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CHARACTERIZATIONS OF REGULAR LOCAL RINGS VIA SYZYGY MODULES OF THE RESIDUE FIELD

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ABSTRACT. Let R be a commutative Noetherian local ring with residue field k. We show that, if a finite direct sum of syzygy modules of k maps onto 'a semidualizing module' or 'a non-zero maximal Cohen-Macaulay module of finite injective dimension,' then R is regular. We also prove that R is regular if and only if some syzygy module of k has a non-zero direct summand of finite injective dimension.

1. Introduction. Throughout this article, unless otherwise specified, all rings are assumed to be commutative Noetherian local rings, and all modules are assumed to be finitely generated. In this article, *R always denotes a local ring with maximal ideal $\mathfrak{m}$ and residue field k*. For every integer $n \geq 0$, we denote the nth *syzygy module* of k by $\Omega_n^R(k)$. Dutta gave the following characterization of regular local rings.

Theorem 1.1 (Dutta [**3,** Corollary 1.3])**.** *The ring R is regular if and only if $\Omega_n^R(k)$ has a non-zero free direct summand for some $n \geq 0$.*

Later, Takahashi generalized Dutta's result by giving a characterization of regular local rings via the existence of a semidualizing direct summand of some syzygy module of the residue field. Next, we recall the definition of a semidualizing module.

Definition 1.2 ([**4**])**.** An R-module M is said to be a *semidualizing module* if the following hold:

(i) the natural homomorphism $R \to \operatorname{Hom}_R(M, M)$ is an isomorphism;
(ii) $\operatorname{Ext}_R^i(M, M) = 0$ for all $i \geq 1$.

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Note that R, itself, is a semidualizing R-module. Thus, the following theorem generalizes the above result of Dutta.

Theorem 1.3 ([**9,** Theorem 4.3])**.** *The ring R is regular if and only if $\Omega_n^R(k)$ has a semidualizing direct summand for some $n \geq 0$.*

If R is a Cohen-Macaulay local ring with canonical module ω, then ω is a semidualizing R-module. Therefore, as an application of Theorem 1.3, Takahashi obtained the following:

Corollary 1.4 ([**9,** Corollary 4.4])**.** *Let R be a Cohen-Macaulay local ring with canonical module ω. Then, R is regular if and only if $\Omega_n^R(k)$ has a direct summand isomorphic to ω for some $n \geq 0$.*

Now, recall that the canonical module (if it exists) over a Cohen-Macaulay local ring has a finite injective dimension. In addition, it is well known that R is regular if and only if k has finite injective dimension. Hence, in this direction, a natural question arises, "if $\Omega_n^R(k)$ has a non-zero direct summand of finite injective dimension for some $n \geq 0$, then is R regular?" In the present study, we see that this question has an affirmative answer.

Kaplansky conjectured that, if some power of the maximal ideal of R is non-zero and of finite projective dimension, then R is regular. Levin and Vasconcelos proved this conjecture [**6,** Theorem 1.1]. In fact, their result is even stronger:

Theorem 1.5. *If M is an R-module such that $\mathfrak{m}M$ is non-zero and of finite projective dimension* (*or of finite injective dimension*)*, then R is regular.*

Motivated by this theorem, Martsinkovsky [**7**] generalized Dutta's result in the following direction.

Theorem 1.6 ([**7,** Proposition 7])**.** *If a finite direct sum of syzygy modules of k maps onto a non-zero R-module of finite projective dimension, then R is regular.*

For a stronger result, we refer the reader to [**1,** Corollary 9]. In this direction, we prove the following result, which considerably strengthens Theorem 1.3. The proof presented here is very simple and elementary.

Theorem I (see Corollary 3.2)**.** If a finite direct sum of syzygy modules of k maps onto a semidualizing R-module, then R is regular.

Furthermore, we raise the following question:

Question 1.7. *If a finite direct sum of syzygy modules of k maps onto a non-zero R-module of finite injective dimension, then is R regular?*

In this article, we give a partial answer to this question as follows:

Theorem II (see Corollary 3.4)**.** If a finite direct sum of syzygy modules of k maps onto a non-zero maximal Cohen-Macaulay R-module L of finite injective dimension, then R is regular.

If R is a Cohen-Macaulay local ring with canonical module ω, then one can take $L = \omega$ in the above theorem.

We obtain one new characterization of regular local rings. It follows from Dutta's result (Theorem 1.1) that R is regular if and only if some syzygy module of k has a non-zero direct summand of finite projective dimension. Here, we prove the following counterpart for the injective dimension.

Theorem III (see Theorem 3.7)**.** The ring R is regular if and only if some syzygy module of k has a non-zero direct summand of finite injective dimension.

Moreover, this result has a dual companion; see Corollary 3.8.

Up until now, we have considered surjective homomorphisms from a finite direct sum of syzygy modules of k to a 'special module.' It may be asked, "what happens if there is an injective homomorphism from a 'special module' to a finite direct sum of syzygy modules of k?" More precisely, if

$$f : L \longrightarrow \bigoplus_{n\in\Lambda} \left(\Omega_n^R(k)\right)^{j_n}$$

is an injective R-module homomorphism, where L is non-zero and of finite projective dimension (or of finite injective dimension), and Λ is a finite collection of non-negative integers, then is the ring R regular? In this situation, we show that R is not necessarily a regular local ring; see Example 3.9.

2. Preliminaries. In this section, we give some preliminaries which we use in order to prove our main results. We start with the following lemma, which gives a relation between the socle of the ring and the annihilator of the syzygy modules.

Lemma 2.1. *Let M be an R-module. Then*:

$$\mathrm{Soc}(R) \subseteq \mathrm{ann}_R\left(\Omega_n^R(M)\right) \quad \textit{for all } n \geq 1.$$

In particular, if $R \neq k$ (i.e., if $\mathfrak{m} \neq 0$), then

$$\mathrm{Soc}(R) \subseteq \mathrm{ann}_R\left(\Omega_n^R(k)\right) \quad \textit{for all } n \geq 0.$$

Proof. Fix $n \geq 1$. If $\Omega_n^R(M) = 0$, then we are done. Thus, we may assume that $\Omega_n^R(M) \neq 0$. Consider the following commutative diagram in the minimal free resolution of M:

$$\begin{tikzcd}
\cdots \arrow[r] & R^{b_n} \arrow[rr, "\delta"] \arrow[dr, two heads, "f"'] & & R^{b_{n-1}} \arrow[r] & \cdots . \\
& & \Omega_n^R(M) \arrow[ur, hook, "g"'] & &
\end{tikzcd}$$

Let $a \in \mathrm{Soc}(R)$, i.e., $a\mathfrak{m} = 0$. Suppose that $x \in \Omega_n^R(M)$. Since f is surjective, there exists a $y \in R^{b_n}$ such that $f(y) = x$. Note that $\delta(ay) = a\delta(y) = 0$ as

$$\delta(R^{b_n}) \subseteq \mathfrak{m}R^{b_{n-1}}$$

and $a\mathfrak{m} = 0$. Therefore,

$$g(ax) = g(f(ay)) = \delta(ay) = 0,$$

which gives $ax = 0$ since g is injective. Hence,

$$\mathrm{Soc}(R) \subseteq \mathrm{ann}_R\left(\Omega_n^R(M)\right) \quad \text{for all } n \geq 1.$$

For the last part, note that, if $\mathfrak{m} \neq 0$, then $\mathrm{Soc}(R) \subseteq \mathfrak{m} = \mathrm{ann}_R(\Omega_0^R(k))$. $\square$

We recall the following result initially obtained by Nagata.

Proposition 2.2 ([**9,** Corollary 5.3])**.** *Let $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ be an R-regular element. Set $\overline{(-)} := (-) \otimes_R R/(x)$. Then*:

$$\overline{\Omega_n^R(k)} \cong \Omega_n^{\overline{R}}(k) \oplus \Omega_{n-1}^{\overline{R}}(k) \quad \textit{for all } n \geq 1.$$

We note that two properties are satisfied by semidualizing modules and maximal Cohen-Macaulay modules of finite injective dimension.

Definition 2.3. Let $\mathcal{P}$ be a property of modules over local rings. We say that $\mathcal{P}$ is a $(*)$-property if $\mathcal{P}$ satisfies the following:

(i) an R-module M satisfies $\mathcal{P}$ implies that the $R/(x)$-module M/xM satisfies $\mathcal{P}$, where x is an R-regular element.
(ii) An R-module M satisfies $\mathcal{P}$ and $\mathrm{depth}(R) = 0$ together imply that $\mathrm{ann}_R(M) = 0$.

Now, we give a few examples of $(*)$-properties.

Example 2.4. The property $\mathcal{P}_1 :=$ 'semidualizing modules over local rings' is a $(*)$-property.

Proof. Let C be a semidualizing R-module. It is shown in [**4,** page 68] that C/xC is a semidualizing $R/(x)$-module, where x is an R-regular element. Since $\mathrm{Hom}_R(C, C) \cong R$, we have $\mathrm{ann}_R(C) = 0$ (without any restriction on $\mathrm{depth}(R)$). $\square$

Here is another example of the $(*)$-property.

Example 2.5. The property $\mathcal{P}_2 :=$ 'non-zero maximal Cohen-Macaulay modules of finite injective dimension over Cohen-Macaulay local rings' is a $(*)$-property.

Proof. Let R be a Cohen-Macaulay local ring, and let L be a non-zero maximal Cohen-Macaulay R-module of finite injective dimension. Suppose that x is an R-regular element. Since L is a maximal Cohen-Macaulay R-module, x is L-regular as well. Therefore, L/xL is a non-zero maximal Cohen-Macaulay module of finite injective dimension over the Cohen-Macaulay local ring $R/(x)$ (see, e.g., [**2,** 3.1.15]).

Now, further assume that $\operatorname{depth}(R) = 0$. Then, R is an Artinian local ring, and $\operatorname{injdim}_R(L) = \operatorname{depth}(R) = 0$. Hence, by [**2,** 3.2.8], we have that $L \cong E^r$, where E is the injective hull of k and $r = \operatorname{rank}_k(\operatorname{Hom}_R(k, L))$. It is well known that $\operatorname{Hom}_R(E, E) \cong R$ since R is an Artinian local ring. Therefore, $\operatorname{ann}_R(L) = \operatorname{ann}_R(E) = 0$. $\square$

3. Main results. Now, we are in a position to prove our main results. First, we prove that, if a finite direct sum of syzygy modules of the residue field maps onto a non-zero module satisfying a $(*)$-property, then the base ring is regular.

Theorem 3.1. *Let $\mathcal{P}$ be a $(*)$-property (see Definition 2.3). Suppose*

$$f : \bigoplus_{n\in\Lambda} \left(\Omega_n^R(k)\right)^{j_n} \longrightarrow L$$
$$(j_n \geq 1 \text{ for each } n \in \Lambda)$$

is a surjective R-module homomorphism, where L $(\neq 0)$ satisfies $\mathcal{P}$. Then, R is regular.

Proof. We prove Theorem 3.1 by using induction on $t := \operatorname{depth}(R)$. First, we assume that $t = 0$. In this case, we claim that $R = k$. If possible, assume that $R \neq k$, i.e., $\mathfrak{m} \neq 0$. Since $\operatorname{depth}(R) = 0$, we have $\operatorname{Soc}(R) \neq 0$. However, by virtue of Lemma 2.1, we obtain that

$$\begin{aligned}\operatorname{Soc}(R) \subseteq \bigcap_{n\in\Lambda} \operatorname{ann}_R\left(\Omega_n^R(k)\right) &= \operatorname{ann}_R\left(\bigoplus_{n\in\Lambda}\left(\Omega_n^R(k)\right)^{j_n}\right)\\ &\subseteq \operatorname{ann}_R(L) \quad \text{with} \quad f : \bigoplus_{n\in\Lambda}\left(\Omega_n^R(k)\right)^{j_n} \longrightarrow L \text{ being surjective}\\ &= 0 \quad \text{since } L \text{ satisfies } \mathcal{P}\text{, which is a } (*)\text{-property.}\end{aligned}$$

This yields a contradiction. Therefore, R $(= k)$ is a regular local ring.

Now, we assume that $t \geq 1$. Suppose the theorem holds true for all such rings of depth smaller than t. Since $\operatorname{depth}(R) \geq 1$, there exists an R-regular element $x \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$. We set $\overline{(-)} := (-) \otimes_R R/(x)$. Clearly,

$$\overline{f} : \bigoplus_{n \in \Lambda} \left(\overline{\Omega_n^R(k)}\right)^{j_n} \longrightarrow \overline{L}$$

is a surjective $\overline{R}$-module homomorphism, where the $\overline{R}$-module $\overline{L}$ $(\neq 0)$ satisfies $\mathcal{P}$ as $\mathcal{P}$ is a $(*)$-property. Since $x \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ is an R-regular element, in view of Proposition 2.2, we obtain that:

$$\bigoplus_{n \in \Lambda} \left(\overline{\Omega_n^R(k)}\right)^{j_n} \cong \bigoplus_{n \in \Lambda} \left(\Omega_n^{\overline{R}}(k) \oplus \Omega_{n-1}^{\overline{R}}(k)\right)^{j_n}$$

by setting $\Omega_{-1}^{\overline{R}}(k) := 0$. Since $\operatorname{depth}(\overline{R}) = t - 1$, by the induction hypothesis, we obtain that $\overline{R}$ is a regular local ring, and hence, R is a regular local ring as $x \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ is an R-regular element. □

As a few applications of Theorem 3.1, we obtain the following necessary and sufficient conditions for a local ring to be regular.

Corollary 3.2. *Suppose that*

$$f : \bigoplus_{n \in \Lambda} \left(\Omega_n^R(k)\right)^{j_n} \longrightarrow L$$

is a surjective R-module homomorphism, where L is a semidualizing R-module. Then, R is regular.

Proof. The corollary follows from Theorem 3.1 and Example 2.4. □

Remark 3.3. We can recover Theorem 1.3 (in particular, Theorem 1.1 since R itself is a semidualizing R-module) as a consequence of Corollary 3.2. In fact, the above result is even stronger than Theorem 1.3.

Now, we give a partial answer to Question 1.7.

Corollary 3.4. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring. Suppose that $f : \bigoplus_{n \in \Lambda} (\Omega_n^R(k))^{j_n} \to L$ is a surjective R-module homomorphism, where L $(\neq 0)$ is a maximal Cohen-Macaulay R-module of finite injective dimension. Then, R is regular.*

Proof. The corollary follows from Theorem 3.1 and Example 2.5. □

Remark 3.5. It is clear from the above corollary that Question 1.7 has an affirmative answer for Artinian local rings.

Let R be a Cohen-Macaulay local ring. Recall that a maximal Cohen-Macaulay R-module ω of type 1 and of finite injective dimension is called the *canonical module* of R. It is well known that the canonical module ω of R is a semidualizing R-module, see e.g., [**2,** 3.3.10]. Thus, both Corollary 3.2 and Corollary 3.4 yield the following result (independently) which strengthens Corollary 1.4.

Corollary 3.6. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring with canonical module ω, and let*

$$f : \bigoplus_{n\in\Lambda} \left(\Omega_n^R(k)\right)^{j_n} \longrightarrow \omega$$

be a surjective R-module homomorphism. Then, R is regular.

Here, we obtain one new characterization of regular local rings. The following characterization is based on the existence of a non-zero direct summand of finite injective dimension of some syzygy module of the residue field.

Theorem 3.7. *The following statements are equivalent*:

(i) *the ring R is regular*;
(ii) *Syzygy module $\Omega_n^R(k)$ has a non-zero direct summand of finite injective dimension for some $n \geq 0$.*

Proof.

(i) $\Rightarrow$ (ii). If R is regular, then $\Omega_0^R(k)$ $(= k)$ itself is a non-zero R-module of finite injective dimension. Hence, the implication follows.

(ii) $\Rightarrow$ (i). Without loss of generality, we may assume that R is complete. Existence of a non-zero (finitely generated) R-module of finite injective dimension ensures that the base ring R is Cohen-Macaulay (see [**2,** 9.6.2, 9.6.4(ii)] and [**8**]). Therefore, we may as well assume that R is a Cohen-Macaulay complete local ring.

Suppose that L is a non-zero direct summand of $\Omega_n^R(k)$ for some $n \geq 0$ such that $\operatorname{injdim}_R(L)$ is finite. We prove the implication by using induction on $d := \dim(R)$. If $d = 0$, then the implication follows from Corollary 3.4.

Now, we assume that $d \geq 1$. Suppose the implication holds true for all such rings of dimension smaller than d. Since R is Cohen-Macaulay and $\dim(R) \geq 1$, there exists an $x \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ which is R-regular. We set $\overline{(-)} := (-) \otimes_R R/(x)$. If $n = 0$, then the direct summand L of $\Omega_0^R(k)$ $(= k)$ must be equal to k, and hence, $\operatorname{injdim}_R(k)$ is finite, which yields that R is regular. Therefore, we may assume that $n \geq 1$. Hence, x is $\Omega_n^R(k)$-regular. Since L is a direct summand of $\Omega_n^R(k)$, x is L-regular as well. This yields that $\operatorname{injdim}_{\overline{R}}(\overline{L})$ is finite.

Next, we fix an indecomposable direct summand L' of $\overline{L}$. Then, $\operatorname{injdim}_{\overline{R}}(L')$ is also finite. Note that the $\overline{R}$-module $\overline{L}$ is a direct summand of $\overline{\Omega_n^R(k)}$. Hence, L' is an indecomposable direct summand of $\overline{\Omega_n^R(k)}$. Since $x \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ is an R-regular element, in view of Proposition 2.2, we have

$$\overline{\Omega_n^R(k)} \cong \Omega_n^{\overline{R}}(k) \oplus \Omega_{n-1}^{\overline{R}}(k).$$

It then follows from the uniqueness of the Krull-Schmidt decomposition ([**5,** Theorem (21.35)]) that L' is isomorphic to a direct summand of $\Omega_n^{\overline{R}}(k)$ or $\Omega_{n-1}^{\overline{R}}(k)$. Since $\dim(\overline{R}) = d-1$, by the induction hypothesis, we obtain that $\overline{R}$ is a regular local ring, and hence, R is a regular local ring as $x \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ is an R-regular element. $\square$

Let M be an R-module. Consider the augmented minimal injective resolution of M:

$$0 \longrightarrow M \xrightarrow{d^{-1}} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \longrightarrow \cdots \longrightarrow I^{n-1} \xrightarrow{d^{n-1}} I^n \longrightarrow \cdots .$$

Recall that the nth *cosyzygy module* of M is defined by

$$\Omega_{-n}^R(M) := \operatorname{Image}(d^{n-1}) \quad \text{for all } n \geq 0.$$

The following result is dual to Theorem 3.7, which gives another characterization of regular local rings via cosyzygy modules of the residue field.

Corollary 3.8. *The following statements are equivalent*:

(i) *the ring R is regular*;
(ii) *the cosyzygy module $\Omega^R_{-n}(k)$ has a non-zero finitely generated direct summand of finite projective dimension for some $n \geq 0$.*

Proof.

(i) $\Rightarrow$ (ii). If R is regular, then $\Omega^R_0(k)$ $(= k)$ has finite projective dimension. Hence, the implication follows.

(ii) $\Rightarrow$ (i). Without loss of generality, we may assume that R is complete. Suppose that $\Omega^R_{-n}(k) \cong P \oplus Q$ for some integer $n \geq 0$, where P is a non-zero finitely generated R-module of finite projective dimension. Consider the following part of the minimal injective resolution of k:

$$\begin{aligned}(3.1) \quad 0 \longrightarrow k \longrightarrow E \longrightarrow E^{\mu_1} \longrightarrow \cdots \\ \longrightarrow E^{\mu_{n-1}} \longrightarrow \Omega^R_{-n}(k) \cong P \oplus Q \longrightarrow 0,\end{aligned}$$

where E is the injective hull of k. Dualizing (3.1) with respect to E and using $\mathrm{Hom}_R(k, E) \cong k$ and $\mathrm{Hom}_R(E, E) \cong R$, cf., [**2,** 3.2.12(a), 3.2.13(a)], we obtain the following part of the minimal free resolution of k:

$$\begin{aligned}0 \longrightarrow \mathrm{Hom}_R(P, E) \oplus \mathrm{Hom}_R(Q, E) \cong \Omega^R_n(k) \longrightarrow R^{\mu_{n-1}} \longrightarrow \cdots \\ \longrightarrow R^{\mu_1} \longrightarrow R \longrightarrow k \longrightarrow 0.\end{aligned}$$

Clearly, $\mathrm{Hom}_R(P, E)$ is non-zero and of finite injective dimension as P is non-zero and of finite projective dimension. Therefore, the implication follows from Theorem 3.7. $\square$

Now, we give an example to ensure that the existence of an injective homomorphism from a 'special module' to a finite direct sum of syzygy modules of the residue field does not necessarily imply that the base ring is regular.

Example 3.9. Let $(R, \mathfrak{m}, k)$ be a d-dimensional Gorenstein local domain. Clearly, $\Omega^R_d(k)$ is a maximal Cohen-Macaulay R-module, see e.g., [**2,** 1.3.7]. Therefore, since R is Gorenstein, $\Omega^R_d(k)$ is a reflexive R-module (by [**2,** 3.3.10]), and hence, it is torsion-free. Then, by mapping 1 to a non-zero element of $\Omega^R_d(k)$, we get an injective R-module homomorphism

$$f : R \longrightarrow \Omega_d^R(k).$$

Note that $\mathrm{injdim}_R(R)$ is finite. However, a Gorenstein local domain need not be a regular local ring.

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