\right| \\ &\leq C||u||_{H^2(\mathbf{R}^d)}||\widetilde{\varphi}||_{H^2(\mathbf{R}^d)} \\ &\leq C_1||u||_{H^2(\mathbf{R}^d)},\end{aligned}$$

because $||\widetilde{\varphi}|||_{H^2(\mathbf{R}^d)} \leq ||\varphi||_{H^{3/2}(\Gamma)} = 1$. Then $||\alpha||_{H^{-3/2}(\Gamma)} \leq C_1||u||_{H^2(\mathbf{R}^d)}$.

Furthermore, we can show that $||\beta||_{H^{-1/2}(\Gamma)} \leq C_2||u||_{H^2(\mathbf{R}^d)}$. Indeed, let $\psi \in H^{1/2}(\Gamma)$ be such that $||\psi||_{H^{1/2}(\Gamma)} = 1$. There exists a $\widetilde{\psi} \in H^2(\mathbf{R}^d)$ such that $\widetilde{\psi}|_\Gamma = 0$ and $\partial\widetilde{\psi}/\partial\nu|_\Gamma = \psi$. Then

$$\begin{aligned}|\langle\beta, \psi\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}| \\ &= \frac{1}{|\kappa_1^2 - \kappa_0^2|}\left|\int_{\mathbf{R}^d\setminus\Gamma}(\Delta u - \kappa_0^2 u)(\Delta\overline{\widetilde{\psi}} - \kappa_1^2\overline{\widetilde{\psi}})\,dx\right| \\ &\leq C||u||_{H^2(\mathbf{R}^d)}||\widetilde{\psi}||_{H^2(\mathbf{R}^d)} \\ &\leq C_2||u||_{H^2(\mathbf{R}^d)},\end{aligned}$$

since $||\widetilde{\psi}|||_{H^2(\mathbf{R}^d)} \leq ||\psi||_{H^{1/2}(\Gamma)} = 1$. Then $||\beta||_{H^{-1/2}(\Gamma)} \leq C_2||u||_{H^2(\mathbf{R}^d)}$.

We now deduce the coercivity of $Z(\mathrm{i}|k|)$.

$$\begin{aligned}
&\left|\left\langle Z(\mathrm{i}|k|)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right\rangle\right| \\
&\qquad = \big|\langle (S_{\mathrm{i}\kappa_1} - S_{\mathrm{i}\kappa_0})\alpha - (K_{\mathrm{i}\kappa_1} - K_{\mathrm{i}\kappa_0})\beta, \alpha\rangle_{H^{-3/2}(\Gamma),H^{3/2}(\Gamma)} \\
&\qquad\quad + \langle -(K'_{\mathrm{i}\kappa_1} - K'_{\mathrm{i}\kappa_0})\alpha + (T_{\mathrm{i}\kappa_1} - T_{\mathrm{i}\kappa_0})\beta, \beta\rangle_{H^{1/2}(\Gamma),H^{-1/2}(\Gamma)}\big| \\
&\qquad \geq \left|\langle u|_\Gamma, \alpha\rangle_{H^{3/2}(\Gamma),H^{-3/2}(\Gamma)} + \left\langle -\frac{\partial u}{\partial \nu}|_\Gamma, \beta\right\rangle_{H^{1/2}(\Gamma),H^{-1/2}(\Gamma)}\right| \\
&\qquad \geq C'||u||^2_{H^2(\mathbf{R}^d)} \\
&\qquad \geq \frac{C'}{C_1}||\alpha||^2_{H^{-3/2}(\Gamma)} + \frac{C'}{C_2}||\beta||^2_{H^{-1/2}(\Gamma)}.
\end{aligned}$$

Then $Z(\mathrm{i}|k|) : H^{-3/2}(\Gamma)\times H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma)\times H^{1/2}(\Gamma)$ is coercive. ☐

Theorem 3.8. *The operator $Z(k) : H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ is Fredholm of index zero and analytic on $k \in \mathbf{C} \setminus \mathbf{R}^-$.*

Proof. The analyticity is a direct consequence of the analyticity of the kernels of the integral operators. We can rewrite

$$Z(k) = -\gamma(k_0, k_1)Z(\mathrm{i}|k|) + T(k),$$

with

$$T(k) := \begin{pmatrix} (S_{k_1} - S_{k_0}) + \gamma(k_0,k_1)(S_{\mathrm{i}|k_1|} - S_{\mathrm{i}|k_0|}) & -(K_{k_1} - K_{k_0}) - \gamma(k_0,k_1)(K_{\mathrm{i}|k_1|} - K_{\mathrm{i}|k_0|}) \\ -(K'_{k_1} - K'_{k_0}) - \gamma(k_0,k_1)(K'_{\mathrm{i}|k_1|} - K'_{\mathrm{i}|k_0|}) & (T_{k_1} - T_{k_0}) + \gamma(k_0,k_1)(T_{\mathrm{i}|k_1|} - T_{\mathrm{i}|k_0|}) \end{pmatrix}.$$

Corollary 3.5 combined with classical trace theorems implies that $T(k) : H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ is compact. Then $Z(k) : H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ is Fredholm. ☐

Theorem 3.9. *Assume that $n \neq 1$. Then the set of transmission eigenvalues is discrete.*

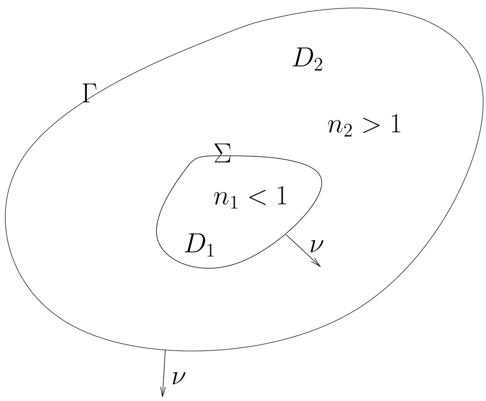

FIGURE 2. Geometry and notation.

Proof. From Lemma 3.7, we deduce that $Z(k)$ is injective for all purely imaginary $k \in \mathbf{C}$. The discreteness of the set of transmission eigenvalues is then a direct consequence of Theorem 3.8 and the use of the analytic Fredholm theory. ◻

4. Cases where the contrasts change sign.

4.1. The case of piecewise constant coefficients and $\mu \neq 1$. Before considering more general cases, we specifically treat here the case where $n - 1$ is piecewise constant and changes sign for $\mu \neq 1$. More precisely, we consider the case where n can have two different constant values $n_1 \neq 1$ and $n_2 \neq 1$ with $(n_1 - 1)(n_2 - 1) < 0$. Let $D_1 \subset D$ be such that $n := n_1$ in D_1 and $n := n_2$ in $D_2 := D \setminus D_1$. We shall assume that $\overline{D_1} \cap \partial D = \varnothing$. We set $\Gamma := \partial D$ and $\Sigma := \partial D_1$ and assume that these surfaces are regular (see Figure 2). We denote by $k_i := k\sqrt{\mu n_i}$ for $i = 1, 2$ and $k_0 := k$.

In this section, in order to differentiate the potentials defined for densities on Γ and Σ, we shall again use the notation introduced in Section 2: SL_k^Γ, DL_k^Γ, S_k^Γ, K_k^Γ, $K_k'^\Gamma$ and T_k^Γ.

4.1.1. Surface integral formulation of ITP. If we define

$$\alpha := \frac{\partial v}{\partial \nu}\Big|_\Gamma = \frac{1}{\mu}\frac{\partial w}{\partial \nu}\Big|_\Gamma \in H^{-1/2}(\Gamma), \qquad \beta := v|_\Gamma = w|_\Gamma \in H^{1/2}(\Gamma),$$
$$\alpha' := \frac{1}{\mu}\frac{\partial w}{\partial \nu}\Big|_\Sigma \in H^{-1/2}(\Sigma), \qquad \beta' := w|_\Sigma \in H^{1/2}(\Sigma).$$

then the solutions to (1) can be written in the form

$$v = \mathrm{SL}^\Gamma_{k_0}\alpha - \mathrm{DL}^\Gamma_{k_0}\beta,$$
$$w = \begin{cases} \mu\mathrm{SL}^\Gamma_{k_2}\alpha - \mathrm{DL}^\Gamma_{k_2}\beta - \mu\mathrm{SL}^\Sigma_{k_2}\alpha' + \mathrm{DL}^\Sigma_{k_2}\beta' & \text{in } D_2,\\ \mu\mathrm{SL}^\Sigma_{k_1}\alpha' - \mathrm{DL}^\Sigma_{k_1}\beta' & \text{in } D_1.\end{cases}$$

We now need to make the difference between the surface potentials defined on Σ or on Γ. To this end, we introduce the integral operators $S^{\Sigma,\Gamma}_{k_i}$, $K^{\Sigma,\Gamma}_{k_i}$, $K'^{\Sigma,\Gamma}_{k_i}$ and $T^{\Sigma,\Gamma}_{k_i}$ defined for regular densities φ on Σ by expressions (2) for all $x \in \Gamma$.

From the boundary conditions of (1) and the continuity of w through Σ, we get

$$\underbrace{\left[\begin{pmatrix} S^\Gamma_{k_0} & -K^\Gamma_{k_0}\\ -K'^\Gamma_{k_0} & T^\Gamma_{k_0}\end{pmatrix} - \begin{pmatrix} \mu S^\Gamma_{k_2} & -K^\Gamma_{k_2}\\ -K'^\Gamma_{k_2} & 1/\mu T^\Gamma_{k_2}\end{pmatrix}\right]}_{Z^\Gamma_{02}(k)}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$
$$+\underbrace{\begin{pmatrix} \mu S^{\Sigma,\Gamma}_{k_2} & -K^{\Sigma,\Gamma}_{k_2}\\ -K'^{\Sigma,\Gamma}_{k_2} & 1/\mu T^{\Sigma,\Gamma}_{k_2}\end{pmatrix}}_{Z^{\Sigma,\Gamma}(k)}\begin{pmatrix}\alpha'\\ \beta'\end{pmatrix} = 0$$

and

$$\underbrace{\left[\begin{pmatrix} \mu S^\Sigma_{k_1} & -K^\Sigma_{k_1}\\ -K'^\Sigma_{k_1} & 1/\mu T^\Sigma_{k_1}\end{pmatrix} + \begin{pmatrix} \mu S^\Sigma_{k_2} & -K^\Sigma_{k_2}\\ -K'^\Sigma_{k_2} & 1/\mu T^\Sigma_{k_2}\end{pmatrix}\right]}_{Z'^\Sigma_{12}(k)}\begin{pmatrix}\alpha'\\ \beta'\end{pmatrix}$$
$$-\underbrace{\begin{pmatrix} \mu S^{\Gamma,\Sigma}_{k_2} & -K^{\Gamma,\Sigma}_{k_2}\\ -K'^{\Gamma,\Sigma}_{k_2} & 1/\mu T^{\Gamma,\Sigma}_{k_2}\end{pmatrix}}_{Z^{\Gamma,\Sigma}(k)}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0.$$

One can remark that the matrix $Z'^{\Sigma}_{12}(k)$ corresponds to the transmission problem

$$(11)\qquad \begin{cases} \Delta w + k_1^2 w = 0 & \text{in } D_1 \\ \Delta v + k_2^2 v = 0 & \text{in } \mathbf{R}^d \backslash \overline{D_1} \\ w - v = h \in H^{1/2}(\Sigma) & \text{on } \Sigma \\ \frac{\partial w}{\partial \nu} - \frac{1}{\mu}\dfrac{\partial v}{\partial \nu} = g \in H^{-1/2}(\Sigma) & \text{on } \Sigma \\ \lim_{r\to\infty} r^{(d-1)/2}\left(\frac{\partial v}{\partial r} - \mathrm{i}kv\right) = 0, & \end{cases}$$

with $w \in H^1(D_1)$ and $v \in H^1_{loc}(\mathbf{R}^d \backslash D_1)$. This is a classical scattering problem that has a unique solution $(w, v) \in H^1(D_1) \times H^1_{loc}(\mathbf{R}^d \backslash D_1)$ (see [**24**]).

Now, since w satisfies the Helmholtz equation in D_1 with wave number k_1 and v is a radiating solution to the Helmholtz equation with wave number k_2, they have an integral representation of the form

$$w(x) = \mathrm{SL}^{\Sigma}_{k_1} \frac{\partial w}{\partial \nu}|_\Sigma(x) - \mathrm{DL}^{\Sigma}_{k_1} w|_\Sigma(x)$$
$$v(x) = -\mathrm{SL}^{\Sigma}_{k_2} \frac{\partial v}{\partial \nu}|_\Sigma(x) - \mathrm{DL}^{\Sigma}_{k_2} v|_\Sigma(x).$$

Using the boundary conditions satisfied by w and v and the jump properties of the potentials, we obtain equivalence between solving (11) and solving the following integral equation:

$$Z'^{\Sigma}_{12}(k) \begin{pmatrix} \frac{1}{\mu}\frac{\partial v}{\partial \nu}|_\Sigma \\ v|_\Sigma \end{pmatrix} = \begin{pmatrix} -\mu S_{k_1} & K_{k_1} + \frac{1}{2}I \\ (K'_{k_1} - \frac{1}{2}I) & -\frac{1}{\mu}T_{k_1} \end{pmatrix} \begin{pmatrix} \frac{1}{\mu}g \\ h \end{pmatrix},$$

where I denotes the identity operator on $H^{1/2}(\Sigma)$.

Consequently, the operator $Z'^{\Sigma}_{12}(k) : H^{-1/2}(\Sigma) \times H^{-1/2}(\Sigma) \to H^{1/2}(\Sigma) \times H^{-1/2}(\Sigma)$ is invertible, and we can rewrite the problem as

$$(12)\qquad \mathcal{Z}(k) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0,$$

where $\mathcal{Z}(k) := Z^{\Gamma}_{02}(k) + Z^{\Sigma,\Gamma}(k) Z'^{\Sigma}_{12}(k)^{-1} Z^{\Gamma,\Sigma}(k)$.

The matrix $Z^{\Gamma}_{02}(k)$ corresponds to the interior transmission problem for n constant equal to n_2 inside D. We will show that $\mathcal{Z}(k)$ is a compact perturbation of $Z^{\Gamma}_{02}(k)$.

The following theorem shows the equivalence between the interior transmission problem and the formulation with $\mathcal{Z}(k)$. Again, to get the equivalence, we need to add the condition that the far field generated by (α, β) vanishes.

Theorem 4.1. *The following two assertions are equivalent.*

(i) *There exists a $(w, v) \in H^1(D) \times H^1(D)$, a nontrivial solution to (1).*

(ii)) *There exist $\alpha \neq 0$ in $H^{-1/2}(\Gamma)$ and $\beta \neq 0$ in $H^{1/2}(\Gamma)$ such that $\mathcal{Z}(k) \left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right) = 0$ and*

$$P_0^\infty \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0.$$

Proof. Let $\alpha \neq 0$ in $H^{-1/2}(\Gamma)$ and $\beta \neq 0$ in $H^{1/2}(\Gamma)$ be such that $\mathcal{Z}(k) \left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right) = 0$. We set

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} := Z_{12}'^{\Sigma}(k)^{-1} Z^{\Gamma,\Sigma}(k) \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Let us define $v = \mathrm{SL}_{k_0}^{\Gamma} \alpha - \mathrm{DL}_{k_0}^{\Gamma} \beta$ in D and

$$w = \begin{cases} \mathrm{SL}_{k_2}^{\Gamma} \alpha - \mathrm{DL}_{k_2}^{\Gamma} \beta - \mathrm{SL}_{k_2}^{\Sigma} \alpha' + \mathrm{DL}_{k_2}^{\Sigma} \beta' & \text{in } D_2, \\ \mathrm{SL}_{k_1}^{\Sigma} \alpha' - \mathrm{DL}_{k_1}^{\Sigma} \beta' & \text{in } D_1. \end{cases}$$

The regularity of the single and double layer potentials shows that $v \in H^1(D)$ and $w \in H^1(D_1) \cap H^1(D_2)$. Furthermore,

$$\begin{aligned} w|_{\Sigma}^{+} &= S_{k_2}^{\Gamma,\Sigma} \alpha - K_{k_2}^{\Gamma,\Sigma} \beta - S_{k_2}^{\Sigma} \alpha' + K_{k_2}^{\Sigma} \beta' + \frac{1}{2} \beta', \\ w|_{\Sigma}^{-} &= S_{k_1}^{\Sigma} \alpha' - K_{k_1}^{\Sigma} \beta' + \frac{1}{2} \beta', \\ \frac{\partial w}{\partial \nu}|_{\Sigma}^{+} &= K_{k_2}'^{\Gamma,\Sigma} \alpha - T_{k_2}^{\Gamma,\Sigma} \beta - K_{k_2}'^{\Sigma} \alpha' + \frac{1}{2} \alpha' + T_{k_2}^{\Sigma} \beta', \\ \frac{\partial w}{\partial \nu}|_{\Sigma}^{-} &= K_{k_1}'^{\Sigma} \alpha' + \frac{1}{2} \alpha' - T_{k_1}^{\Sigma} \beta', \end{aligned}$$

and then

$$\begin{pmatrix} w|_\Sigma^+ - w|_\Sigma^- \\ \frac{\partial w}{\partial \nu}|_\Sigma^+ - \frac{\partial w}{\partial \nu}|_\Sigma^- \end{pmatrix} = Z^{\Gamma,\Sigma}(k)\begin{pmatrix}\beta \\ \alpha\end{pmatrix} - Z'^{\Sigma}_{12}(k)\begin{pmatrix}\beta' \\ \alpha'\end{pmatrix} = 0,$$

by definition of α' and β'. We deduce that $w \in H^1(D)$.

Now we must show that $v \neq 0$ or $w \neq 0$. Assume that $P_0^\infty \binom{\alpha}{\beta} = 0$. From Rellich's lemma, we deduce that $v = 0$ in $\mathbf{R}^d \backslash D$. Assume that $v = 0$ also in D. We have, in particular, that $[v]_\Gamma = [\partial v/\partial \nu]_\Gamma = 0$ and, from the jump properties of the single and double layer potentials, we also have that $[v]_\Gamma = -\beta$ and $[\partial v/\partial \nu]_\Gamma = -\alpha$. This contradicts the fact that $(\alpha, \beta) \neq (0,0)$. Then $v \neq 0$ in D and, as a consequence, we also have that $w \neq 0$ in D. $\square$

Theorem 4.2. *The operator $\mathcal{Z}(k) : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is Fredholm of index zero and analytic on $k \in \mathbf{C}\backslash\mathbf{R}^-$.*

Proof. From Theorem 2.3, the operator $Z^{\Gamma}_{02}(k)$ is Fredholm from $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ into $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$. Finally, the operators $Z^{\Sigma,\Gamma}(k) : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Sigma) \times H^{-1/2}(\Sigma)$ and $Z^{\Gamma,\Sigma}(k) : H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ are compact due to the regularity of the kernels and $Z'^{\Sigma}_{12}(k)^{-1} : H^{1/2}(\Sigma) \times H^{-1/2}(\Sigma) \to H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)$ is continuous. This shows that $Z^{\Sigma,\Gamma}(k) Z'^{\Sigma}_{12}(k)^{-1} Z^{\Gamma,\Sigma}(k) : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is compact. Consequently, the operator $\mathcal{Z}(k) : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is Fredholm. $\square$

4.1.2. Discreteness of the set of transmission eigenvalues.

Lemma 4.3. *Assume that $\mu - 1$ and $1 - n_2$ are both either positive or negative. Then the operator $\mathcal{Z}(\mathrm{i}k) : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is injective for $k \in \mathbf{R}^{*+}$.*

Proof. Assume that $\mathcal{Z}(\mathrm{i}k)\binom{\alpha}{\beta} = 0$. Let us define $v := \mathrm{SL}^{\Gamma}_{\mathrm{i}k_0}\alpha - \mathrm{DL}^{\Gamma}_{\mathrm{i}k_0}\beta$ in $\mathbf{R}^d \setminus \Gamma$ and

$$w := \begin{cases} \mu \mathrm{SL}^{\Gamma}_{\mathrm{i}k_2}\alpha - \mathrm{DL}^{\Gamma}_{\mathrm{i}k_2}\beta - \mu\mathrm{SL}^{\Sigma}_{\mathrm{i}k_2}\alpha' + \mathrm{DL}^{\Sigma}_{\mathrm{i}k_2}\beta' & \text{in } \mathbf{R}^d \setminus (\overline{D}_1 \cup \Gamma) \\ \mu\mathrm{SL}^{\Sigma}_{\mathrm{i}k_1}\alpha' - \mathrm{DL}^{\Sigma}_{\mathrm{i}k_1}\beta' & \text{in } D_1, \end{cases}$$

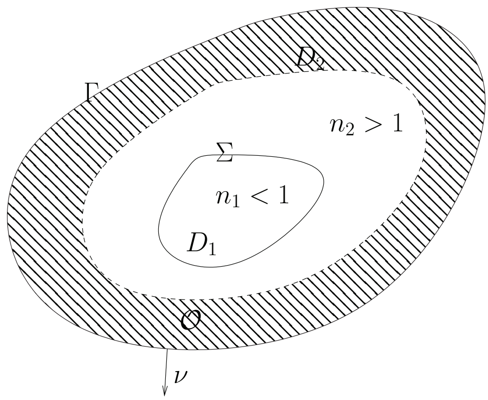

FIGURE 3. Illustration of the cutoff domain $\mathcal{O}$.

where $\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} := Z_{12}'^{\Sigma}(k)^{-1} Z^{\Gamma,\Sigma}(k) \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. The relation $\mathcal{Z}(\mathrm{i}k)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$ implies that, on Γ, we have $w^\pm = v^\pm$ and $(1/\mu)[(\partial w^\pm)/(\partial \nu)] = \partial v^\pm/\partial \nu$. Consequently, the pair $(w,v) \in (H^1(\mathbf{R}^d \setminus \overline{D}) \cup H^1(D)) \times (H^1(\mathbf{R}^d \setminus \overline{D}) \cup H^1(D))$ is a solution to

$$\begin{cases} \Delta v - k^2 v = 0 & \text{in } \mathbf{R}^d \setminus \Gamma \\ \nabla \cdot \frac{1}{\mu} \nabla w - k^2 n_2 w = 0 & \text{in } \mathbf{R}^d \setminus (\overline{D_1} \cup \Gamma) \\ \nabla \cdot \frac{1}{\mu} \nabla w - k^2 n_1 w = 0 & \text{in } D_1 \\ w^\pm = v^\pm & \text{on } \Gamma \\ \frac{1}{\mu}\frac{\partial w^\pm}{\partial \nu} = \frac{\partial v^\pm}{\partial \nu} & \text{on } \Gamma. \end{cases}$$

Let us define the Hilbert space

$$\mathbf{H} := \big\{ (w,v) \in \big(H^1(\mathbf{R}^d \setminus \overline{D}) \cup H^1(D)\big) \\ \times \big(H^1(\mathbf{R}^d \setminus \overline{D}) \cup H^1(D)\big) \mid w^\pm = v^\pm \text{ on } \Gamma \big\}.$$

Let $(\varphi, \psi) \in \mathbf{H}$. Multiplying the equation satisfied by v by ψ and the equations satisfied by w by φ, integrating by parts on both sides of Γ and using the jump relations on γ, we get the following variational formulation: find (w,v) in $\mathbf{H}$ such that $a_k((w,v),(\varphi,\psi)) = 0$ for all

(φ, ψ) in $\mathbf{H}$, where

$$a_k((w,v),(\varphi,\psi)) := \int_{\mathbf{R}^d\setminus\Gamma}\left(\frac{1}{\mu}\nabla w\cdot\nabla\overline{\varphi}+k^2nw\overline{\varphi}\right)dx \\ -\int_{\mathbf{R}^d\setminus\Gamma}\left(\nabla v\cdot\nabla\overline{\psi}+k^2v\overline{\psi}\right)\,dx.$$

Let $\mathcal{O}$ be a neighborhood of Γ such that $\mathcal{O}\cap D_1=\varnothing$ (see Figure 3).

Let us define the cutoff function χ with compact support in $\mathcal{O}$ such that $\chi=1$ on Γ. There exists a $C>0$ such that $||\nabla\chi||_\infty\le C$.

1. Case where $\mu<1$ and $n_2>1$. Let us define the isomorphism

$$\begin{aligned} T:\mathbf{H}\times\mathbf{H} &\longrightarrow \mathbf{H}\times\mathbf{H} \\ (w,v) &\longmapsto (w,-v+2\chi w). \end{aligned}$$

$$\begin{aligned} a_k((w,v),T(w,v)) &= \int_{\mathbf{R}^d\setminus\Gamma}\left(\frac{1}{\mu}|\nabla w|^2+k^2n|w|^2+|\nabla v|^2+k^2|v|^2\right)dx \\ &\quad -2\int_\Omega\nabla v\cdot\nabla(\chi\overline{w})-2k^2\int_\Omega\chi v\overline{w} \\ &\ge \frac{1}{\mu}||\nabla w||^2_{L^2(\mathbf{R}^d)}+k^2n_1||w||^2_{L^2(D_1)} \\ &\quad +k^2n_2||w||^2_{L^2(\mathbf{R}^d\setminus\overline{D}_1)}+||\nabla v||^2_{L^2(\mathbf{R}^d)} \\ &\quad +k^2||v||^2_{L^2(\mathbf{R}^d)}-\gamma||\nabla v||^2_{L^2(\mathbf{R}^d)} \\ &\quad -\frac{1}{\gamma}||\nabla w||^2_{L^2(\mathbf{R}^d)}-C\eta||\nabla v||^2_{L^2(\mathbf{R}^d)} \\ &\quad -\frac{C}{\eta}||w||^2_{L^2(\mathbf{R}^d\setminus\overline{D}_1)}-k^2\delta||v||^2_{L^2(\mathbf{R}^d)} \\ &\quad -\frac{k^2}{\delta}||w||^2_{L^2(\mathbf{R}^d\setminus\overline{D}_1)} \\ &= (1-\gamma-C\eta)\,||\nabla v||^2_{L^2(\mathbf{R}^d)} \\ &\quad +k^2(1-\delta)||v||^2_{L^2(\mathbf{R}^d)}+\left(\frac{1}{\mu}-\frac{1}{\gamma}\right)||\nabla w||^2_{L^2(\mathbf{R}^d)} \\ &\quad +k^2n_1||w||^2_{L^2(D_1)}+\left(k^2\left(n_2-\frac{1}{\delta}\right)-\frac{C}{\eta}\right)||w||^2_{L^2(\mathbf{R}^d\setminus\overline{D}_1)}. \end{aligned}$$

Let $\mu < \gamma < 1$, $1/n_2 < \delta < 1$ and η be such that $1 - \gamma - C\eta > 0$ is fixed. Then, if k is large enough to have $k^2(n_2 - (1/\delta) - (C/\eta) > 0$, we deduce that a_k is T-coercive. As a consequence, $w = 0$ and $v = 0$ are the only solutions. From the equality $[v]_\Gamma = -\beta$ and $[\partial v/\partial \nu]_\Gamma = -\alpha$, we get that $\alpha = \beta = 0$ and finally $\mathcal{Z}(\mathrm{i}k) : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is injective.

2. Case where $\mu > 1$ and $n_2 < 1$. Let us define the isomorphism

$$\begin{aligned} T : \mathbf{H} \times \mathbf{H} &\longrightarrow \mathbf{H} \times \mathbf{H} \\ (w, v) &\longmapsto (w - 2\chi v, -v). \end{aligned}$$

$$\begin{aligned}
a_k((w,v),T(w,v)) &= \int_{\mathbf{R}^d} \left(|\nabla v|^2 + k^2|v|^2 + \frac{1}{\mu}|\nabla w|^2 + k^2 n|w|^2 \right) dx \\
&\quad - 2\int_{\mathcal{O}} \frac{1}{\mu}\nabla w \cdot \nabla(\chi\overline{v}) - 2k^2 \int_{\mathcal{O}} n\chi w\overline{v} \\
&\geq ||\nabla v||^2_{L^2(\mathbf{R}^d)} + k^2||v||^2_{L^2(\mathbf{R}^d)} \\
&\quad + \mu\left\|\frac{1}{\mu}\nabla w\right\|^2_{L^2(\mathbf{R}^d)} + k^2 n_1||w||^2_{L^2(D_1)} \\
&\quad - \frac{1}{\gamma}||\nabla v||^2_{L^2(\mathbf{R}^d)} - \gamma\left\|\frac{1}{\mu}\nabla w\right\|^2_{L^2(\mathbf{R}^d)} \\
&\quad - \frac{C}{\eta}||v||^2_{L^2(\mathbf{R}^d)} - \delta C\left\|\frac{1}{\mu}\nabla w\right\|^2_{L^2(\mathbf{R}^d\setminus\overline{D}_1)} \\
&\quad - \frac{k^2}{\eta}||v||^2_{L^2(\mathbf{R}^d)} - k^2\eta||n_2 w||^2_{L^2(\mathbf{R}^d\setminus\overline{D}_1)} \\
&\quad + \frac{k^2}{n_2}||n_2 w||^2_{L^2(\mathbf{R}^d\setminus\overline{D}_1)} \\
&= \left(1 - \frac{1}{\gamma}\right)||\nabla v||^2_{L^2(\mathbf{R}^d)} \\
&\quad + \left(k^2\left(1 - \frac{1}{\eta}\right) - \frac{C}{\delta}\right)||v||^2_{L^2(\mathbf{R}^d)} + k^2 n_1||w||^2_{L^2(D_1)} \\
&\quad + (\mu - \gamma - \delta C)\,||\frac{1}{\mu}\nabla w||^2_{L^2(\mathbf{R}^d)} \\
&\quad + k^2\left(\frac{1}{n_2} - \eta\right)||n_2 w||^2_{L^2(\mathbf{R}^d\setminus\overline{D}_1)}.
\end{aligned}$$

Let $1 < \gamma < \mu$, $1 < \eta < 1/n_2$ and δ be such that $\mu - \gamma - \delta C > 0$ fixed. Then, for k large enough to have $k^2(1 - (1/\eta)) - (C/\delta) > 0$, a_k is T-coercive. Similarly we deduce that $\mathcal{Z}(\mathrm{i}k) : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is injective. $\square$

Theorem 4.4. *Assume that $\mu - 1$ and $1 - n_2$ are both either positive or negative. Then, the set of transmission eigenvalues is discrete.*

Proof. This is a direct consequence of Theorem 4.2 and Lemma 4.3 using the analytic Fredholm theory. $\square$

4.2. The inhomogeneous case. The results derived in the previous subsection for the case of piecewise constant coefficients can be easily generalized to the case where μ and n are constant in a neighborhood $O \subset D$ of the boundary Γ. In this case one can consider Σ to be a regular surface lying in O so that the region between Σ and Γ is connected (for instance, in the case of regular boundary Γ, one can choose $\Sigma = \Gamma - \delta\nu$ where δ is a sufficiently small parameter). The previous analysis then holds true if one replaces $\Phi_{k_1}(\cdot, y)$ with the fundamental solution $\mathbf{G}(\cdot, y) \in H^1_{loc}(\mathbf{R}^d) \setminus \{y\}$ of

$$\nabla \cdot \frac{1}{\mu}\nabla \mathbf{G}(\cdot, y) + k^2 n \mathbf{G}(\cdot, y) = -\delta_y \quad \text{in } \mathbf{R}^d,$$

in the distributional sense and satisfying the Sommerfeld radiation condition (we extend μ and n outside D by their constant values in O). Since, for all $y \in \mathbf{R}^d$,

$$x \longmapsto \mathbf{G}(x, y) - \Phi_{k_2}(x, y)$$

satisfies the Helmholtz equation (with constant coefficients) in O, then this function is a C^∞ function in O. By symmetry, the same holds for $y \mapsto \mathbf{G}(x, y) - \Phi_{k_2}(x, y)$. Therefore, the (previously introduced) potentials defined on Σ with $\mathbf{G}$ replacing Φ_{k_1} keep exactly the same mapping properties. The invertibility of $Z'^{\Sigma}_{12}(k)$ is ensured as long as the forward scattering problem associated with μ and n is well posed. The latter is, for instance, true if one assumes n and μ to be bounded functions with nonnegative imaginary parts and positive definite real parts. Finally, the proof of injectivity of $\mathcal{Z}(ik)$ for positive k can be reproduced, with minor obvious modifications, in the current setting.

We can therefore state the following theorem:

Theorem 4.5. *Assume that n and μ are bounded functions with nonnegative imaginary parts and positive definite real parts, and further assume that $\mu - 1$ and $1 - n$ are either positive or negative constants in a neighborhood of Γ. Then, the set of transmission eigenvalues is discrete.*

4.3. The case of $\mu = 1$. Indeed our analysis for the case $\mu \neq 1$ extends to the case $\mu = 1$ when one uses the appropriate function spaces. For instance, following exactly the same procedure as in the previous section and applying the analysis done for the case $\mu = 1$ and $n = cte \neq 1$ in the neighborhood of Γ, then with

$$\mathcal{Z}(k) := Z_{02}^{\Gamma}(k) + Z^{\Sigma,\Gamma}(k) Z_{12}^{\prime\Sigma}(k)^{-1} Z^{\Gamma,\Sigma}(k)$$

we have:

Lemma 4.6. *Assume that $\mu = 1$ and n is a bounded function with nonnegative imaginary part, and further assume that $1 - n$ is either a positive or negative constant in a neighborhood of Γ. Then, the operator $\mathcal{Z}(k) : H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ is Fredholm and analytic on $k \in \mathbf{C} \setminus \mathbf{R}^-$.*

The only missing point here is to prove injectivity when $n - 1$ changes sign. We refer to [**25**] where injectivity is proved for purely imaginary wavenumbers with large enough modulus.

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