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A SHORT NOTE ON MINIMAL PRIME IDEALS

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ABSTRACT. In this note we give a characterization of the rings with finitely many minimal prime ideals.

Throughout this note, R is a commutative ring with identity. It is well known that if R is Noetherian (i.e., any ideal of R is finitely generated), then the set of minimal prime ideals of R, $\mathrm{Min}(R)$, is finite.

In [**1**], Anderson abandoned the Noetherianness and showed the following theorem:

Theorem A. *If each minimal prime ideal of R is finitely generated ideal, then* $\mathrm{Min}(R)$ *is finite.*

Let I be a proper ideal of R. Then the *radical* of I is defined to be the intersection of all prime ideals of R containing I.

In [**2**], Gilmer and Heinzer proved the following theorem that is an extension of Anderson's theorem:

Theorem B. *If each minimal prime ideal of R is the radical of a finitely generated ideal, then* $\mathrm{Min}(R)$ *is finite.*

The following example shows that the converse of Theorem B (and, by extension, of Theorem A) is false.

Example. Let $R = \mathbf{R}[x_1, x_2, \ldots]$, $I = \langle x_1x_2, x_1x_3, \ldots\rangle$, $\mathfrak{p}_0 = \langle x_2, x_3, \ldots\rangle$ and $\mathfrak{p}_1 = \langle x_1\rangle$. Then it is easy to see that $\mathrm{Min}(R/I) = \{\mathfrak{p}_0/I, \mathfrak{p}_1/I\}$ and $\mathfrak{p}_0/I$ is not a radical of a finitely generated ideal.

In this note we extend Theorem B (Theorem A) and bring a characterization of the rings with finitely many minimal prime ideals:

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Theorem. *The following are equivalent*:

(a) *The set* $\mathrm{Min}(R)$ *is finite.*
(b) *For any* $\mathfrak{p} \in \mathrm{Min}(R)$, *there exists a finitely generated ideal* $\mathfrak{p}^*$ *of* R *such that* $\mathfrak{p}^* \subseteq \mathfrak{p}$ *and* $\mathrm{Min}(R/\mathfrak{p}^*)$ *is finite.*

Proof. (a) $\Rightarrow$(b). Follows by taking $\mathfrak{p}^* = 0$.
(b) $\Rightarrow$(a). Let S denote the collection of finitely generated ideals I of R such that $\mathrm{Min}(R/I)$ is finite.

Set

$$T = \{J \mid J \text{ is an ideal of } R \text{ such that } I \nsubseteq J \text{ for any } I \in S\}.$$

We show that $0 \notin T$. Suppose to the contrary that $0 \in T$. Since the collection T is nonempty and elements of S are finitely generated, T is inductive and hence by Zorn's lemma has a maximal element $\mathfrak{q}$. We show that $\mathfrak{q}$ is a prime ideal of R. If $\mathfrak{q}$ is not prime, then there exist a, $b \in R \setminus \mathfrak{q}$ such that $ab \in \mathfrak{q}$. Therefore, there exist I_1, $I_2 \in S$ such that $I_1 \subseteq \mathfrak{q} + Ra$ and $I_2 \subseteq \mathfrak{q} + Rb$. So, we have

$$I_1 I_2 \subseteq (\mathfrak{q} + Ra)(\mathfrak{q} + Rb) \subseteq \mathfrak{q}^2 + \mathfrak{q}Rb + \mathfrak{q}Ra + Rab \subseteq \mathfrak{q}.$$

Since $|\mathrm{Min}(R/I_1 I_2)| \leq |\mathrm{Min}(R/I_1) \cup \mathrm{Min}(R/I_2)|$, it follows that $I_1 I_2 \in S$, which is a contradiction. Thus, $\mathfrak{q}$ is a prime ideal of R. Now let $\mathfrak{p}$ be a minimal prime ideal of R such that $\mathfrak{p} \subseteq \mathfrak{q}$. By hypothesis, there is an element $\mathfrak{p}^* \in S$ such that $\mathfrak{p}^* \subseteq \mathfrak{p}$. Thus, $\mathfrak{q} \notin T$, and this is also a contradiction. Therefore, $0 \notin T$, and hence $0 \in S$, it follows that $\mathrm{Min}(R)$ is finite. Thus, the theorem is completely proved. $\square$

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