JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 25, Number 1, Spring 2013

A REAL ANALYTICITY RESULT FOR A NONLINEAR INTEGRAL OPERATOR

MASSIMO LANZA DE CRISTOFORIS AND PAOLO MUSOLINO

Communicated by Jürgen Appell

ABSTRACT. We consider a nonlinear integral operator which involves a Nemytskij type operator and which appears in the applications as a pull-back of layer potential operators. We prove an analyticity result in Schauder spaces by splitting the operator into the composition of a nonlinear operator acting into Roumieu classes and a composition operator.

1. Introduction. In this paper, we consider integral operators of the form

$$H^\sharp_G[\psi,\phi,z,f](t)=\int_Y G(\psi(t),\phi(y),z)f(y)\,d\mu_y \quad \text{for all } t\in\mathbf{M}, \tag{1.1}$$

where $\mathbf{M}$, Y are compact manifolds imbedded into $\mathbf{R}^n$, and z ranges in a Banach space of parameters $\mathcal{K}$, ψ and ϕ are functions in the Schauder spaces $C^{m,\alpha}(\mathbf{M},\mathbf{R}^{h_1})$ and $C^{m,\alpha}(Y,\mathbf{R}^{h_2})$ for some $m\in\mathbf{N}$ and $\alpha\in\,]0,1]$, respectively, $f\in L^1(Y)$, and G a real analytic function defined on an open subset of $\mathbf{R}^{h_1}\times\mathbf{R}^{h_2}\times\mathcal{K}$.

Typically, integrals as (1.1) appear in the analysis of pull-backs of layer potentials, where f plays the role of moment or density of the layer potential, and z may be a parameter as the wave number when dealing with acoustic potentials (cf., e.g., [**3**, Lemma 5.1], [**9**, Appendix B], [**11**], [**12**, Section 5], [**13**, Lemma 3.9].)

Here the question is whether the (nonlinear) map $H^\sharp_G$ from the set, say $\mathcal{G}$, of (ψ,ϕ,z,f) in $C^{m,\alpha}(\mathbf{M},\mathbf{R}^{h_1})\times C^{m,\alpha}(Y,\mathbf{R}^{h_2})\times\mathcal{K}\times L^1(Y)$ for which the composition with G in (1.1) makes sense to the space $C^{m,\alpha}(\mathbf{M})$ is real analytic.

2010 AMS *Mathematics subject classification.* Primary 47H30.
Keywords and phrases. Analyticity, integral operators, superposition operators, Roumieu classes, Schauder spaces.
The second named author acknowledges the financial support of the 'Fondazione Gini' of Padova.
Received by the editors on August 9, 2011, and in revised form on April 2, 2012.
DOI:10.1216/JIE-2013-25-1-21

Pull-backs of layer potentials appear in the analysis of a large number of linear or nonlinear boundary value problems when a variation of the domain is involved. So, if we need to perform a local analysis of such problems, we need to understand the regularity of operators such as (1.1). Here we are showing the real analyticity, which implies in particular Frechét differentiability of all orders, a property which enables us to invoke basic theorems such as the Implicit Function theorem in Banach spaces or a variety of bifurcation results. Here we understand that a real analytic operator is an infinitely many times Frechét differentiable map which admits around each point a normal convergent Taylor expansion. For the definition and properties of analytic operators, we refer to Prodi and Ambrosetti [**17**, page 89] or Deimling [**4**, page 150].

One way to solve the problem in case $\mathcal{K}$ is finite dimensional and $\mathbf{M}$ and Y are smooth enough, would be to identify a triple $(\psi, \phi, z) \in C^{m,\alpha}(\mathbf{M}, \mathbf{R}^{h_1}) \times C^{m,\alpha}(Y, \mathbf{R}^{h_2}) \times \mathcal{K}$ with the map in $C^{m,\alpha}(\mathbf{M} \times Y, \mathbf{R}^{h_1} \times \mathbf{R}^{h_2} \times \mathcal{K})$ which takes a pair $(x, y) \in \mathbf{M} \times Y$ to the triple $(\psi(x), \phi(y), z)$, and then to deduce the analyticity of $H_G^\sharp$ by the analyticity of the composition operator from $C^{m,\alpha}(\mathbf{M} \times Y, \mathbf{R}^{h_1} \times \mathbf{R}^{h_2} \times \mathcal{K})$ to $C^{m,\alpha}(\mathbf{M} \times Y)$ which takes Ξ to $G(\Xi)$ and of the bilinear and continuous map from $C^{m,\alpha}(\mathbf{M} \times Y) \times L^1(Y)$ to $C^{m,\alpha}(\mathbf{M})$ which takes a pair (g, f) to $\int_Y g(x, y) f(y)\, d\mu_y$ (cf. [**14**, Lemma 4.8].) The analyticity of the above composition operator follows by the known analyticity of composition operators generated by analytic functions and acting in $C^{m,\alpha}(\mathbf{M} \times Y, \mathbf{R}^{h_1} \times \mathbf{R}^{h_2} \times \mathcal{K})$ (cf., Böhme and Tomi [**1**, page 10], Henry [**6**, page 29], Valent [**20**, Theorem 5.2, page 44]). Such an approach, however, carries the inconvenience of requiring ϕ to be of class $C^{m,\alpha}$, of requiring $\mathcal{K}$ to be finite dimensional and of requiring Y to be a compact manifold with a certain degree of smoothness. In this paper, we consider an approach that considerably reduces the regularity to be imposed on ϕ and which allows $\mathcal{K}$ to be infinite dimensional and Y to be a topological space with a measure μ defined on a σ-algebra containing the Borel sets of Y. In particular, ϕ is now required to be an element of the space $C_b^0(Y, \mathbf{R}^{h_2})$ of bounded continuous functions from Y to $\mathbf{R}^{h_2}$. First, we consider the integral operator

$$H_G[\phi, z, f](x) = \int_Y G(x, \phi(y), z) f(y)\, d\mu_y \qquad (1.2)$$
$$\text{for all } x \in \operatorname{cl}\Omega,$$

where Ω is a certain open bounded subset of $\mathbf{R}^{h_1}$ such that the composition with the function G in (1.2) makes sense, and we note that

$$H_G^\sharp[\psi, \phi, z, f](t) = H_G[\phi, z, f] \circ \psi(t) \quad \text{for all } t \in \mathbf{M}, \tag{1.3}$$

provided that $\psi(\mathbf{M}) \subseteq \Omega$. Then we can deduce our analyticity results by combining the following two ingredients.

(a) Analyticity results for the composition '$\circ$.'

(b) Analyticity results for $H_G[\cdot, \cdot, \cdot]$.

We first consider (a). We want to consider the composition operator which takes a pair (ζ, ψ) to the composite map $\zeta \circ \psi$ (so that then we can take $\zeta = H_G[\phi, z, f]$ as in (1.3)). If we expect the composition '$\circ$' to be analytic with both ψ and $\zeta \circ \psi$ in a Schauder space, then the map which takes ψ to $\zeta \circ \psi$ should be analytic in a Schauder space. Then one can easily see that ζ must be analytic. Then Preciso [**15, 16**] has shown that an effective choice of the function space for ζ in order that the composition '$\circ$' be analytic with both ψ and $\zeta \circ \psi$ in a Schauder space is a Roumieu class $C^0_{\omega,\rho}(\mathrm{cl}\,\Omega)$ of analytic functions defined in $\mathrm{cl}\,\Omega$ (see (2.1)).

Then we show that the nonlinear map $H_G[\cdot, \cdot, \cdot]$ which takes a triple (ϕ, z, f) in $C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K} \times L^1(Y)$ for which the composition with G in (1.2) makes sense to the function $H_G[\phi, z, f](\cdot)$ of $C^0_{\omega,\rho}(\mathrm{cl}\,\Omega)$ defined by the right hand side of (1.2) is real analytic for a suitable choice of $\rho > 0$ (see Theorem 3.1). Then, by combining such a result with the above-mentioned results of Preciso, we deduce the analyticity of $H_G^\sharp$ from the set of (ψ, ϕ, z, f) of $C^{m,\alpha}(\mathbf{M}, \mathbf{R}^{h_1}) \times C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K} \times L^1(Y)$ for which the composition with G in (1.1) makes sense to the space $C^{m,\alpha}(\mathbf{M})$ (see Proposition 4.1).

We note that the result for H_G we have mentioned above considerably generalizes a corresponding result where the target space of H_G is $C^r(\mathrm{cl}\,\Omega)$ for $r \in \mathbf{N}$, which has found several applications in the references mentioned at the beginning of the introduction, and which we now deduce as a Corollary (see Corollary 3.14).

This paper is organized as follows. In Section 2, we introduce some notation. In Section 3, we introduce our main result on H_G. Then, in Section 4, we introduce our main result on $H_G^\sharp$. In the Appendix we present a result of Preciso.

2. Notation. We denote the norm on a normed space $\mathcal{X}$ by $\|\cdot\|_{\mathcal{X}}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. We endow the space $\mathcal{X} \times \mathcal{Y}$ with the norm defined by $\|(x,y)\|_{\mathcal{X}\times\mathcal{Y}} \equiv \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$ for all $(x,y) \in \mathcal{X} \times \mathcal{Y}$, while we use the Euclidean norm for $\mathbf{R}^n$. If $x \in \mathcal{X}$ and $R > 0$, we denote by $\mathbf{B}_{\mathcal{X}}(x,R)$ the ball $\{y \in \mathcal{X} : \|x-y\|_{\mathcal{X}} < R\}$. For standard definitions of calculus in normed spaces, we refer to Cartan [**2**]. The symbol $\mathbf{N}$ denotes the set of natural numbers including 0. If $k \in \mathbf{N} \setminus \{0\}$, and if $\mathcal{X}_1, \dots, \mathcal{X}_k, \mathcal{Y}$ are normed spaces, then we denote by $\mathcal{L}^{(k)}(\mathcal{X}_1 \times \cdots \times \mathcal{X}_k, \mathcal{Y})$ the space of k-multilinear and continuous maps from $\mathcal{X}_1 \times \cdots \times \mathcal{X}_k$ to $\mathcal{Y}$, endowed with its usual norm. Let $n \in \mathbf{N} \setminus \{0\}$. Let $\mathbf{D} \subseteq \mathbf{R}^n$. Then $\mathrm{cl}\,\mathbf{D}$ denotes the closure of $\mathbf{D}$ and $\partial\mathbf{D}$ denotes the boundary of $\mathbf{D}$. For all $R > 0$, $x \in \mathbf{R}^n$, x_j denotes the jth coordinate of x, $|x|$ denotes the Euclidean modulus of x in $\mathbf{R}^n$, and $\mathbf{B}_n(x,R)$ denotes the ball $\{y \in \mathbf{R}^n : |x-y| < R\}$. Let Ω be an open subset of $\mathbf{R}^n$. The space of m times continuously differentiable real-valued functions on Ω is denoted by $C^m(\Omega, \mathbf{R})$, or more simply by $C^m(\Omega)$. Let $r \in \mathbf{N} \setminus \{0\}$. Let $f \in (C^m(\Omega))^r$. The sth component of f is denoted f_s, and Df denotes the Jacobian matrix $(\partial f_s/\partial x_l)_{s=1,\dots,r,\ l=1,\dots,n}$. Let $\eta \equiv (\eta_1, \dots, \eta_n) \in \mathbf{N}^n$, $|\eta| \equiv \eta_1 + \cdots + \eta_n$. Then $D^\eta f$ denotes $\partial^{|\eta|} f/(\partial x_1^{\eta_1} \cdots \partial x_n^{\eta_n})$. The subspace of $C^m(\Omega)$ of those functions f whose derivatives $D^\eta f$ of order $|\eta| \leq m$ can be extended with continuity to $\mathrm{cl}\,\Omega$ is denoted $C^m(\mathrm{cl}\,\Omega)$. The subspace of $C^m(\mathrm{cl}\,\Omega)$ whose functions have mth order derivatives that are Hölder continuous with exponent $\alpha \in]0,1]$ is denoted $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ (cf., e.g., Gilbarg and Trudinger [**5**]). Let $\mathbf{D} \subseteq \mathbf{R}^r$. Then $C^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbf{D})$ denotes $\{f \in (C^{m,\alpha}(\mathrm{cl}\,\Omega))^r : f(\mathrm{cl}\,\Omega) \subseteq \mathbf{D}\}$. If Y is a topological space, then $C^0_b(Y, \mathbf{R}^r)$ denotes the space of bounded continuous functions from Y to $\mathbf{R}^r$, endowed with the norm defined by $\|f\|_{C^0_b(Y,\mathbf{R}^r)} \equiv \sup_{y\in Y} |f(y)|$, for all $f \in C^0_b(Y, \mathbf{R}^r)$.

Now let Ω be a bounded open subset of $\mathbf{R}^n$. Then $C^m(\mathrm{cl}\,\Omega)$ and $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ are endowed with their usual norm and are well known to be Banach spaces (cf., e.g., Troianiello [**19**, subsection 1.2.1]). We say that a bounded open subset Ω of $\mathbf{R}^n$ is of class C^m or of class $C^{m,\alpha}$, if it is a manifold with boundary imbedded in $\mathbf{R}^n$ of class C^m or $C^{m,\alpha}$, respectively (cf., e.g., Gilbarg and Trudinger [**5**, subsection 6.2]). For standard properties of functions in Schauder spaces, we refer the reader to Gilbarg and Trudinger [**5**] and to Troianiello [**19**] (see also [**7**, Section 2, Lemma 3.1, 4.26, Theorem 4.28], [**13**, Section 2]).

If $\mathbf{M}$ is a manifold imbedded in $\mathbf{R}^n$ of class $C^{m,\alpha}$, with $m \geq 1$, $\alpha \in]0,1]$, one can define the Schauder spaces also on $\mathbf{M}$ by exploiting the local parametrizations. In particular, if Ω is a bounded open set of class $C^{m,\alpha}$ then one can consider the spaces $C^{k,\alpha}(\partial\Omega)$ on $\partial\Omega$ for $0 \leq k \leq m$.

If Y is a set, $\mathcal{M}$ a σ-algebra of parts of Y and μ a measure on $\mathcal{M}$, we retain the standard notation for the space $L^1(Y)$ of (equivalence classes of) μ-integrable functions.

We note that, throughout the paper, "analytic" means "real analytic." For the definition and properties of analytic operators, we refer to Prodi and Ambrosetti [**17**, page 89] or Deimling [**4**, page 150].

For all bounded open subsets Ω of $\mathbf{R}^n$ and $\rho > 0$, we set

$$C^0_{\omega,\rho}(\mathrm{cl}\,\Omega) \equiv \left\{ u \in C^\infty(\mathrm{cl}\,\Omega) \colon \sup_{\beta\in\mathbf{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\mathrm{cl}\,\Omega)} < +\infty \right\}, \tag{2.1}$$

and

$$\|u\|_{C^0_{\omega,\rho}(\mathrm{cl}\,\Omega)} \equiv \sup_{\beta\in\mathbf{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\mathrm{cl}\,\Omega)}, \quad \text{for all } u \in C^0_{\omega,\rho}(\mathrm{cl}\,\Omega),$$

where $|\beta| \equiv \beta_1 + \cdots + \beta_n$, for all $\beta \equiv (\beta_1, \ldots, \beta_n) \in \mathbf{N}^n$. Here the letter ω indicates that we have a space of analytic functions. As pointed out by Roumieu himself, the Roumieu class $(C^0_{\omega,\rho}(\mathrm{cl}\,\Omega), \|\cdot\|_{C^0_{\omega,\rho}(\mathrm{cl}\,\Omega)})$ is a Banach space (cf. [**18**]).

For every bounded open connected subset Ω of $\mathbf{R}^n$, we set

$$c[\Omega] \equiv \sup\left\{ \frac{\lambda(x,y)}{|x-y|} \colon x, y \in \Omega, x \neq y \right\},$$

where

$$\lambda(x,y) \equiv \inf\left\{ \int_0^1 |\zeta'(s)|\, ds \colon \zeta \in C^1([0,1],\Omega),\ \zeta(0) = x,\ \zeta(1) = y \right\}.$$

If $c[\Omega] < +\infty$, then Ω is said to be regular in the sense of Whitney. It is well known that if Ω is a bounded open connected subset of $\mathbf{R}^n$ of class C^1, then $c[\Omega] < +\infty$.

Let $\mathcal{K}$ be a Banach space. Let $h_1, h_2 \in \mathbf{N} \setminus \{0\}$. Let $\mathcal{W}$ be an open subset of $\mathbf{R}^{h_1} \times \mathbf{R}^{h_2} \times \mathcal{K}$. If $k \in \mathbf{N}$ and F is a function defined on $\mathcal{W}$, we denote by $\partial^k_{(\eta,z)} F(x,\eta,z)$ the kth order partial differential of $F(x,\eta,z)$ with respect to the second and third arguments. Similarly, if $\alpha \in \mathbf{N}^{h_1}$, we denote by $D^\alpha_x F(x,\eta,z)$ the partial derivative of multi-index α of $F(x,\eta,z)$ with respect to the first argument.

3. Analyticity of an integral operator in Roumieu classes. In order to write the formulas in a concise way, we choose to put a '$\hat{\ }$' symbol on a term which we wish to suppress. So, for example, if $1 \leq l \leq s$,

$$(\xi_1, \dots, \widehat{\xi_l}, \dots, \xi_s) \quad \text{denotes} \quad (\xi_1, \dots, \xi_{l-1}, \xi_{l+1}, \dots, \xi_s).$$

Then we have the following.

Theorem 3.1. *Let $h_1, h_2 \in \mathbf{N} \setminus \{0\}$. Let Ω be a bounded open subset of $\mathbf{R}^{h_1}$. Let Y be a topological space. Let $\mathcal{M}$ be a σ-algebra of parts of Y containing the Borel sets of Y. Let μ be measure on $\mathcal{M}$. Let $\mathcal{K}$ be a Banach space. Let $\mathcal{W}$ be an open subset of $\mathbf{R}^{h_1} \times \mathbf{R}^{h_2} \times \mathcal{K}$ such that there exists $(\eta^*, z^*) \in \mathbf{R}^{h_2} \times \mathcal{K}$ such that*

$$\mathrm{cl}\, \Omega \times \{\eta^*\} \times \{z^*\} \subseteq \mathcal{W}.$$

Let

$$\mathcal{F} \equiv \Big\{ (\phi, z) \in C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K} \colon \mathrm{cl}\, \Omega \times \mathrm{cl}\, \phi(Y) \times \{z\} \subseteq \mathcal{W} \Big\}.$$

Let G be a real analytic map from $\mathcal{W}$ to $\mathbf{R}$. Let H_G be the map from $\mathcal{F} \times L^1(Y)$ to $C^0(\mathrm{cl}\, \Omega)$ defined by

$$H_G[\phi, z, f](x) \equiv \int_Y G(x, \phi(y), z) f(y)\, d\mu_y \quad \text{for all } x \in \mathrm{cl}\, \Omega,$$

for all $(\phi, z, f) \in \mathcal{F} \times L^1(Y)$. Then, for each pair $(\phi_0, z_0) \in \mathcal{F}$, there exist an open neighborhood $\mathcal{U}_0$ of (ϕ_0, z_0) in $\mathcal{F}$ and $\rho_0 \in]0, +\infty[$, such that H_G is real analytic from $\mathcal{U}_0 \times L^1(Y)$ to $C^0_{\omega,\rho}(\mathrm{cl}\, \Omega)$ for all $\rho \in]0, \rho_0[$.

Moreover, if $k \in \mathbf{N} \setminus \{0\}$, then the kth order differential of H_G at $(\phi, z, f) \in \mathcal{U}_0 \times L^1(Y)$ is delivered by the following formula:

$$
\begin{aligned}
(3.2) \quad & d^k H_G[\phi, z, f]\big((u^{[1]}, w^{[1]}, v^{[1]}), \dots, (u^{[k]}, w^{[k]}, v^{[k]})\big)(x) \\
&= \int_Y \partial^k_{(\eta,z)} G(x, \phi(y), z)[(u^{[1]}(y), w^{[1]}), \dots , \\
&\qquad\qquad (u^{[k]}(y), w^{[k]})] f(y)\, d\mu_y \\
&\quad + \sum_{l=1}^{k} \int_Y \partial^{k-1}_{(\eta,z)} G(x, \phi(y), z)[(u^{[1]}(y), w^{[1]}), \dots , \\
&\qquad\qquad \widehat{(u^{[l]}(y), w^{[l]})}, \dots \\
&\qquad\qquad \dots , (u^{[k]}(y), w^{[k]})] v^{[l]}(y)\, d\mu_y \quad \text{for all } x \in \operatorname{cl}\Omega,
\end{aligned}
$$

for all $(u^{[1]}, w^{[1]}, v^{[1]}), \dots , (u^{[k]}, w^{[k]}, v^{[k]}) \in C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K} \times L^1(Y)$, where we understand that the right hand side equals

$$
\int_Y \partial_{(\eta,z)} G(x, \phi(y), z)[(u^{[1]}(y), w^{[1]})] f(y)\, d\mu_y + \int_Y G(x, \phi(y), z) v^{[1]}(y)\, d\mu_y
$$

if $k = 1$.

Proof. We first note that $\mathcal{F} \neq \varnothing$. Indeed, if we define the function $\phi_{\eta^*} \in C^0_b(Y, \mathbf{R}^{h_2})$ by setting

$$\phi_{\eta^*}(y) \equiv \eta^* \quad \text{for all } y \in Y,$$

then clearly $(\phi_{\eta^*}, z^*) \in \mathcal{F}$. Let $(\phi_0, z_0) \in \mathcal{F}$. Let

$$\widetilde{\delta}_0 \equiv \frac{1}{4} \min \Big\{ \operatorname{dist} \big\{ \operatorname{cl}\Omega \times \operatorname{cl}\phi_0(Y) \times \{z_0\}, \partial\mathcal{W} \big\}, 1 \Big\},$$

where dist denotes the distance in $\mathbf{R}^{h_1} \times \mathbf{R}^{h_2} \times \mathcal{K}$. Then we note that

$$\operatorname{cl}\mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0, \widetilde{\delta}_0) \times \operatorname{cl}\mathbf{B}_{\mathcal{K}}(z_0, \widetilde{\delta}_0) \subseteq \mathcal{F}.$$

We first prove that there exist $\rho_0 > 0$ and $0 < \delta_0 < \widetilde{\delta}_0$ such that H_G maps $\mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0, \delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0, \delta_0) \times L^1(Y)$ to $C^0_{\omega,\rho}(\operatorname{cl}\Omega)$ for all

$\rho \in]0, \rho_0[$. Actually, we shall prove a stronger statement which we need below in the proof. To do so, we introduce some notation. If $k \in \mathbf{N} \setminus \{0\}$, then we denote by

$$H_{\partial^k_{(\eta,z)}G}[\phi, z, f][(u^{[1]}, w^{[1]}), \dots, (u^{[k]}, w^{[k]})]$$

the function from $\mathrm{cl}\,\Omega$ to $\mathbf{R}$ defined by

$$\begin{aligned} H_{\partial^k_{(\eta,z)}G}[\phi, z, f][(u^{[1]}, w^{[1]}), \dots, (u^{[k]}, w^{[k]})](x) \\ \equiv \int_Y \partial^k_{(\eta,z)} G(x, \phi(y), z)[(u^{[1]}(y), w^{[1]}), \dots, \\ (u^{[k]}(y), w^{[k]})] f(y)\, d\mu_y \quad \text{for all } x \in \mathrm{cl}\,\Omega, \end{aligned}$$

for all $(u^{[1]}, w^{[1]}), \dots, (u^{[k]}, w^{[k]}) \in C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K}$, and for all (ϕ, z, f) in $\mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0, \widetilde{\delta}_0) \times \mathbf{B}_{\mathcal{K}}(z_0, \widetilde{\delta}_0) \times L^1(Y)$. We prove that there exist $\delta_0 \in \left]0, \widetilde{\delta}_0\right[$ and $\rho_0 \in]0, +\infty[$ such that $H_G[\phi, z, f]$ belongs to $C^0_{\omega,\rho}(\mathrm{cl}\,\Omega)$, and such that $H_{\partial^k_{(\eta,z)}G}[\phi, z, f][\cdot, \dots, \cdot]$ is an element of $\mathcal{L}^{(k)}\big((C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K})^k, C^0_{\omega,\rho}(\mathrm{cl}\,\Omega)\big)$ for all $\rho \in]0, \rho_0[$, and for all $(\phi, z, f) \in \mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0, \delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0, \delta_0) \times L^1(Y)$, and for all $k \in \mathbf{N} \setminus \{0\}$.

By standard theorems of differentiability of integrals dependent upon a parameter, we have

$$\begin{aligned} D^\beta_x H_{\partial^k_{(\eta,z)}G}[\phi, z, f][(u^{[1]}, w^{[1]}), \dots, (u^{[k]}, w^{[k]})](x) \\ = \int_Y D^\beta_x \partial^k_{(\eta,z)} G(x, \phi(y), z)[(u^{[1]}(y), w^{[1]}), \dots, \\ (u^{[k]}(y), w^{[k]})] f(y)\, d\mu_y \quad \text{for all } x \in \mathrm{cl}\,\Omega, \end{aligned}$$

for all $(\phi, z, f) \in \mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0, \widetilde{\delta}_0) \times \mathbf{B}_{\mathcal{K}}(z_0, \widetilde{\delta}_0) \times L^1(Y)$, for all $(u^{[1]}, w^{[1]}), \dots, (u^{[k]}, w^{[k]}) \in C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K}$, and for all $\beta \in \mathbf{N}^{h_1}$.

We now show that there exist $c_0 > 0$ and $\delta_0 \in \left]0, \widetilde{\delta}_0\right[$ such that

(3.3)
$$|D^\beta_x G(x, \eta, z)| \le c_0^{1+|\beta|} |\beta|!,$$

(3.4)
$$\|D^\beta_x \partial^k_{(\eta,z)} G(x, \eta, z)\|_{\mathcal{L}^{(k)}((\mathbf{R}^{h_2} \times \mathcal{K})^k, \mathbf{R})} \le c_0^{1+|\beta|+k} (|\beta| + k)!,$$

for all $(x,\eta,z) \in \mathrm{cl}\,\Omega \times (\mathrm{cl}\,\phi_0(Y) + \mathrm{cl}\,\mathbf{B}_{h_2}(0,\delta_0)) \times \mathrm{cl}\,\mathbf{B}_{\mathcal{K}}(z_0,\delta_0)$, and for all $\beta \in \mathbf{N}^{h_1}$, and for all $k \in \mathbf{N} \setminus \{0\}$. Indeed, let $(\widetilde{x},\widetilde{\eta}) \in \mathrm{cl}\,\Omega \times \big(\mathrm{cl}\,\phi_0(Y) + \mathrm{cl}\,\mathbf{B}_{h_2}(0,\widetilde{\delta}_0)\big)$. By assumption of analyticity of G, there exist $c_{(\widetilde{x},\widetilde{\eta},z_0)} > 0$ and $0 < \delta_{(\widetilde{x},\widetilde{\eta},z_0)} < \widetilde{\delta}_0$, such that

$$\big\|D_x^\beta \partial_{(\eta,z)}^k G(x,\eta,z)\big\|_{\mathcal{L}^{(k)}((\mathbf{R}^{h_2}\times\mathcal{K})^k,\mathbf{R})} \le c_{(\widetilde{x},\widetilde{\eta},z_0)} \frac{(|\beta|+k)!}{\delta_{(\widetilde{x},\widetilde{\eta},z_0)}^{|\beta|+k}},$$

for all $(x,\eta,z) \in \mathrm{cl}\,\Omega \times (\mathrm{cl}\,\phi_0(Y) + \mathrm{cl}\,\mathbf{B}_{h_2}(0,\widetilde{\delta}_0)) \times \mathrm{cl}\,\mathbf{B}_{\mathcal{K}}(z_0,\widetilde{\delta}_0)$ such that

$$\max\big\{\|x-\widetilde{x}\|_{\mathbf{R}^{h_1}}, \|\eta-\widetilde{\eta}\|_{\mathbf{R}^{h_2}}, \|z-z_0\|_{\mathcal{K}}\big\} \le \delta_{(\widetilde{x},\widetilde{\eta},z_0)},$$

for all $\beta \in \mathbf{N}^{h_1}$, and for all $k \in \mathbf{N} \setminus \{0\}$. Since $\mathrm{cl}\,\Omega \times (\mathrm{cl}\,\phi_0(Y) + \mathrm{cl}\,\mathbf{B}_{h_2}(0,\widetilde{\delta}_0))$ is compact in $\mathbf{R}^{h_1} \times \mathbf{R}^{h_2}$, there exist

$$(x_1,\eta_1),\dots,(x_m,\eta_m) \in \mathrm{cl}\,\Omega \times \big(\mathrm{cl}\,\phi_0(Y) + \mathrm{cl}\,\mathbf{B}_{h_2}(0,\widetilde{\delta}_0)\big),$$

such that

$$\begin{aligned}\mathrm{cl}\,\Omega \times \big(\mathrm{cl}\,\phi_0(Y) + \mathrm{cl}\,\mathbf{B}_{h_2}(0,\widetilde{\delta}_0)\big) \\ \subseteq \bigcup_{j=1}^m \Big(\mathbf{B}_{h_1}(x_j,\delta_{(x_j,\eta_j,z_0)}) \times \mathbf{B}_{h_2}(\eta_j,\delta_{(x_j,\eta_j,z_0)})\Big).\end{aligned}$$

Now we set

$$\delta_0 \equiv \min\big\{\delta_{(x_j,\eta_j,z_0)} : j = 1,\dots,m\big\},$$
$$c_0' \equiv \max\big\{c_{(x_j,\eta_j,z_0)} : j = 1,\dots,m\big\}.$$

If $(x,\eta) \in \mathrm{cl}\,\Omega \times (\mathrm{cl}\,\phi_0(Y) + \mathrm{cl}\,\mathbf{B}_{h_2}(0,\delta_0)\big) \subseteq \mathrm{cl}\,\Omega \times (\mathrm{cl}\,\phi_0(Y) + \mathrm{cl}\,\mathbf{B}_{h_2}(0,\widetilde{\delta}_0))$, then there exists $j_0 \in \{1,\dots,m\}$ such that

$$(x,\eta) \in \mathbf{B}_{h_1}(x_{j_0},\delta_{(x_{j_0},\eta_{j_0},z_0)}) \times \mathbf{B}_{h_2}(\eta_{j_0},\delta_{(x_{j_0},\eta_{j_0},z_0)}),$$

and, accordingly,

$$\big\|D_x^\beta \partial_{(\eta,z)}^k G(x,\eta,z)\big\|_{\mathcal{L}^{(k)}((\mathbf{R}^{h_2}\times\mathcal{K})^k,\mathbf{R})} \le c_0' \frac{(|\beta|+k)!}{(\delta_0)^{|\beta|+k}},$$

for all $z \in \mathbf{B}_{\mathcal{K}}(z_0, \delta_0)$, for all $\beta \in \mathbf{N}^{h_1}$, and for all $k \in \mathbf{N} \setminus \{0\}$. Thus, inequality (3.4) follows with

$$c_0 \equiv \max\left\{c_0', \frac{1}{\delta_0}\right\}.$$

Then inequality (3.3) follows precisely by the same argument with a possibly larger constant c_0 and a possibly smaller constant δ_0. Then we have

$$\begin{aligned}
(3.5)\quad & \frac{\rho^{|\beta|}}{|\beta|!}\left|\int_Y D_x^\beta \partial_{(\eta,z)}^k G(x,\phi(y),z)[(u^{[1]}(y),w^{[1]}),\dots,\right. \\
&\qquad\qquad\qquad \left.(u^{[k]}(y),w^{[k]})]f(y)\,d\mu_y\right| \\
&\leq \frac{\rho^{|\beta|}}{|\beta|!} c_0^{1+|\beta|+k}(|\beta|+k)!\left(\prod_{j=1}^k \|(u^{[j]},w^{[j]})\|_{C_b^0(Y,\mathbf{R}^{h_2})\times\mathcal{K}}\right)\|f\|_{L^1(Y)} \\
&\leq (c_0\rho)^{|\beta|} c_0^{1+k}(|\beta|+1)\cdot\ldots\cdot \\
&\qquad (|\beta|+k)\left(\prod_{j=1}^k \|(u^{[j]},w^{[j]})\|_{C_b^0(Y,\mathbf{R}^{h_2})\times\mathcal{K}}\right)\|f\|_{L^1(Y)} \\
&\qquad \text{for all } x \in \operatorname{cl}\Omega, \\
&\qquad \text{for all } (\phi,z,f) \in \mathbf{B}_{C_b^0(Y,\mathbf{R}^{h_2})}(\phi_0,\delta_0)\times\mathbf{B}_{\mathcal{K}}(z_0,\delta_0)\times L^1(Y), \\
&\qquad \text{for all } (u^{[1]},w^{[1]}),\dots,(u^{[k]},w^{[k]}) \in C_b^0(Y,\mathbf{R}^{h_2})\times\mathcal{K},
\end{aligned}$$

for all $\beta \in \mathbf{N}^{h_1}$, for all $k \in \mathbf{N}\setminus\{0\}$, and for all $\rho \in]0,+\infty[$.

Next we set

$$(3.6)\qquad c_{\rho,k} \equiv \sup_{h\in\mathbf{N}}(h+1)\cdot\ldots\cdot(h+k)(c_0\rho)^h \quad \text{for all } k \in \mathbf{N}\setminus\{0\},$$

for all $\rho \in]0,+\infty[$. Clearly,

$$\begin{aligned}
c_{\rho,k} &\leq \max\Big\{\sup_{h\leq k}(h+1)\cdot\ldots\cdot(h+k)(c_0\rho)^h, \sup_{h\geq k}(h+1)\cdot\ldots \\
&\qquad\qquad \cdot(h+k)(c_0\rho)^h\Big\} \\
&\leq \max\Big\{(2k)^k\sup_{h\leq k}(c_0\rho)^h, \sup_{h\geq k}(2h)^k(c_0\rho)^h\Big\},
\end{aligned}$$

for all $k \in \mathbf{N} \setminus \{0\}$ and $\rho \in]0, +\infty[$. If $0 < \rho < 1/c_0$, we have $(2k)^k (c_0\rho)^h \leq (2k)^k$, and $\sup_{h \geq k} (2h)^k (c_0\rho)^h < +\infty$. Hence, we conclude that $c_{\rho,k} < +\infty$ for each $k \in \mathbf{N} \setminus \{0\}$ and for each $\rho \in]0, 1/c_0[$.

As a consequence, we have

$$
\begin{aligned}
(3.7) \quad & \|H_{\partial^k_{(\eta,z)} G}[\phi, z, f][(u^{[1]}, w^{[1]}), \dots, (u^{[k]}, w^{[k]})]\|_{C^0_{\omega,\rho}(\mathrm{cl}\,\Omega)} \\
& \qquad \leq c_0^{1+k} c_{\rho,k} \left(\prod_{j=1}^{k} \|(u^{[j]}, w^{[j]})\|_{C^0_b(Y,\mathbf{R}^{h_2}) \times \mathcal{K}} \right) \|f\|_{L^1(Y)} \\
& \qquad \text{for all } (\phi, z, f) \in \mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0, \delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0, \delta_0) \times L^1(Y), \\
& \qquad \text{for all } (u^{[1]}, w^{[1]}), \dots, (u^{[k]}, w^{[k]}) \in C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K},
\end{aligned}
$$

for all $k \in \mathbf{N} \setminus \{0\}$, for all $\rho \in]0, 1/c_0[$, and accordingly

$$
\begin{aligned}
& \|H_{\partial^k_{(\eta,z)} G}[\phi, z, f]\|_{\mathcal{L}^{(k)}((C^0_b(Y,\mathbf{R}^{h_2}) \times \mathcal{K})^k, C^0_{\omega,\rho}(\mathrm{cl}\,\Omega))} \\
& \qquad \leq c_0^{1+k} c_{\rho,k} \|f\|_{L^1(Y)} \\
& \qquad \text{for all } (\phi, z, f) \in \mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0, \delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0, \delta_0) \times L^1(Y),
\end{aligned}
$$

for all $k \in \mathbf{N} \setminus \{0\}$ and for all $\rho \in]0, 1/c_0[$. In particular,

$$
H_{\partial^k_{(\eta,z)} G}[\phi, z, \cdot] \in \mathcal{L}\Big(L^1(Y), \mathcal{L}^{(k)}\big((C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K})^k, C^0_{\omega,\rho}(\mathrm{cl}\,\Omega)\big)\Big),
$$

for all $(\phi, z) \in \mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0, \delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0, \delta_0)$, for each $k \in \mathbf{N} \setminus \{0\}$, and for each $\rho \in]0, 1/c_0[$.

In the case $k = 0$ the same argument implies that

$$
\begin{aligned}
\|H_G[\phi, z, f]\|_{C^0_{\omega,\rho}(\mathrm{cl}\,\Omega)} &\leq c_0 \|f\|_{L^1(Y)}, \\
H_G[\phi, z, \cdot] &\in \mathcal{L}(L^1(Y), C^0_{\omega,\rho}(\mathrm{cl}\,\Omega)),
\end{aligned}
$$

for all $(\phi, z, f) \in \mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0, \delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0, \delta_0) \times L^1(Y)$, and for each $\rho \in]0, 1/c_0[$.

Our next goal is to show that $H_{\partial^k_{(\eta,z)} G}[\cdot, \cdot, \cdot]$ is of class C^1 from the set $\mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0, \delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0, \delta_0) \times L^1(Y)$ to $\mathcal{L}^{(k)}((C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K})^k, C^0_{\omega,\rho}(\mathrm{cl}\,\Omega))$ for all $k \in \mathbf{N} \setminus \{0\}$ and $\rho \in]0, 1/c_0[$, and that $H_G[\cdot, \cdot, \cdot]$

is of class C^1 from the set $\mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0,\delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0,\delta_0) \times L^1(Y)$ to $C^0_{\omega,\rho}(\mathrm{cl}\,\Omega)$ for all $\rho \in]0,1/c_0[$. Actually, we prove a stronger statement.

If $k \in \mathbf{N} \setminus \{0\}$, we introduce the map $K_{\partial^k_{(\eta,z)}G}[\cdot,\cdot]$ from

$$\mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0,\delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0,\delta_0)$$

to $\mathcal{L}^{(k+1)}((C^0_b(Y,\mathbf{R}^{h_2}) \times \mathcal{K})^k \times L^1(Y), C^0_{\omega,\rho}(\mathrm{cl}\,\Omega))$ for $\rho \in]0,1/c_0[$, defined by

$$\begin{aligned}
K_{\partial^k_{(\eta,z)}G}&[\phi,z][(u^{[1]},w^{[1]}),\dots,(u^{[k]},w^{[k]}),f] \\
&\equiv H_{\partial^k_{(\eta,z)}G}[\phi,z,f][(u^{[1]},w^{[1]}),\dots,(u^{[k]},w^{[k]})] \\
&\text{for all } (\phi,z,f) \in \mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0,\delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0,\delta_0) \times L^1(Y), \\
&\text{for all } (u^{[1]},w^{[1]}),\dots,(u^{[k]},w^{[k]}) \in C^0_b(Y,\mathbf{R}^{h_2}) \times \mathcal{K}.
\end{aligned}$$

If $k = 0$, we introduce the map $K_G[\cdot,\cdot]$ from $\mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0,\delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0,\delta_0)$ to $\mathcal{L}(L^1(Y), C^0_{\omega,\rho}(\mathrm{cl}\,\Omega))$ for $\rho \in]0,1/c_0[$ defined by

$$K_G[\phi,z][f] \equiv H_G[\phi,z,f]$$
$$\text{for all } (\phi,z,f) \in \mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0,\delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0,\delta_0) \times L^1(Y).$$

Then we prove below $K_{\partial^k_{(\eta,z)}G}[\cdot,\cdot]$ is of class C^1 from $\mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0,\delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0,\delta_0)$ to $\mathcal{L}^{(k+1)}((C^0_b(Y,\mathbf{R}^{h_2}) \times \mathcal{K})^k \times L^1(Y), C^0_{\omega,\rho}(\mathrm{cl}\,\Omega))$ for $\rho \in]0,1/c_0[$, where we understand that $(C^0_b(Y,\mathbf{R}^{h_2}) \times \mathcal{K})^k$ is omitted if $k = 0$. To do so, we note that the equalities

$$\begin{aligned}
H_{\partial^k_{(\eta,z)}G}[\phi,z,f][\cdot,\dots,\cdot] &= K_{\partial^k_{(\eta,z)}G}[\phi,z][\cdot,\dots,\cdot,f] \quad \text{for all } k \in \mathbf{N} \setminus \{0\}, \\
H_G[\phi,z,f] &= K_G[\phi,z][f],
\end{aligned}$$

for all $(\phi,z,f) \in \mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0,\delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0,\delta_0) \times L^1(Y)$ hold. Then we note that the bilinear map which takes a pair (L,f) in $\mathcal{L}^{(k+1)}((C^0_b(Y,\mathbf{R}^{h_2}) \times \mathcal{K})^k \times L^1(Y), C^0_{\omega,\rho}(\mathrm{cl}\,\Omega)) \times L^1(Y)$ to $L[\cdot,\dots,\cdot,f]$ in $\mathcal{L}^{(k)}((C^0_b(Y,\mathbf{R}^{h_2}) \times \mathcal{K})^k, C^0_{\omega,\rho}(\mathrm{cl}\,\Omega))$ if $k > 0$ and in $C^0_{\omega,\rho}(\mathrm{cl}\,\Omega)$ if $k = 0$ is continuous. Hence, we conclude that $H_{\partial^k_{(\eta,z)}G}$ is of class C^1 if $K_{\partial^k_{(\eta,z)}G}$ is of class C^1 and if $\rho \in]0,1/c_0[$.

We now prove that, if $(\phi, z) \in \mathbf{B}_{C_b^0(Y,\mathbf{R}^{h_2})}(\phi_0, \delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0, \delta_0)$, then $K_{\partial^k_{(\eta,z)} G}$ is differentiable at (ϕ, z) and that $dK_{\partial^k_{(\eta,z)} G}[\phi, z]$, which is an element of

$$\mathcal{L}\Big(C_b^0(Y, \mathbf{R}^{h_2}) \times \mathcal{K}, \mathcal{L}^{(k+1)}\big((C_b^0(Y, \mathbf{R}^{h_2}) \times \mathcal{K})^k \times L^1(Y), C^0_{\omega,\rho}(\mathrm{cl}\,\Omega)\big)\Big),$$

is delivered by the formula

$$\begin{aligned}
(3.8)\quad & dK_{\partial^k_{(\eta,z)} G}[\phi, z](u, w)[(u^{[1]}, w^{[1]}), \dots, (u^{[k]}, w^{[k]}), f](x) \\
&\qquad = \int_Y \partial^{k+1}_{(\eta,z)} G(x, \phi(y), z)[(u^{[1]}(y), w^{[1]}), \dots, \\
&\qquad\qquad\qquad (u^{[k]}(y), w^{[k]}), (u(y), w)] f(y)\, d\mu_y \\
&\qquad = K_{\partial^{k+1}_{(\eta,z)} G}[\phi, z][(u^{[1]}, w^{[1]}), \dots, \\
&\qquad\qquad (u^{[k]}, w^{[k]}), (u, w), f](x), \\
&\qquad\quad \text{for all } x \in \mathrm{cl}\,\Omega, \text{for all } (u^{[1]}, w^{[1]}), \dots, \\
&\qquad\qquad (u^{[k]}, w^{[k]}) \in C_b^0(Y, \mathbf{R}^{h_2}) \times \mathcal{K}, \quad \text{for all } f \in L^1(Y),
\end{aligned}$$

for all $(u, w) \in C_b^0(Y, \mathbf{R}^{h_2}) \times \mathcal{K}$, and for all $(\phi, z) \in \mathbf{B}_{C_b^0(Y,\mathbf{R}^{h_2})}(\phi_0, \delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0, \delta_0)$ and for $\rho \in \,]0, 1/c_0[$. So let $(\phi, z) \in \mathbf{B}_{C_b^0(Y,\mathbf{R}^{h_2})}(\phi_0, \delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0, \delta_0)$. Let $(u, w) \in C_b^0(Y, \mathbf{R}^{h_2}) \times \mathcal{K}$ be such that $\|u\|_{C_b^0(Y,\mathbf{R}^{h_2})} < \delta_0 - \|\phi - \phi_0\|_{C_b^0(Y,\mathbf{R}^{h_2})}$ and $\|w\|_{\mathcal{K}} < \delta_0 - \|z - z_0\|_{\mathcal{K}}$. Then we note that

$$\begin{aligned}
& K_{\partial^k_{(\eta,z)} G}[\phi + u, z + w][(u^{[1]}, w^{[1]}), \dots, (u^{[k]}, w^{[k]}), f](x) \\
& - K_{\partial^k_{(\eta,z)} G}[\phi, z][(u^{[1]}, w^{[1]}), \dots, (u^{[k]}, w^{[k]}), f](x) \\
& \quad - \int_Y \partial^{k+1}_{(\eta,z)} G(x, \phi(y), z)[(u^{[1]}(y), w^{[1]}), \dots, \\
& \qquad\qquad\qquad (u^{[k]}(y), w^{[k]}), (u(y), w)] f(y)\, d\mu_y \\
& = \int_Y \Big\{ \partial^k_{(\eta,z)} G(x, \phi(y) + u(y), z + w)[(u^{[1]}(y), w^{[1]}), \dots, \\
& \qquad (u^{[k]}(y), w^{[k]})] \\
& \qquad - \partial^k_{(\eta,z)} G(x, \phi(y), z)[(u^{[1]}(y), w^{[1]}), \dots, (u^{[k]}(y), w^{[k]})] \\
& \qquad - \partial^{k+1}_{(\eta,z)} G(x, \phi(y), z)[(u^{[1]}(y), w^{[1]}), \dots,
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad (u^{[k]}(y), w^{[k]}), (u(y), w)]\Big\} f(y)\, d\mu_y \\
&= \int_Y \Big\{ \int_0^1 \Big[\partial^{k+1}_{(\eta,z)} G(x, \phi(y) + r u(y), z + r w)[(u^{[1]}(y), w^{[1]}), \dots \\
&\qquad \dots, (u^{[k]}(y), w^{[k]}), (u(y), w)] \\
&\qquad - \partial^{k+1}_{(\eta,z)} G(x, \phi(y), z)[(u^{[1]}(y), w^{[1]}), \dots, \\
&\qquad\qquad\qquad (u^{[k]}(y), w^{[k]}), (u(y), w)]\Big]\, dr \Big\} f(y)\, d\mu_y \\
&= \int_Y \Big\{ \int_0^1 \int_0^1 \partial^{k+2}_{(\eta,z)} G(x, \phi(y) + r\tau u(y), z + r\tau w)[(u^{[1]}(y), w^{[1]}), \dots \\
&\qquad\qquad \dots, (u^{[k]}(y), w^{[k]}), (u(y), w), (u(y), w)] r\, d\tau\, dr \Big\} f(y)\, d\mu_y
\end{aligned}
$$

for all $x \in \mathrm{cl}\,\Omega$, for all $(u^{[1]}, w^{[1]}), \dots, (u^{[k]}, w^{[k]}) \in C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K}$, for all $f \in L^1(Y)$.

We note that, in accordance with our notation, if $k = 0$ the arguments $(u^{[1]}, w^{[1]}), \dots, (u^{[k]}, w^{[k]})$ should be omitted. Now let $\beta \in \mathbf{N}^{h_1}$. By standard theorems of differentiation of integrals dependent upon a parameter, we have

$$
\begin{aligned}
(3.9)\quad & \frac{\rho^{|\beta|}}{|\beta|!} D^\beta_x \Big\{ K_{\partial^k_{(\eta,z)} G}[\phi + u, z + w][(u^{[1]}, w^{[1]}), \dots, \\
&(u^{[k]}, w^{[k]}), f](x) - K_{\partial^k_{(\eta,z)} G}[\phi, z][(u^{[1]}, w^{[1]}), \dots, \\
&(u^{[k]}, w^{[k]}), f](x) - \int_Y \partial^{k+1}_{(\eta,z)} G(x, \phi(y), z)[(u^{[1]}(y), w^{[1]}), \dots, \\
&\qquad\qquad (u^{[k]}(y), w^{[k]}), (u(y), w)] f(y)\, d\mu_y \Big\} \\
&= \int_Y \Big\{ \int_0^1 \int_0^1 \frac{\rho^{|\beta|}}{|\beta|!} D^\beta_x \partial^{k+2}_{(\eta,z)} G(x, \phi(y) + r\tau u(y), z + r\tau w) \\
&\qquad [(u^{[1]}(y), w^{[1]}), \dots, (u^{[k]}(y), w^{[k]}), \\
&\qquad (u(y), w), (u(y), w)] r\, d\tau\, dr \Big\} f(y)\, d\mu_y \quad \text{for all } x \in \mathrm{cl}\,\Omega,
\end{aligned}
$$

for all $(u^{[1]}, w^{[1]}), \dots, (u^{[k]}, w^{[k]}) \in C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K}$,
for all $f \in L^1(Y)$,

for all $\rho \in]0, 1/c_0[$. Then we note that the absolute value of the right hand side of (3.9) is less than or equal to

$$\begin{aligned}
&\frac{\rho^{|\beta|}}{|\beta|!} c_0^{|\beta|+k+3}(|\beta|+k+2)!\left(\prod_{j=1}^{k} \|(u^{[j]}, w^{[j]})\|_{C_b^0(Y,\mathbf{R}^{h_2})\times\mathcal{K}}\right) \\
&\qquad\qquad \times \|(u,w)\|^2_{C_b^0(Y,\mathbf{R}^{h_2})\times\mathcal{K}} \|f\|_{L^1(Y)} \\
&\qquad\qquad \leq c_{\rho,k+2} c_0^{k+3}\left(\prod_{j=1}^{k} \|(u^{[j]}, w^{[j]})\|_{C_b^0(Y,\mathbf{R}^{h_2})\times\mathcal{K}}\right) \\
&\qquad\qquad\quad \times \|(u,w)\|^2_{C_b^0(Y,\mathbf{R}^{h_2})\times\mathcal{K}} \|f\|_{L^1(Y)} \\
&\qquad\qquad \text{for all } (u^{[1]}, w^{[1]}), \dots, (u^{[k]}, w^{[k]}) \in C_b^0(Y, \mathbf{R}^{h_2}) \times \mathcal{K}, \\
&\qquad\qquad \text{for all } f \in L^1(Y),
\end{aligned}$$

for all $(\phi, z) \in \mathbf{B}_{C_b^0(Y,\mathbf{R}^{h_2})}(\phi_0, \delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0, \delta_0)$, and for all $(u, w) \in C_b^0(Y, \mathbf{R}^{h_2}) \times \mathcal{K}$ such that $\|u\|_{C_b^0(Y,\mathbf{R}^{h_2})} < \delta_0 - \|\phi - \phi_0\|_{C_b^0(Y,\mathbf{R}^{h_2})}$ and $\|w\|_{\mathcal{K}} < \delta_0 - \|z - z_0\|_{\mathcal{K}}$, and for all $\rho \in]0, 1/c_0[$ (see also (3.4)–(3.6)). Accordingly,

$$\begin{aligned}
&\Big\| K_{\partial^k_{(\eta,z)} G}[\phi + u, z + w][(u^{[1]}, w^{[1]}), \dots, (u^{[k]}, w^{[k]}), f] \\
&\quad - K_{\partial^k_{(\eta,z)} G}[\phi, z][(u^{[1]}, w^{[1]}), \dots, (u^{[k]}, w^{[k]}), f] \\
&\quad - K_{\partial^{k+1}_{(\eta,z)} G}[\phi, z][(u^{[1]}, w^{[1]}), \dots, (u^{[k]}, w^{[k]}), (u, w), f] \Big\|_{C^0_{\omega,\rho}(\mathrm{cl}\,\Omega)} \\
&\leq c_{\rho,k+2} c_0^{k+3}\left(\prod_{j=1}^{k} \|(u^{[j]}, w^{[j]})\|_{C_b^0(Y,\mathbf{R}^{h_2})\times\mathcal{K}}\right) \|(u,w)\|^2_{C_b^0(Y,\mathbf{R}^{h_2})\times\mathcal{K}} \\
&\qquad \times \|f\|_{L^1(Y)} \\
&\text{for all } (u^{[1]}, w^{[1]}), \dots, (u^{[k]}, w^{[k]}) \in C_b^0(Y, \mathbf{R}^{h_2}) \times \mathcal{K}, \\
&\text{for all } f \in L^1(Y),
\end{aligned}$$

for all $(\phi, z) \in \mathbf{B}_{C_b^0(Y,\mathbf{R}^{h_2})}(\phi_0, \delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0, \delta_0)$, for all $(u, w) \in C_b^0(Y, \mathbf{R}^{h_2}) \times \mathcal{K}$ such that $\|u\|_{C_b^0(Y,\mathbf{R}^{h_2})} < \delta_0 - \|\phi - \phi_0\|_{C_b^0(Y,\mathbf{R}^{h_2})}$ and $\|w\|_{\mathcal{K}} < \delta_0 - \|z - z_0\|_{\mathcal{K}}$, and for all $\rho \in]0, 1/c_0[$. Such an inequality proves the differentiability of $K_{\partial^k_{(\eta,z)} G}$ from $\mathbf{B}_{C_b^0(Y,\mathbf{R}^{h_2})}(\phi_0, \delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0, \delta_0)$ to $\mathcal{L}^{(k+1)}((C_b^0(Y, \mathbf{R}^{h_2}) \times \mathcal{K})^k \times L^1(Y), C^0_{\omega,\rho}(\mathrm{cl}\,\Omega))$ for all

$\rho \in]0, 1/c_0[$, and formula (3.8). We note that, in accordance with our notation, if $k = 0$ the product with $j = 1, \dots, k$ in parentheses on the right hand side of the above inequality is omitted.

Next we want to show $dK_{\partial^k_{(\eta,z)} G}$ is continuous in $\mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0, \delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0, \delta_0)$. To do so, we can exploit the argument above to prove the differentiability of $K_{\partial^k_{(\eta,z)} G}$ and prove that $K_{\partial^{k+1}_{(\eta,z)} G}$ is differentiable in $\mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0, \delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0, \delta_0)$. Then formula (3.8) shows that $dK_{\partial^k_{(\eta,z)} G}$ is differentiable and accordingly continuous. As a consequence, we can conclude that $K_{\partial^k_{(\eta,z)} G}$ is of class C^1 from the set $\mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0, \delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0, \delta_0)$ to $\mathcal{L}^{(k+1)}((C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K})^k \times L^1(Y), C^0_{\omega,\rho}(\mathrm{cl}\,\Omega))$ for all $\rho \in]0, 1/c_0[$ and that, as we have mentioned above, $H_{\partial^k_{(\eta,z)} G}$ is of class C^1 in $\mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0, \delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0, \delta_0) \times L^1(Y)$. Then, by the chain rule and by the rule of differentiation for bilinear maps, we immediately deduce that

$$
\begin{aligned}
(3.10)\quad & dH_{\partial^k_{(\eta,z)} G}[\phi, z, f](u, w, v)[\cdot, \dots, \cdot] \\
&= K_{\partial^{k+1}_{(\eta,z)} G}[\phi, z][\cdot, \dots, \cdot, (u, w), f] + K_{\partial^k_{(\eta,z)} G}[\phi, z][\cdot, \dots, \cdot, v] \\
&\qquad\qquad \text{for all } (u, w, v) \in C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K} \times L^1(Y),
\end{aligned}
$$

for all $(\phi, z, f) \in \mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0, \delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0, \delta_0) \times L^1(Y)$ and for all $k \in \mathbf{N}$ (see also (3.8)).

We now show that H_G is of class C^k from $\mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0, \delta_0) \times \mathbf{B}_{\mathcal{K}}(z_0, \delta_0) \times L^1(Y)$ to $C^0_{\omega,\rho}(\mathrm{cl}\,\Omega)$ for all $\rho \in]0, 1/c_0[$, for all $k \in \mathbf{N} \setminus \{0\}$, and that the kth order differential is delivered by formula (3.2) for all $k \in \mathbf{N} \setminus \{0\}$. We argue by induction on k. To do so, we first introduce some notation. If $k \in \mathbf{N} \setminus \{0\}$ and $l \in \{1, \dots, k\}$, then we denote by $T_{k,l}$ the linear and continuous map from $(C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K} \times L^1(Y))^k$ to $(C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K})^{k-1} \times L^1(Y)$, defined by

$$
\begin{aligned}
& T_{k,l}\big[(u^{[1]}, w^{[1]}, v^{[1]}), \dots, (u^{[k]}, w^{[k]}, v^{[k]})\big] \\
&\quad \equiv \big((u^{[1]}, w^{[1]}), \dots, \widehat{(u^{[l]}, w^{[l]})}, \dots, (u^{[k]}, w^{[k]}), v^{[l]}\big) \quad \text{if } k > 1, \\
&\qquad\qquad T_{k,1}[(u^{[1]}, w^{[1]}, v^{[1]})] \equiv v^{[1]} \quad \text{if } k = 1,
\end{aligned}
$$

for all $((u^{[1]}, w^{[1]}, v^{[1]}), \dots, (u^{[k]}, w^{[k]}, v^{[k]})) \in (C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K} \times L^1(Y))^k$. If $k \in \mathbf{N} \setminus \{0\}$, we also introduce the linear and continuous map $\widetilde{T}_k$ from the space $(C^0_b(Y,\mathbf{R}^{h_2})\times\mathcal{K}\times L^1(Y))^k$ to $(C^0_b(Y,\mathbf{R}^{h_2})\times\mathcal{K})^k$, defined by

$$\widetilde{T}_k\Big[(u^{[1]},w^{[1]},v^{[1]}),\dots,(u^{[k]},w^{[k]},v^{[k]})\Big]\equiv\big((u^{[1]},w^{[1]}),\dots,(u^{[k]},w^{[k]})\big),$$

for all $((u^{[1]},w^{[1]},v^{[1]}),\dots,(u^{[k]},w^{[k]},v^{[k]}))\in(C^0_b(Y,\mathbf{R}^{h_2})\times\mathcal{K}\times L^1(Y))^k$. A simple computation shows that the map from $\mathcal{L}^{(k)}((C^0_b(Y,\mathbf{R}^{h_2})\times\mathcal{K})^k, C^0_{\omega,\rho}(\mathrm{cl}\,\Omega))$ to $\mathcal{L}^{(k)}((C^0_b(Y,\mathbf{R}^{h_2})\times\mathcal{K}\times L^1(Y))^k, C^0_{\omega,\rho}(\mathrm{cl}\,\Omega))$ which takes L to $L\circ\widetilde{T}_k$ is linear and continuous, for all $k\in\mathbf{N}\setminus\{0\}$ and $\rho\in\,]0,+\infty[$. Similarly, the map from $\mathcal{L}^{(k)}((C^0_b(Y,\mathbf{R}^{h_2})\times\mathcal{K})^{k-1}\times L^1(Y), C^0_{\omega,\rho}(\mathrm{cl}\,\Omega))$ to $\mathcal{L}^{(k)}((C^0_b(Y,\mathbf{R}^{h_2})\times\mathcal{K}\times L^1(Y))^k, C^0_{\omega,\rho}(\mathrm{cl}\,\Omega))$ which takes L to $L\circ T_{k,l}$ is linear and continuous, for each $l\in\{1,\dots,k\}$ and for all $k\in\mathbf{N}\setminus\{0\}$ and $\rho\in\,]0,+\infty[$.

We have already proved that H_G is of class C^1 and that (3.2) holds for $k=1$. We now assume that H_G is of class C^k and that (3.2) holds for k, and we prove that H_G is of class C^{k+1} and that (3.2) holds for $k+1$. By inductive assumption, we have

$$\begin{aligned}(3.11)\quad & d^k H_G[\phi,z,f]\big((u^{[1]},w^{[1]},v^{[1]}),\dots,(u^{[k]},w^{[k]},v^{[k]})\big)\\ &= H_{\partial^k_{(\eta,z)}G}[\phi,z,f]\circ\widetilde{T}_k\big[(u^{[1]},w^{[1]},v^{[1]}),\dots,(u^{[k]},w^{[k]},v^{[k]})\big]\\ &\quad+\sum_{l=1}^{k}K_{\partial^{k-1}_{(\eta,z)}G}[\phi,z]\circ T_{k,l}\big[(u^{[1]},w^{[1]},v^{[1]}),\dots,(u^{[k]},w^{[k]},v^{[k]})\big]\end{aligned}$$

for all $(u^{[1]},w^{[1]},v^{[1]}),\dots,(u^{[k]},w^{[k]},v^{[k]})\in C^0_b(Y,\mathbf{R}^{h_2})\times\mathcal{K}\times L^1(Y)$ and for all $(\phi,z,f)\in\mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0,\delta_0)\times\mathbf{B}_{\mathcal{K}}(z_0,\delta_0)\times L^1(Y)$. Since $H_{\partial^k_{(\eta,z)}G}[\cdot,\cdot,\cdot]$ and $K_{\partial^{k-1}_{(\eta,z)}G}[\cdot,\cdot]$ are continuously differentiable in $\mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0,\delta_0)\times\mathbf{B}_{\mathcal{K}}(z_0,\delta_0)\times L^1(Y)$ and in $\mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0,\delta_0)\times\mathbf{B}_{\mathcal{K}}(z_0,\delta_0)$, respectively, we can conclude that $d^kH_G[\cdot,\cdot,\cdot]$ is of class C^1 on $\mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0,\delta_0)\times\mathbf{B}_{\mathcal{K}}(z_0,\delta_0)\times L^1(Y)$ and that, accordingly, the function H_G is of class C^{k+1} from $\mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})}(\phi_0,\delta_0)\times\mathbf{B}_{\mathcal{K}}(z_0,\delta_0)\times L^1(Y)$ to $C^0_{\omega,\rho}(\mathrm{cl}\,\Omega)$ for all $\rho\in\,]0,1/c_0[$.

We now compute the formula for $d^{k+1}H_G[\phi,z,f]$. To do so, we fix

$$(u^{[1]},w^{[1]},v^{[1]}),\dots,(u^{[k+1]},w^{[k+1]},v^{[k+1]})\in C^0_b(Y,\mathbf{R}^{h_2})\times\mathcal{K}\times L^1(Y).$$

Then by formulas (3.8), (3.10), (3.11), we have

$$
\begin{aligned}
d^{k+1} H_G[\phi, z, f]\big((u^{[1]}, w^{[1]}, v^{[1]}), \dots, (u^{[k+1]}, w^{[k+1]}, v^{[k+1]})\big) \\
= K_{\partial^{k+1}_{(\eta,z)} G}[\phi, z][(u^{[1]}, w^{[1]}), \dots, (u^{[k+1]}, w^{[k+1]}), f] \\
+ K_{\partial^{k}_{(\eta,z)} G}[\phi, z][(u^{[1]}, w^{[1]}), \dots, (u^{[k]}, w^{[k]}), v^{[k+1]}] \\
+ \sum_{l=1}^{k} K_{\partial^{k}_{(\eta,z)} G}[\phi, z] \circ T_{k+1,l}\big[(u^{[1]}, w^{[1]}, v^{[1]}), \dots, \\
(u^{[k+1]}, w^{[k+1]}, v^{[k+1]})\big] \\
= H_{\partial^{k+1}_{(\eta,z)} G}[\phi, z, f] \circ \widetilde{T}_{k+1}\big[(u^{[1]}, w^{[1]}, v^{[1]}), \dots, \\
(u^{[k+1]}, w^{[k+1]}, v^{[k+1]})\big] \\
+ \sum_{l=1}^{k+1} K_{\partial^{k}_{(\eta,z)} G}[\phi, z] \circ T_{k+1,l}\big[(u^{[1]}, w^{[1]}, v^{[1]}), \dots, \\
(u^{[k+1]}, w^{[k+1]}, v^{[k+1]})\big]
\end{aligned}
$$

which proves formula (3.2) for k replaced by $(k+1)$.

Finally, we prove that H_G is a real analytic function from the open ball $\mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})\times\mathcal{K}\times L^1(Y)}((\phi_0, z_0, f_0), \delta_0')$ to $C^0_{\omega,\rho}(\mathrm{cl}\,\Omega)$ for all $\rho \in]0, 1/(2c_0)[$, where

$$
\delta_0' \equiv \min\Big\{\delta_0, \frac{1}{2c_0}\Big\}.
$$

To do so, we exploit the classical Cauchy estimates. We must prove that there exists $C > 0$ such that for each $\rho \in]0, 1/(2c_0)[$, we have

(3.12)
$$
\|d^k H_G[\phi, z, f]\|_{\mathcal{L}^{(k)}((C^0_b(Y,\mathbf{R}^{h_2})\times\mathcal{K}\times L^1(Y))^k, C^0_{\omega,\rho}(\mathrm{cl}\,\Omega))} \leq C\frac{k!}{(1/2c_0)^k},
$$

for all $k \in \mathbf{N}\setminus\{0,1\}$ and for all (ϕ, z, f) in $\mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})\times\mathcal{K}\times L^1(Y)}((\phi_0, z_0, f_0), \delta_0')$.

By formulas (3.2) and (3.7), we have

$$
\begin{aligned}
(3.13)\quad & \|d^k H_G[\phi,z,f]\big((u^{[1]},w^{[1]},v^{[1]}),\dots,(u^{[k]},w^{[k]},v^{[k]})\big)\|_{C^0_{\omega,\rho}(\mathrm{cl}\,\Omega)} \\
\le\ & c_0^{1+k} c_{\rho,k}\Big(\prod_{j=1}^{k}\|(u^{[j]},w^{[j]})\|_{C^0_b(Y,\mathbf{R}^{h_2})\times\mathcal{K}}\Big)\|f\|_{L^1(Y)} \\
& + \sum_{l=1}^{k}\Big(c_0^{1+k-1}c_{\rho,k-1}\Big(\prod_{j\in\{1,\dots,k\}\setminus\{l\}}\|(u^{[j]},w^{[j]})\|_{C^0_b(Y,\mathbf{R}^{h_2})\times\mathcal{K}}\Big)\|v^{[l]}\|_{L^1(Y)}\Big) \\
\le\ & c_0^{1+k} c_{\rho,k}\Big(\prod_{j=1}^{k}\|(u^{[j]},w^{[j]})\|_{C^0_b(Y,\mathbf{R}^{h_2})\times\mathcal{K}}\Big)\|f\|_{L^1(Y)} \\
& + k c_0^{k} c_{\rho,k-1}\prod_{j=1}^{k}\Big(\|(u^{[j]},w^{[j]})\|_{C^0_b(Y,\mathbf{R}^{h_2})\times\mathcal{K}} + \|v^{[j]}\|_{L^1(Y)}\Big) \\
\le\ & c_0^{1+k} c_{\rho,k}\Big(\prod_{j=1}^{k}\|(u^{[j]},w^{[j]})\|_{C^0_b(Y,\mathbf{R}^{h_2})\times\mathcal{K}}\Big)\|f\|_{L^1(Y)} \\
& + k c_0^{k} c_{\rho,k-1}\prod_{j=1}^{k}\|(u^{[j]},w^{[j]},v^{[j]})\|_{C^0_b(Y,\mathbf{R}^{h_2})\times\mathcal{K}\times L^1(Y)},
\end{aligned}
$$

for all $(u^{[1]},w^{[1]},v^{[1]}),\dots,(u^{[k]},w^{[k]},v^{[k]})$ in $C^0_b(Y,\mathbf{R}^{h_2})\times\mathcal{K}\times L^1(Y)$, for all (ϕ,z,f) in $\mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2})\times\mathcal{K}\times L^1(Y)}((\phi_0,z_0,f_0),\delta_0')$, and for all $\rho\in\,]0,1/(2c_0)[$. Next we note that

$$
\begin{aligned}
(h+1)\cdot\ldots\cdot(h+k)(c_0\rho)^h &= \binom{h+k}{k}k!(c_0\rho)^h \le 2^{h+k}k!(c_0\rho)^h \\
&= (2c_0\rho)^h k! 2^k \quad \text{for all } h,k\in\mathbf{N},\ k>0,
\end{aligned}
$$

and that, accordingly,

$$
c_{\rho,k}\le k!2^k \quad \text{for all } k\in\mathbf{N}\setminus\{0\},
$$

for all $\rho\in\,]0,1/(2c_0)[$. Then the right hand side of (3.13) is less than or equal to

$$\begin{aligned} k! c_0^k 2^k \Big(c_0 \|f\|_{L^1(Y)} + \frac{1}{2} \Big) \Big(\prod_{j=1}^{k} \|(u^{[j]}, w^{[j]}, v^{[j]})\|_{C^0_b(Y,\mathbf{R}^{h_2}) \times \mathcal{K} \times L^1(Y)} \Big) \\ \leq \frac{k!}{(1/2c_0)^k} \Big(c_0 \|f_0\|_{L^1(Y)} + c_0 \|f - f_0\|_{L^1(Y)} + \frac{1}{2} \Big) \\ \times \Big(\prod_{j=1}^{k} \|(u^{[j]}, w^{[j]}, v^{[j]})\|_{C^0_b(Y,\mathbf{R}^{h_2}) \times \mathcal{K} \times L^1(Y)} \Big) \\ \leq \frac{k!}{(1/2c_0)^k} \Big(c_0 \|f_0\|_{L^1(Y)} + c_0 \frac{1}{2c_0} + \frac{1}{2} \Big) \\ \times \Big(\prod_{j=1}^{k} \|(u^{[j]}, w^{[j]}, v^{[j]})\|_{C^0_b(Y,\mathbf{R}^{h_2}) \times \mathcal{K} \times L^1(Y)} \Big) \end{aligned}$$

for all $(u^{[1]}, w^{[1]}, v^{[1]}), \dots, (u^{[k]}, w^{[k]}, v^{[k]})$ in $C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K} \times L^1(Y)$, for all (ϕ, z, f) in $\mathbf{B}_{C^0_b(Y,\mathbf{R}^{h_2}) \times \mathcal{K} \times L^1(Y)}((\phi_0, z_0, f_0), \delta'_0)$, and for all $\rho \in]0, 1/(2c_0)[$. Hence, (3.12) holds with $C \equiv c_0 \|f_0\|_{L^1(Y)} + 1$, and the proof is complete. □

Then we have the following corollary (see also [**9**, Proposition 6.1]).

Corollary 3.14. *Let $r \in \mathbf{N}$, $h_1, h_2 \in \mathbf{N} \setminus \{0\}$. Let Ω be a bounded open subset of $\mathbf{R}^{h_1}$. Let Y be a topological space. Let $\mathcal{M}$ be a σ-algebra of parts of Y containing the Borel sets of Y. Let μ be a measure on $\mathcal{M}$. Let $\mathcal{K}$ be a Banach space. Let $\mathcal{W}$ be an open subset of $\mathbf{R}^{h_1} \times \mathbf{R}^{h_2} \times \mathcal{K}$ such that there exists $(\eta^*, z^*) \in \mathbf{R}^{h_2} \times \mathcal{K}$ such that*

$$\mathrm{cl}\,\Omega \times \{\eta^*\} \times \{z^*\} \subseteq \mathcal{W}.$$

Let

$$\mathcal{F} \equiv \Big\{ (\phi, z) \in C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K} \colon \mathrm{cl}\,\Omega \times \mathrm{cl}\,\phi(Y) \times \{z\} \subseteq \mathcal{W} \Big\}.$$

Let G be a real analytic map from $\mathcal{W}$ to $\mathbf{R}$. Then the map H_G from $\mathcal{F} \times L^1(Y)$ to $C^r(\mathrm{cl}\,\Omega)$ defined by

$$H_G[\phi, z, f](x) \equiv \int_Y G(x, \phi(y), z) f(y)\, d\mu_y \quad \text{for all } x \in \mathrm{cl}\,\Omega,$$

for all $(\phi, z, f) \in \mathcal{F} \times L^1(Y)$ is real analytic.

Proof. It clearly suffices to prove that, for each $(\phi_0, z_0) \in \mathcal{F}$, there exists an open neighborhood $\mathcal{U}_0$ of (ϕ_0, z_0) in $\mathcal{F}$, such that H_G is real analytic from $\mathcal{U}_0 \times L^1(Y)$ to $C^r(\mathrm{cl}\,\Omega)$. So, let $(\phi_0, z_0) \in \mathcal{F}$. By applying Theorem 3.1, we deduce that there exist an open neighbourhood $\mathcal{U}_0$ of (ϕ_0, z_0) in $\mathcal{F}$ and $\rho \in\,]0, +\infty[$, such that H_G is real analytic from $\mathcal{U}_0 \times L^1(Y)$ to $C^0_{\omega,\rho}(\mathrm{cl}\,\Omega)$. Then we note that the embedding of $C^0_{\omega,\rho}(\mathrm{cl}\,\Omega)$ in $C^r(\mathrm{cl}\,\Omega)$ is linear and continuous, and thus real analytic. Hence, H_G is real analytic from $\mathcal{U}_0 \times L^1(Y)$ to $C^r(\mathrm{cl}\,\Omega)$, and thus the proof is complete. $\square$

We note that the same argument of the proof of Theorem 3.1 can be exploited to prove that H_G is of class C^k from $\mathcal{F} \times L^1(Y)$ to a Schauder space if G satisfies appropriate differentiability conditions but is not necessarily analytic.

4. Analyticity of an integral operator in Schauder spaces. We are now ready to prove our main result on $H^\sharp_G$.

Proposition 4.1. *Let $h_1, h_2 \in \mathbf{N} \setminus \{0\}$. Let $m \in \mathbf{N}$, $\alpha \in\,]0,1]$. Let Y be a topological space. Let $\mathcal{M}$ be a σ-algebra of parts of Y containing the Borel sets of Y. Let μ be measure on $\mathcal{M}$. Let $\mathcal{K}$ be a Banach space. Let $\mathcal{W}$ be a nonempty open subset of $\mathbf{R}^{h_1} \times \mathbf{R}^{h_2} \times \mathcal{K}$. Let G be a real analytic map from $\mathcal{W}$ to $\mathbf{R}$. Then the following statements hold.*

(i) *Let $n \in \mathbf{N} \setminus \{0\}$. Let Ω_1 be a bounded open connected subset of $\mathbf{R}^n$. Let Ω_1 be regular in the sense of Whitney. Let*

$$\widetilde{\mathcal{F}} \equiv \Big\{ (\Psi, \phi, z) \in C^{m,\alpha}(\mathrm{cl}\,\Omega_1, \mathbf{R}^{h_1}) \times C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K} : \\ \Psi(\mathrm{cl}\,\Omega_1) \times \mathrm{cl}\,\phi(Y) \times \{z\} \subseteq \mathcal{W} \Big\}.$$

Then the map $\widetilde{H}_G$ from $\widetilde{\mathcal{F}} \times L^1(Y)$ to $C^{m,\alpha}(\mathrm{cl}\,\Omega_1)$ defined by

$$\widetilde{H}_G[\Psi, \phi, z, f](t) \equiv \int_Y G(\Psi(t), \phi(y), z) f(y)\, d\mu_y \quad \textit{for all } t \in \mathrm{cl}\,\Omega_1,$$

for all $(\Psi, \phi, z, f) \in \widetilde{\mathcal{F}} \times L^1(Y)$, is real analytic.

(ii) *Let* $n, s \in \mathbf{N}$, $1 \leq s < n$. *Let* $\mathbf{M}$ *be a compact manifold of class* $C^{\max\{1,m\},\alpha}$ *embedded into* $\mathbf{R}^n$ *and of dimension* s. *Let*

$$\mathcal{F}^\sharp \equiv \Big\{(\psi, \phi, z) \in C^{m,\alpha}(\mathbf{M}, \mathbf{R}^{h_1}) \times C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K} : \\ \psi(\mathbf{M}) \times \mathrm{cl}\,\phi(Y) \times \{z\} \subseteq \mathcal{W}\Big\}.$$

Then the map $H^\sharp_G$ *from* $\mathcal{F}^\sharp \times L^1(Y)$ *to* $C^{m,\alpha}(\mathbf{M})$ *defined by*

$$H^\sharp_G[\psi, \phi, z, f](t) \equiv \int_Y G(\psi(t), \phi(y), z) f(y)\, d\mu_y \quad \textit{for all } t \in \mathbf{M},$$

for all $(\psi, \phi, z, f) \in \mathcal{F}^\sharp \times L^1(Y)$, *is real analytic.*

Proof. We first consider statement (i). We first prove that $\widetilde{\mathcal{F}} \neq \varnothing$. Let $(x^*, \eta^*, z^*) \in \mathcal{W}$. Let the functions $\Psi_{x^*} \in C^{m,\alpha}(\mathrm{cl}\,\Omega_1, \mathbf{R}^{h_1})$ and $\phi_{\eta^*} \in C^0_b(Y, \mathbf{R}^{h_2})$ be defined by setting $\Psi_{x^*}(t) \equiv x^*$ for all $t \in \mathrm{cl}\,\Omega_1$ and $\phi_{\eta^*}(y) \equiv \eta^*$ for all $y \in Y$. Then we have $(\Psi_{x^*}, \phi_{\eta^*}, z^*) \in \widetilde{\mathcal{F}}$.

Since the maps Ψ and ϕ are continuous and bounded, and $\mathcal{W}$ is open, then the set $\widetilde{\mathcal{F}}$ is easily seen to be open in $C^{m,\alpha}(\mathrm{cl}\,\Omega_1, \mathbf{R}^{h_1}) \times C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K}$. Then it suffices to show that, for each $(\Psi_0, \phi_0, z_0) \in \widetilde{\mathcal{F}}$ there exists an open neighborhood $\mathcal{V}_0$ of (Ψ_0, ϕ_0, z_0) in $\widetilde{\mathcal{F}}$ such that $\widetilde{H}_G$ is real analytic from $\mathcal{V}_0 \times L^1(Y)$ to $C^{m,\alpha}(\mathrm{cl}\,\Omega_1)$.

By assumption, $\Psi_0(\mathrm{cl}\,\Omega_1) \times \mathrm{cl}\,\phi_0(Y) \times \{z_0\}$ has a positive distance δ_0 from the complement of $\mathcal{W}$. Thus, possibly shrinking δ_0, we can assume that the closure of

$$(\Psi_0(\mathrm{cl}\,\Omega_1) + \mathbf{B}_{h_1}(0, \delta_0)) \times \mathrm{cl}\,\phi_0(Y) \times \{z_0\}$$

is contained in $\mathcal{W}$. Then we set

$$\Omega \equiv \Psi_0(\mathrm{cl}\,\Omega_1) + \mathbf{B}_{h_1}(0, \delta_0),$$

and we note that Ω is a bounded open connected subset of $\mathbf{R}^{h_1}$, and that

$$\mathrm{cl}\,\Omega \times \mathrm{cl}\,\phi_0(Y) \times \{z_0\} \subseteq \mathcal{W}.$$

By Theorem 3.1, there exist an open neighborhood $\mathcal{U}_0$ of (ϕ_0, z_0) in $\mathcal{F}$ and $\rho_0 \in]0, +\infty[$ such that H_G is real analytic from $\mathcal{U}_0 \times L^1(Y)$ to $C^0_{\omega,\rho}(\mathrm{cl}\,\Omega)$ for all $\rho \in]0, \rho_0[$. Then we set

$$\mathcal{V}_0 \equiv C^{m,\alpha}(\mathrm{cl}\,\Omega_1, \Omega) \times \mathcal{U}_0,$$

and we note that

$$\widetilde{H}_G[\Psi, \phi, z, f](t) = H_G[\phi, z, f] \circ \Psi(t) \quad \text{for all } t \in \mathrm{cl}\,\Omega_1,$$

for all $(\Psi, \phi, z, f) \in \mathcal{V}_0 \times L^1(Y)$. Then, by Theorem 3.1 and Proposition 5.2 of the Appendix on the composition operator, we immediately deduce that $\widetilde{H}_G$ is real analytic from $\mathcal{V}_0 \times L^1(Y)$ to $C^{m,\alpha}(\mathrm{cl}\,\Omega_1)$.

We now prove statement (ii). We first prove that $\mathcal{F}^\sharp \neq \varnothing$. Let $(x^*, \eta^*, z^*) \in \mathcal{W}$. Let the functions $\psi_{x^*} \in C^{m,\alpha}(\mathbf{M}, \mathbf{R}^{h_1})$ and $\phi_{\eta^*} \in C^0_b(Y, \mathbf{R}^{h_2})$ be defined by setting $\psi_{x^*}(t) \equiv x^*$ for all $t \in \mathbf{M}$ and $\phi_{\eta^*}(y) \equiv \eta^*$ for all $y \in Y$. Then we have $(\psi_{x^*}, \phi_{\eta^*}, z^*) \in \mathcal{F}^\sharp$.

Since the maps ψ and ϕ are continuous and bounded, and $\mathcal{W}$ is open, the set $\mathcal{F}^\sharp$ is easily seen to be open in $C^{m,\alpha}(\mathbf{M}, \mathbf{R}^{h_1}) \times C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K}$. Then it suffices to show that, for each $(\psi_0, \phi_0, z_0) \in \mathcal{F}^\sharp$, there exists an open neighborhood $\mathcal{V}_0$ of (ψ_0, ϕ_0, z_0) in $\mathcal{F}^\sharp$ such that $H^\sharp_G$ is real analytic from $\mathcal{V}_0 \times L^1(Y)$ to $C^{m,\alpha}(\mathbf{M})$.

By assumption, $\psi_0(\mathbf{M}) \times \mathrm{cl}\,\phi_0(Y) \times \{z_0\}$ has a positive distance δ_0 from the complement of $\mathcal{W}$. Then, possibly shrinking δ_0, we can assume that the closure of

$$(\psi_0(\mathbf{M}) + \mathbf{B}_{h_1}(0, \delta_0)) \times \mathrm{cl}\,\phi_0(Y) \times \{z_0\}$$

is contained in $\mathcal{W}$. Let $R > 0$ be such that $\mathbf{M} \subseteq \mathbf{B}_n(0, R)$. Let $\mathbf{E}$ be an extension operator from $C^{m,\alpha}(\mathbf{M}, \mathbf{R}^{h_1})$ to $C^{m,\alpha}(\mathrm{cl}\,\mathbf{B}_n(0, R), \mathbf{R}^{h_1})$ (see Lemma 5.1 of the Appendix.) By the uniform continuity of $\mathbf{E}[\psi_0]$, there exists $\eta > 0$ such that

$$|\mathbf{E}[\psi_0](x_1) - \mathbf{E}[\psi_0](x_2)| < \delta_0$$

whenever

$$x_1, x_2 \in \mathrm{cl}\,\mathbf{B}_n(0, R), \ |x_1 - x_2| < \eta.$$

Possibly shrinking η, we can assume that $\mathbf{M} + \mathbf{B}_n(0,\eta) \subseteq \mathbf{B}_n(0,R)$. Next we take an open bounded subset Ω_1 of class C^∞ of $\mathbf{R}^n$ such that

$$\mathbf{M} \subseteq \Omega_1 \subseteq \mathrm{cl}\,\Omega_1 \subseteq \mathbf{M} + \mathbf{B}_n(0,\eta),$$

and we still denote by $\mathbf{E}$ the extension operator $\mathbf{E}$ composed with the restriction operator from $C^{m,\alpha}(\mathrm{cl}\,\mathbf{B}_n(0,R), \mathbf{R}^{h_1})$ to $C^{m,\alpha}(\mathrm{cl}\,\Omega_1, \mathbf{R}^{h_1})$. Obviously,

$$\mathbf{E}[\psi_0](\mathrm{cl}\,\Omega_1) \subseteq \psi_0(\mathbf{M}) + \mathbf{B}_{h_1}(0,\delta_0).$$

Then we set

$$\mathcal{V}_1 \equiv \Big\{(\Psi,\phi,z) \in C^{m,\alpha}(\mathrm{cl}\,\Omega_1, \mathbf{R}^{h_1}) \times C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K} : \\ \Psi(\mathrm{cl}\,\Omega_1) \times \mathrm{cl}\,\phi(Y) \times \{z\} \subseteq \mathcal{W}\Big\}.$$

As we have seen,

$$\mathbf{E}[\psi_0](\mathrm{cl}\,\Omega_1) \times \mathrm{cl}\,\phi_0(Y) \times \{z_0\} \\ \subseteq (\psi_0(\mathbf{M}) + \mathbf{B}_{h_1}(0,\delta_0)) \times \mathrm{cl}\,\phi_0(Y) \times \{z_0\} \subseteq \mathcal{W}.$$

Since Ω_1 has a finite number of connected components, statement (i) implies that $\widetilde{H}_G$ is analytic in $\mathcal{V}_1 \times L^1(Y)$. Then we set

$$\mathcal{V}_0 \equiv \left\{(\psi,\phi,z) \in C^{m,\alpha}(\mathbf{M}, \mathbf{R}^{h_1}) \times C^0_b(Y, \mathbf{R}^{h_2}) \times \mathcal{K} :\ (\mathbf{E}[\psi],\phi,z) \in \mathcal{V}_1\right\}.$$

Since $\mathbf{E}$ is continuous and $\mathcal{V}_1$ is open, then $\mathcal{V}_0$ is an open neighborhood of (ψ_0,ϕ_0,z_0). Since $H^\sharp_G[\psi,\phi,z,f](t) = \widetilde{H}_G[\mathbf{E}[\psi],\phi,z,f](t)$ for all $t \in \mathbf{M}$, we conclude that $H^\sharp_G$ is analytic in $\mathcal{V}_0 \times L^1(Y)$. $\square$

APPENDIX

5. First we introduce the following classical extension lemma. A proof can be effected as in Troianiello [**19**, Proof of Lemma 1.5, page 16].

Lemma 5.1. *Let $n, s \in \mathbf{N}$, $1 \leq s < n$. Let $m \in \mathbf{N}$, $\alpha \in]0,1]$. Let $\mathbf{M}$ be a compact manifold of class $C^{\max\{1,m\},\alpha}$ embedded into $\mathbf{R}^n$*

of dimension s. Let Ω be a bounded open subset of $\mathbf{R}^n$ containing $\mathbf{M}$. Then there exists a linear and continuous extension operator $\mathbf{E}$ from $C^{m,\alpha}(\mathbf{M})$ to $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ such that the support of $\mathbf{E}[\psi]$ is compact and contained in Ω for all $\psi \in C^{m,\alpha}(\mathbf{M})$.

Then we introduce the following slight variant of Preciso [**15**, Proposition 4.2.16, page 51] [**16**, Proposition 1.1, page 101] on the real analyticity of a composition operator. See also [**8**, Proposition 2.17, Remark 2.19] and the slight variant of the argument of Preciso of the proof of [**10**, Proposition 9, page 214].

Proposition 5.2. *Let $h, k \in \mathbf{N} \setminus \{0\}$, $m \in \mathbf{N}$. Let $\alpha \in]0,1]$, $\rho > 0$. Let Ω, Ω' be bounded open connected subsets of $\mathbf{R}^h$ and $\mathbf{R}^k$, respectively. Let Ω' be regular in the sense of Whitney. Then the operator T defined by*

$$T[\zeta, \psi] \equiv \zeta \circ \psi$$

for all $(\zeta, \psi) \in C^0_{\omega,\rho}(\mathrm{cl}\,\Omega) \times C^{m,\alpha}(\mathrm{cl}\,\Omega', \Omega)$ is real analytic from the open subset $C^0_{\omega,\rho}(\mathrm{cl}\,\Omega) \times C^{m,\alpha}(\mathrm{cl}\,\Omega', \Omega)$ of $C^0_{\omega,\rho}(\mathrm{cl}\,\Omega) \times C^{m,\alpha}(\mathrm{cl}\,\Omega', \mathbf{R}^h)$ to $C^{m,\alpha}(\mathrm{cl}\,\Omega')$.

REFERENCES

1. R. Böhme and F. Tomi, *Zur Struktur der Lösungsmenge des Plateauproblems*, Math. Z. **133** (1973), 1–29.

2. H. Cartan, *Differential calculus*, Hermann, Paris; translated from the French: Houghton Mifflin Co., Boston, Mass., 1971.

3. M. Dalla Riva and M.Lanza de Cristoforis, *A perturbation result for the layer potentials of general second order differential operators with constant coefficients*, J. Appl. Funct. Anal. **5** (2010), 10–30.

4. K. Deimling, *Nonlinear functional analysis*, Springer-Verlag, Berlin, 1985.

5. D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 1983.

6. D. Henry, *Topics in nonlinear analysis*, Trabalho de Matemática **192**, Brasilia, 1982.

7. M. Lanza de Cristoforis, *Properties and pathologies of the composition and inversion operators in Schauder spaces*, Acc. Naz. delle Sci. detta dei XL **15** (1991), 93–109.

8. ———, *Differentiability properties of a composition operator*, Rend. Circ. Mat. Palermo **56** (1998), 157–165.

9. M. Lanza de Cristoforis, *Asymptotic behavior of the solutions of a nonlinear Robin problem for the Laplace operator in a domain with a small hole. A functional analytic approach*, Complex Var. Elliptic Equat. **52** (2007), 945–977.

10. ———, *Perturbation problems in potential theory, A functional analytic approach*, J. Appl. Funct. Anal. **2** (2007), 197–222.

11. ———, *Asymptotic behavior of the solutions of a nonlinear transmission problem for the Laplace operator in a domain with a small hole. A functional analytic approach*, Complex Var. Elliptic Equat. **55** (2010), 269–303.

12. M. Lanza de Cristoforis and P. Musolino, *A perturbation result for periodic layer potentials of general second order differential operators with constant coefficients*, Far East J. Math. Sci. **52** (2011), 75–120.

13. M. Lanza de Cristoforis and L. Rossi, *Real analytic dependence of simple and double layer potentials upon perturbation of the support and of the density*, J. Integral Equations Appl. **16** (2004), 137–174.

14. ———, *Real analytic dependence of simple and double layer potentials for the Helmholtz equation upon perturbation of the support and of the density*, Analytic methods of analysis and differential equations, A.A. Kilbas and S.V. Rogosin, eds., Cambridge Scientific Publishers, Cambridge (UK), 2008.

15. L. Preciso, *Perturbation analysis of the conformal sewing problem and related problems*, Ph.D. dissertation, University of Padova, 1998.

16. ———, *Regularity of the composition and of the inversion operator and perturbation analysis of the conformal sewing problem in Roumieu type spaces*, Nat. Acad. Sci. Belarus, Proc. Inst. Math. **5** (2000), 99–104.

17. G. Prodi and A. Ambrosetti, *Analisi non lineare*, Editrice Tecnico Scientifica, Pisa, 1973.

18. C. Roumieu, *Sur quelques extensions de la notion de distribution*, Ann. Sci. École Norm. Sup. **77** (1960), 41–121.

19. G.M. Troianiello, *Elliptic differential equations and obstacle problems*, Plenum Press, New York, 1987.

20. T. Valent, *Boundary value problems of finite elasticity. Local theorems on existence, uniqueness and analytic dependence on data*, Springer-Verlag, New York, 1988.

Department of Mathematics, Università degli studi di Padova, Via Trieste 63, 35121 Padova, Italy
Email address: mldc@math.unipd.it

Institut de Recherche Mathématiques de Rennes (IRMAR), Université de Rennes 1, 263 Avenue du General Leclerc, 35042 Rennes, France
Email address: musolinopaolo@gmail.com