JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 22, Number 3, Fall 2010

KERNEL PERTURBATIONS FOR VOLTERRA CONVOLUTION INTEGRAL EQUATIONS

F.R. DE HOOG AND R.S. ANDERSSEN

Communicated by Kendall Atkinson

Dedicated to Chuck Groetsch

ABSTRACT. This paper investigates how errors in the kernels of Volterra convolution integral equations affect their solutions. The situation is examined for both first and second kind equations with smooth kernels. Various error estimates are derived through the use of an analogue of the interconversion equation of linear viscoelasticity.

1. Introduction. In the analysis of causal processes, the first and second kind convolution Volterra equations

$$(1)\quad (k * y)(t) = \int_0^t k(t-s)y(s)ds = f(t), \qquad f(0) = 0,\ 0 \le t \le \infty,$$

and

$$(2)\quad \begin{aligned}(\lambda y + K * y)(t) = \lambda y(t) + \int_0^t K(t-s)y(s)ds = F(t),\\ 0 \le t \le \infty, \quad \lambda \neq 0,\end{aligned}$$

commonly arise as the appropriate models. In such applications, either the kernel k (K and λ for the second kind form) and the inhomogeneous term (output) f (F) are given with the solution (input) y to be determined or the input y and the output f (F) are given with the kernel k (K, for a given λ for the second kind form) to be determined. Representative examples include, for the first kind equations, the modeling of viscoelastic processes [**7, 11**], exponential forgetting [**4, 21**], hydrological applications (Jakeman and Young in

Received by the editors on September 20, 2009, and in revised form on February 16, 2010.

DOI:10.1216/JIE-2010-22-3-427

[**5, 22**]). Further, applications are discussed in [**15**, Chapter 2] and in Groetsch [**10, 11**]. For second kind equations, applications include the renewal equations ([**14**, Chapter 2] and [**6, 7**]).

In this paper, the effect of perturbations in the kernel k on the input y and the output f are analyzed. The results appear to be new but are limited to situations where the kernel k is positive, monotone and $\lim_{t\to\infty} k(t) > 0$. This, for example, holds for the Boltzmann model for the stress-strain response of a linear viscoelastic solids, and the defect version of the renewal equation when rewritten in its equivalent first kind form.

Traditionally, as explained in Linz [**15, 16**], stability results, in various forms, have been derived for first kind Volterra equations by initially differentiating them to obtain their second kind equivalent (assuming that $k(0) \neq 0$) and then using the second kind form directly or via their resolvent kernel solution. Here, the process is reversed and the effect of perturbations in the kernel k are derived for first kind convolution Volterra equations directly. Second kind Volterra equation (2) can be rewritten in an equivalent first kind form (1) using the transformations

$$(3) \qquad k(t) = \lambda + \int_0^t K(\eta)d\eta, \qquad f(t) = \int_0^t F(s)ds, \quad \lambda = k(0).$$

For the analysis of integral equations with difference kernels defined on finite intervals, a related strategy (of analyzing second kind equations in terms of their first kind equivalents) is the basis for the deliberations of Sakhnovich [**20**]. However, the analysis developed below, because it is based on a generalization of the interconversion equation of rheology, as examined in [**1, 2, 3**], is both new and independent from that found in [**20**].

Understanding the effect on the solution of a first kind convolution Volterra equation of perturbations in its kernel is important from both a theoretical and practical perspective. There exist situations, in rheology and river flow modeling, where the explicit structure of the kernel k is not defined by the problem context, but is chosen on the basis of heuristic or experimental considerations using partial least squares or neural network methodologies. It is then necessary to understand the effect of model errors as reflected in perturbations in the kernel k.

In linear viscoelasticity, because of its analytic structure, the interconversion relationships [**7, 11, 12**]

$$(4)\qquad \begin{aligned}(G * J)(t) &= \int_0^t G(t-s)J(s)\,ds = t,\\ (J * G)(t) &= \int_0^t J(t-s)G(s)\,ds = t,\end{aligned}$$

where $G(t)$ and $J(t)$ denote, respectively, the relaxation and creep (retardation) moduli, have been utilized by various authors [**1**–**3, 16, 17**] to analyze rheological processes. The importance of the interconversion equation relates to the fact that, in a rheology laboratory, it is necessary to determine both $G(t)$ and $J(t)$ for the molecular characterization of a viscoelastic material [**18**]. Through the utilization of (4), it is only necessary to go to the time and expense of measuring either $G(t)$ or $J(t)$ and determining the other by solving (4). A summary of the historical background to this problem can be found in [**1, 2, 3**]. This leads naturally to the question "What is the optimal strategy: measure $G(t)$ and solve (4) for $J(t)$ or measure $J(t)$ and solve (4) for $G(t)$?". It has been known, on the basis of the numerical experimentation of Mead [**16**], that the creep-to-relaxation interconversion going from $J(t)$ to $G(t)$ tends to be unstable, while the relaxation-to-creep interconversion going from $G(t)$ to $J(t)$ is less unstable. Though it was anticipated that this observation was an essential feature of the interconversion relationship (4) rather than a numerical artefact of Mead's calculations [**15**], theoretical confirmation of this fact has only recently been published [**1, 2, 3**].

The purpose of this paper is to show how the application of this interconversion relationship, in resolving rheological stability issues [**1, 2, 3**], can be generalized for the analysis, with respect to kernel perturbations, of the stability of the solutions of first kind Volterra integral equations. In particular, if, for a given kernel k, there exists a function h such that

$$(5)\qquad k * h = \int_0^t k(t-s)h(s)\,ds = \int_0^t h(t-s)k(s)\,ds = h * k = t,$$

then:

Lemma 1. *From the convolution Volterra equation* (1) *and the relationship* (5), *it follows that*

$$y(t) = \frac{d^2}{dt^2}\,(h * f)\,. \tag{6}$$

Proof. Forming the convolution of h with equation (1) yields

$$h * k * y = k * h * y = t * y = h * f.$$

Differentiation of $t * y = h * f$ twice with respect to t yields the required result. □

This is the key relationship that will be utilized in the sequel.

An interesting feature of equation (5) is that its structure is identical to that of the interconversion relationship (4). However, for a given k, it is not necessary to know the explicit form for h, but only that there exists a unique h which satisfies (5) with appropriate properties. The formal solution for h is given by

$$h(t) = \mathcal{L}^{-1}\left(\frac{\mathcal{L}[t]}{\mathcal{L}[k(t)]}\right),$$

where $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote, respectively, the Laplace transform and its inverse. For a given $k(t)$, asymptotic relationships between h and $\mathcal{L}[h(t)]$ can be identified using appropriate Tauberian theorems (e.g., [**23**, Theorem 5.2] in Zayed). However, in the sequel, the special structure of (5) is exploited directly to derive the required asymptotic behavior of $k(t)$ and $h(t)$.

Because it allows an explicit identity for the solution of the first kind Volterra equation (1) to be derived, the relationship (5) plays a role similar to the resolvent kernel relationship for second kind Volterra equations. For the second kind equation (2), the resolvent kernel $R(t)$ satisfies

$$\lambda^2 R + \lambda K * R = -K, \tag{7}$$

and the solution of (2) is given by

$$y = \lambda^{-1} F + R * F.$$

Using the transformation (3), which identifies the first kind counterpart (1) of the second kind equation (2), it follows that $K = k'$ and $k(0) = \lambda$. The differentiation of (5) twice, after using these last two relationships and multiplying through with λ, yields

$$\lambda^2 h' + K + h' * K = 0,$$

which, when compared with the resolvent equation (7), establishes that $h' = R$. For the first kind equations, Gripenberg et al. [**8**] introduced the first kind resolvent g satisfying

$$k * g = 1.$$

Since, under appropriate regularity, the differentiation of (5) yields $h(0)k(t) + k * h' = 1$, it follows that $g(t) = h(0)\delta(t) + h'$. However, the analysis and results given below are substantially different in structure and content from the material in [**8**].

In the sequel, it will be shown how the identity (6) can be used to derive various stability results which highlight the effect of perturbations in the kernel k of equation (1). The paper has been organized in the following manner. Preliminaries are given in Section 2. The stability results are stated and proved in Section 3.

2. Preliminaries. The dash superscript will be regularly used to denote differentiation with respect to t; namely, $a' = da/dt$.

2.1 Basic properties of Volterra convolutions. In this subsection, it is assumed that the appropriate regularity conditions hold which guarantee the validity of the manipulations performed. In the remainder of the paper, they will be stated explicitly.

The following lemmas, for double convolutions, form the basis for the proofs of the stability results given in Section 3.

Lemma 2.

$$\frac{d}{dt}((a * b * c)(t)) = a(0)(b * c)(t) + a' * b * c(t), \tag{8}$$

and

$$\frac{d^2}{dt^2}((a*b*c)(t)) = a(0)b(0)c(t) + a(0)(b'*c)(t) + b(0)(a'*c)(t) + (a'*b'*c)(t). \tag{9}$$

Proof. Equation (8) is an immediate consequence of differentiating a first kind Volterra convolution equation where a is taken to be the kernel and $b*c$ the solution. Exploiting the fact that $a*b = b*a$ and again using the mentioned differentiation procedure, equation (9) follows. □

Lemma 3.

$$|a*b*c| \le (|a|*1)(|b|*1)\|c\|_\infty, \tag{10}$$

$$\left|\frac{d}{dt}(a*b*c)\right| \le (|a(0)| + |a'|*1)(|b|*1)\|c\|_\infty, \tag{11}$$

and

$$\left|\frac{d^2}{dt^2}(a*b*c)\right| \le (|a(0)| + |a'|*1)(|b(0)| + |b'|*1)\|c\|_\infty. \tag{12}$$

Proof. The inequality (10) follows from recalling the fact that $|a*b*c| \le |a*b*1| \cdot \|c\|_\infty$ and then applying

$$|a*b*1| \le (|a|*1)(|b|*1).$$

The inequalities (11) and (12) are a consequence of applying this procedure to the first and second derivative relationships in Lemma 2. □

2.2. Monotonicity properties of k and h. In deriving the stability estimates in the next section of the paper, the monotonicity properties of k and h will play a key role.

Lemma 4. *Let $k(t) \in C^1[0,\infty)$, $k > 0$, $k' \le 0$, $k(\infty) > 0$. Then, $h(t) \in C^1[0,\infty)$, $h > 0$, $h' > 0$ with $k(0)h(0) = 1$, $k(t)h(t) \le 1$ and $k(\infty)h(\infty) = 1$.*

Proof. The differentiation of (5) with respect to t gives

$$k(0)h(t) = 1 - \int_0^t k'(t-s)h(s)\,ds = 1 - \int_0^t h(t-s)k'(s)ds, \qquad 0 \le t < \infty. \tag{13}$$

It follows from equation (13) that h starts out being positive. If it is assumed that it goes negative, then, on continuity grounds, it must first go to zero. If this is assumed to happen at $t = t_0$, then

$$1 = \int_0^{t_0} h(t_0 - s)k'(s)\,ds,$$

which is a contradiction. Differentiation of the relationship (13) then gives

$$k(0)h'(t) = -h(0)k'(t) - \int_0^t h'(t-s)k'(s)\,ds, \qquad 0 \le t < \infty. \tag{14}$$

The same logic above that established the positivity of h can now be used to establish the positivity of $h'(t)$ and thereby prove the positive monotone increasing behavior of $h(t)$. As a consequence of the increasing positive behavior of $h(t)$ and the fact that $k' \le 0$, it follows from (13) that

$$1 = k(0)h(t) + k' * h \ge k(0)h(t) + h(t)\int_0^t k'(s)\,ds = h(t)k(t).$$

Hence, $h(t) \le 1/k(t) < 1/k(\infty)$ is bounded. This, along with the monotonicity of $h(t)$, establishes that the limit, as $t \to \infty$, of $h(t)$ exists. Integration of (14) yields

$$\begin{aligned}\int_0^\infty (k(0)h'(\tau) + h(0)k'(\tau) + k' * h'(\tau))\,d\tau \\ = k(0)\int_0^\infty h'(\tau)\,d\tau + h(0)\int_0^\infty k'(\tau)\,d\tau \\ + \int_0^\infty h'(\tau)\,d\tau \int_0^\infty k'(\tau)\,d\tau = 0,\end{aligned}$$

which simplifies to yield $k(\infty)h(\infty) = 1$. □

Note. Using $h(0)k(0) = 1$, the relationship (14) can be rewritten as

$$k(0)^2 h'(t) + k(0) \int_0^t h'(t-s)k'(s)\,ds = -k'(t),$$

which corresponds to the resolvent kernel relationship (7) with $k' = K$ and $h' = R$. □

However, when k is positive increasing, it does not follow that h is positive and decreasing. In fact, the simple choice of $k(t) = 1+2S(t-1)$, with $S(t)$ denoting the Unit Step (Heaviside) function, is indicative of what can occur, since, because $k'(t) = 2\delta(t-1)$, with $\delta(t)$ denoting the Dirac delta function, it follows that $h(t) = 1 - 2S(t-1)$. Because of the time delay $t-1$ on the right hand side compared with t on the left, $h(t)$ switches between the values ± 1 as t increases. A smooth exemplification of this result can be constructed using $k(t) = 2 + \tanh((t-1)/\varepsilon)$ since, as $\varepsilon \to 0$, the simple Heaviside Unit Step function choice is recovered.

Consequently, some suitable regularity must be imposed on the growth of the kernel k in order to ensure that h is decreasing. For example, using the results contained in Chapter VIII of Gross [**11**] (cf. [**1**]), it can be shown that, if

$$k(t) = k(\infty) - \int_0^\infty \exp(-t/\tau)\frac{L(\tau)}{\tau}\,d\tau, \quad L(t) \geq 0,$$

then there exists a $G(t) \geq 0$ such that

$$h(t) = h(\infty) + \int_0^\infty \exp(-t/\tau)\frac{G(\tau)}{\tau}\,d\tau,$$

and $k * h = t$, which guarantees that h is positive and decreasing. In fact, more generally, the complete monotonicity of $k'(t)$ guarantees that $h(t)$ is completely monotone [**15**].

2.3. Other properties of k and h. For the estimates derived in Section 3, bounds are required for $\|h'\|_1$ and $h(0) + \|h'\|_1$ in terms of k.

Lemma 5. *For positive decreasing $k \in C^1[0,\infty)$ and $k(\infty) > 0$,*

$$h(0) + \|h'\|_1 = \frac{1}{k(\infty)}, \qquad \|h'\|_\infty \leq \frac{\|k'\|_\infty}{k(0)k(\infty)}. \tag{15}$$

Proof. The first result is an immediate consequence of the monotonically increasing behavior of $h(t)$ and $h(\infty)k(\infty) = 1$ of Lemma 4 and

$$\|h'\|_1 = h(\infty) - h(0).$$

The second result depends on first noting that taking the L_∞-norm of equation (14) gives

$$k(0)\|h'\|_\infty \le h(0)\|k'\|_\infty + \|k'\|_\infty\|h'\|_1.$$

The required result follows on using the above value of $\|h'\|_1$ and then rearranging the terms in the resulting relationship. □

Lemma 6. *For positive decreasing* $h \in C^1[0,\infty)$ *and* $h(\infty) > 0$,

$$h(0) + \|h'\|_1 < \frac{2}{k(0)}. \tag{16}$$

Proof. It follows from Lemma 4, interchanging the roles of h and k, that k is positive and increasing. Since, for a positive monotonically decreasing $h(t)$,

$$\|h'\|_1 = h(0) - h(\infty),$$

the result follows on noting that

$$h(0) + \|h'\|_1 = 2h(0) - h(\infty) = \frac{2}{k(0)} - \frac{1}{k(\infty)} < \frac{2}{k(0)}. \qquad \square$$

3. Stability results. As noted in the Introduction, the goal of this paper is a study of the effect on the solution of the first kind Volterra convolution integral equation (1) of perturbations in the kernel k and the right hand side f. The estimates are given in the following six theorems.

3.1. Consequences of perturbing the kernel. Corresponding to equation (1), the effect of kernel perturbed on its solution is assumed to take the form

$$(k + \gamma) * (y + \varepsilon) = f, \tag{17}$$

from which it follows that

$$k * \varepsilon = -y * \gamma - \varepsilon * \gamma. \tag{18}$$

Applying Lemma 1 to this last relationship yields

$$\varepsilon = -\frac{d^2}{dt^2}((h * y) * \gamma)) - \frac{d^2}{dt^2}((h * \gamma) * \varepsilon)). \tag{19}$$

The first two stability results are derived for the situation where the kernel k is positive and decreasing.

Theorem 1. *For positive decreasing $k \in C^1[0,\infty)$ and $k(\infty) > 0$, with $\zeta < 1$,*

$$\begin{aligned} \|\varepsilon\|_\infty &\le \frac{\beta}{1-\zeta}\|\gamma\|_\infty, \\ \beta &= \frac{|y(0)| + \|y'\|_1}{k(\infty)}, \\ \zeta &= \frac{|\gamma(0)| + \|\gamma'\|_1}{k(\infty)}. \end{aligned} \tag{20}$$

Proof. From (19), it follows, on using equation (12) of Lemma 3, that

$$\begin{aligned} |\varepsilon| \le\ & (h(0) + |h'| * 1)(|y(0)| + |y'| * 1)\|\gamma\|_\infty \\ & + (h(0) + |h'| * 1)(|\gamma(0)| + |\gamma'| * 1)\|\varepsilon\|_\infty, \end{aligned}$$

and, hence, on recalling that the maximum value that $|f| * 1$ can have is $\|f\|_1$,

$$\begin{aligned} \|\varepsilon\|_\infty \le\ & (h(0) + \|h'\|_1)(|y(0)| + \|y'\|_1)\|\gamma\|_\infty \\ & + (h(0) + \|h'\|_1)(|\gamma(0)| + \|\gamma'\|_1)\|\varepsilon\|_\infty. \end{aligned}$$

Using Lemma 5 then gives

$$(1-\zeta)\|\varepsilon\|_\infty \le \beta\|y\|_\infty,$$

from which (20) is an immediate consequence. □

The importance of this result is that the bound on the perturbation $\|\varepsilon\|_\infty$ in the solution resulting from a perturbation $\|\gamma\|_\infty$ applied to the kernel k is given in terms of the maximum value $\|\gamma\|_\infty$ of γ. This appears to be new.

On rewriting equation (19) in the alternative form

$$\varepsilon = -\frac{d^2}{dt^2}((h * \gamma) * y) - \frac{d^2}{dt^2}((h * \gamma) * \varepsilon), \tag{21}$$

and, using the same logic as for the proof of Lemma 5 and Theorem 1, one obtains

Theorem 2. *For positive decreasing $k \in C^1[0,\infty)$ and $k(\infty) > 0$, with $\zeta < 1$,*

$$\|\varepsilon\|_\infty \le \frac{\zeta}{1-\zeta}\|y\|_\infty, \tag{22}$$
$$\zeta = \frac{|\gamma(0)| + \|\gamma'\|_1}{k(\infty)}.$$

This estimate takes the more traditional form where the bound on the error $\|\varepsilon\|_\infty$ in the solution y is given in terms of an estimate of the maximum size of the solution $\|y\|_\infty$.

Both estimates (20) and (22) are useful depending on the circumstances. The estimate (20) is the appropriate choice when the perturbation γ in the kernel k is large compared with the size of the solution y, whereas (22) is the appropriate choice in the more commonly occurring situation where the perturbation in k is small, especially when compared with the size of y.

The counterparts of Theorems 1 and 2 for positive increasing $k(t)$ and positive decreasing $h(t)$ become, using Lemma 6,

Theorem 3. *For positive decreasing $h \in C^1[0,\infty)$ and $h(\infty) > 0$, with $\theta < 1$,*

$$\|\varepsilon\|_\infty \le \frac{\eta}{1-\theta}\|\gamma\|_\infty,$$
$$\eta = \frac{2(|y(0)| + \|y'\|_1)}{k(0)},$$
$$\theta = \frac{2(|\gamma(0)| + \|\gamma'\|_1)}{k(0)}.$$

Theorem 4. *For positive decreasing* $h \in C^1[0, \infty)$ *and* $h(\infty) > 0$, *with* $\theta < 1$,

$$\|\varepsilon\|_\infty \leq \frac{\theta}{1-\theta}\|y\|_\infty,$$
$$\theta = \frac{2(|\gamma(0)| + \|\gamma'\|_1)}{k(0)}.$$

3.2. Consequences of perturbing the non-homogeneous term f**.** Corresponding to equation (1), because of its linearity, the effect on the solution y of perturbing the right hand side f reduces to an examination of the effect of f on y

$$k * y = f. \tag{23}$$

Applying Lemma 1 to this last relationship yields

$$\begin{aligned} y &= \frac{d^2}{dt^2}(h * f) = h(0)f'(t) + h' * f', \\ |y| &= |\frac{d^2}{dt^2}(h * f)| \leq (h(0) + |h'| * 1)\|f'\|_\infty. \end{aligned} \tag{24}$$

Applying to equation (24) similar logic as utilized in the above proofs, it follows that

Theorem 5. *For positive decreasing* $k \in C^1[0, \infty)$ *and* $k(\infty) > 0$,

$$\|y\|_\infty \leq \frac{1}{k(\infty)}\left(\|f'\|_\infty\right).$$

Proof. Taking the L_∞-norm of equation (24) gives

$$\begin{aligned} \|y\|_\infty &\leq (h(0) + \|h'\|_1)\|f'\|_\infty, \\ &\leq h(\infty)\|f'\|_\infty = \frac{\|f'\|_\infty}{k(\infty)}. \end{aligned} \tag{25}$$

The required inequality now follows using Lemma 5. □

Theorem 6. *For positive decreasing $h \in C^1[0,\infty)$ and $h(\infty) > 0$,*

$$\|y\|_\infty \le \frac{2}{k(0)}\|f'\|_\infty.$$

Proof. Again, the starting point is the relationship

$$\begin{aligned} \|y\|_\infty &\le (h(0) + \|h'\|_1)\|f'\|_\infty, \\ &\le (2h(0) - h(\infty))\,\|f'\|_\infty. \end{aligned} \tag{26}$$

The required result now follows from Lemma 6. □

Both the above $\|y\|_\infty$ estimates require knowledge about both $\|k'\|_\infty$ and $\|f'\|_\infty$, which reflects the level of improperly posedness associated with first kind Volterra equations.

3.3. Extension to second kind equations. As explained in the Introduction, the second kind equation (2) can be rewritten in first kind form $k * y = f$ with k and f defined in equation (3). In terms of the notation of equation (3), let the perturbed form of (2) be

$$\lambda(y + \varepsilon) + (K + \Gamma) * (y + \varepsilon) = F.$$

This is equivalent to the perturbed form of equation (17) applied to equation (2) in conjunction with (3) and

$$\gamma(t) = \int_0^t \Gamma(\tau)\,d\tau.$$

Thus,

$$\|\gamma\|_\infty \le \|\Gamma\|_1, \qquad \|\gamma'\|_1 = \|\Gamma\|_1.$$

Consequently, the stability results given in Theorems 1, 2 and 5 hold for $K \in C[0,\infty)$, $K \le 0$ and $k(\infty) = \lambda + \int_0^\infty K(\tau)\,d\tau$. Because the solution y and the perturbation γ remain the same for the equivalent form of $k * y = f$, the estimates for $\|\varepsilon\|_\infty$ of Theorems 1 and 2, and for $\|y\|_\infty$ of Theorem 5, become the corresponding estimates for the second kind equations.

For example, using the above results, the estimates for $\|\varepsilon\|_\infty$ of Theorems 1 and 2 become, respectively,

$$\|\varepsilon\|_\infty \le \frac{|y(0)| + \|y'\|_1}{\lambda + \int_0^\infty K(\tau)\, d\tau - \|\Gamma\|_1} \|\Gamma\|_1,$$

and

$$\|\varepsilon\|_\infty \le \frac{\|\Gamma\|_1}{\lambda + \int_0^\infty K(\tau)\, d\tau - \|\Gamma\|_1} \|y\|_\infty.$$

4. Concluding remarks. The linear viscoelastic convolution interconversion relationship of equation (4) quantifies analytically the relationship between the relaxation and retardation moduli $G(t)$ and $J(t)$, respectively. On assuming that such a relationship holds more generally for a first kind convolution Volterra equation $k * y = f$ and takes the form $k * h = t$, it has been established that this generalization of the interconversion relationship represents, for Volterra equations, a first kind counterpart of the resolvent kernel relationship for second kind equations. It allows $k * y = f$ to be solved analytically (Lemma 1), monotonic properties of and relationships between k and h to be derived and various stability estimates to be established. In particular, new estimates for the effect on the solution y of perturbations in the kernel k are derived. In addition, it is shown how the corresponding perturbation analysis for the second kind equations can be derived as a special case of the first kind equation results.

Acknowledgments. Both authors wish to acknowledge with sincere thanks the usefulness of discussions that they have had with Russell Davies and Rick Loy before and during the preparation of this paper. The advice of the referees is acknowledged.

REFERENCES

1. R.S. Anderssen, A.R. Davies and F.R. de Hoog, *On the interconversion integral equation for relaxation and creep*, ANZIAM J. **48** (2007), C249–C266.

2. ———, *On the sensitivity of interconversion between relaxation and creep*, Rheologica Acta **47** (2008), 159–167.

3. ———, *On the Volterra integral equation relating creep and relaxation*, Inverse Problems **24** (2008), 035009 (p. 13).

4. R.A. Berk, *Statistical learning from a regression perspective*, Springer, New York, 2008.

5. F.R. de Hoog, R.S. Anderssen and M.A. Lukas, *The application and numerical solution of integral equations*, Sijthoff & Noordhoff, The Netherlands, 1980.

6. W. Feller, *An introduction to probability theory and its applications*, Volume II, John Wiley & Sons, New York, 1966.

7. J. Ferry, *Viscoelastic properties of polymers*, Wiley, New York, 1980.

8. G. Gripenberg, S.-O. Londen and O. Staffans, *Volterra integral and functional equations*, Encycl. Math. Appl. **34**, Cambridge University Press, Cambridge, 1990.

9. C.W. Groetsch, *A simple numerical model for nonlinear warming of a slab*, J. Comp. Appl. Math. **38** (1990), 149–156.

10. ———, *Integral equations of the first kind, inverse problems and regularization: A crash course*, J. Phys: Conference Series **73** (2007), 012001.

11. B. Gross, *Mathematical structure of the theories of viscoelasticity*, Hermann, Paris, 1953.

12. I. L. Hopkins and R. W. Hamming. On creep and relaxation. J. Appl. Phys., 28:906909, 1957.

13. P. Linz, *A survey of methods for the solution of Volterra equations of the first kind*, in *The application and numerical solution of integral equations*, R.S. Anderssen, F.R. de Hoog and M.A. Lukas, eds., 1980.

14. ———, *Analytic and numerical methods for volterra equations*, SIAM Studies in Applied Mathematics, Philadelphia, 1985.

15. R.J. Loy and R.S. Anderssen, *Linear viscoelastic interconversion relationships*, in preparation.

16. D.W. Mead, *Numerical interconversion of linear viscoelastic material functions*, J. Rheol. **38** (1994), 1769–1795.

17. A. Nikonov, A.R. Davies and I. Emri, *The determination of creep and relaxation functions from a single experiment*, J. Rheol. **49** (2005), 1193–1211.

18. D.J. Plazek and I. Echeverria, *Don't cry for me Charlie Bbrown, or with compliance comes comprehension*, J. Rheol. **44** (2000), 831–841.

19. G. Psarrakos, *Asymptotic results for heavy-tailed distributions using defective renewal equations*, Stat. Prob. Lett. **79** (2009), 774–779.

20. L.A. Sakhnovich, *Integral equations with difference kernels on finite intervals*, Birkhauser Verlag, Berlin, 1996.

21. J.E. Slotine and W. Li, *Applied nonlinear control*, Prentice Hall, Englewood Cliffs, New Jersey, 1991.

22. N. Su, F. Liu and J. Barringer, *Forecasting floods at variable catchment scales using stochastics systems hydro-geographic models and digital topographic data*, Proc. MODSIM 1997, pages 16951700.

23. A.I. Zayed, *Handbook of function and generalized function transformations*, CRC Press, Boca Raton, 1996.

CSIRO Mathematics, Informatics and Statistics, GPO Box 664, Canberra ACT 2601, Australia
Email address: Frank.Dehoog@csiro.au

CSIRO Mathematics, Informatics and Statistics, GPO Box 664, Canberra ACT 2601, Australia
Email address: Bob.Anderssen@csiro.au