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ON THE REGULARITY OF CONFIGURATIONS OF $\mathbf{F}_q$-RATIONAL POINTS IN PROJECTIVE SPACE

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Dedicated to Jürgen Herzog on the occasion of his 70th birthday

ABSTRACT. We are interested in the smallest number $s = s(n, q)$ such that, for any given n distinct $\mathbf{F}_q$-rational points $P_1, \dots, P_n \in \mathbf{P}^{n-1}$, there exists a hypersurface H of degree s and defined over $\mathbf{F}_q$ such that $P_1, \dots, P_{n-1} \in H, P_n \notin H$. Alternately, $s(n, q)$ is the maximal Castelnuovo-Mumford regularity of a set of n $\mathbf{F}_q$-rational points in some projective space. Finally, $s(n, q)$ is the index of stability of certain one-dimensional local Cohen-Macaulay rings.

1. Introduction. Let K be a field and $\mathbf{P}^k(K)$ the set of all K-rational points in the k-dimensional projective space over K $(k \geq 1)$. We consider subsets $\mathfrak{X} \subset \mathbf{P}^k(K)$ with $\deg \mathfrak{X} = |\mathfrak{X}| =: n \geq 1$.

Let $R = K[X_0, \dots, X_k]$ be the polynomial ring over K in the variables $X_0, \dots, X_k$, and let

$$I_{\mathfrak{X}} := (\{F \in R \text{ homogenous} \mid F(P) = 0 \text{ for all } P \in \mathfrak{X}\})$$

be the homogenous vanishing ideal of $\mathfrak{X}$. Then $S := R/I_{\mathfrak{X}}$ is a standard graded ring. The Hilbert function $H_{\mathfrak{X}}$ of $\mathfrak{X}$ is defined as

$$H_{\mathfrak{X}}(d) = \dim_K S_d \quad (d \in \mathbf{N}),$$

where S_d is the homogenous component of degree d of S. As is well known, there is a number $r_{\mathfrak{X}}$, such that $H_{\mathfrak{X}}(d) = n$ for $d \geq r_{\mathfrak{X}}$ and $H_{\mathfrak{X}}(r_{\mathfrak{X}} - 1) < n$. It is called the *regularity* (Castelnuovo-Mumford regularity) of $\mathfrak{X}$. For $0 \leq d \leq r_{\mathfrak{X}}$, the function $H_{\mathfrak{X}}$ is strictly increasing; hence, $r_{\mathfrak{X}} \leq n - 1$.

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We are mainly interested in the case where $K = \mathbf{F}_q$ is a finite field with q elements. Systems $\mathfrak{X} \subset \mathbf{P}^k(\mathbf{F}_q)$ play a role, for example, in algebraic coding theory, see [**3, 4**]. We set

$$s(n,q) := \operatorname{Max}\{r_{\mathfrak{X}} \mid \mathfrak{X} \subset \mathbf{P}^k(\mathbf{F}_q) \text{ with } \deg \mathfrak{X} = n \text{ and arbitrary } k \geq 1\}.$$

Obviously, $s(1,q) = 0$. The function $s(n,q)$ has the following geometric description.

1.1. *Remark.* $s(n,q)$ is the smallest number s such that, for any given n distinct points $P_1, \dots, P_n \in \mathbf{P}^{n-1}(\mathbf{F}_q)$, there exists in $\mathbf{P}^{n-1}$ a hypersurface H of degree s and defined over $\mathbf{F}_q$ with $P_1, \dots, P_{n-1} \in H$, $P_n \notin H$.

A simple explanation will be given in Section 5.

About regularity, the following facts are known. Let $\overline{\mathbf{F}_q}$ be the algebraic closure of $\mathbf{F}_q$. Choose in $\overline{S} := \overline{\mathbf{F}_q} \otimes_{\mathbf{F}_q} S$ a homogenous non-zerodivisor z of degree 1. Then

$$\Delta H_{\mathfrak{X}}(d) := H_{\mathfrak{X}}(d) - H_{\mathfrak{X}}(d-1) = \dim_{\overline{\mathbf{F}_q}} \overline{S_d}/z\overline{S}_{d-1} \quad (d \geq 1)$$

and $r_{\mathfrak{X}}$ is the degree of the highest non-vanishing component of $\overline{S}/(z)$.

For $\mathfrak{Y} \subset \mathbf{P}^k(\mathbf{F}_q)$ with $\mathfrak{X} \subset \mathfrak{Y}$ and $|\mathfrak{Y}| = |\mathfrak{X}| + 1$ we have $r_{\mathfrak{X}} \leq r_{\mathfrak{Y}} \leq r_{\mathfrak{X}} + 1$ ([**2**, 2.1e]). This implies

1.2. Lemma. $s(n,q) \leq s(n+1,q) \leq s(n,q) + 1$ $(n \geq 1)$.

We may extend $s(n,q)$ to a step function $s(x,q)$, $x \in \mathbf{R}$, $x \geq 1$, with its jump discontinuities being the $n \in \mathbf{N}$ with $s(n,q) = s(n-1,q) + 1$. By the initial value $s(1,q) = 0$ and the jump discontinuities $a_1 < a_2 < \cdots$, the function s is completely determined:

$$s(x,q) = i \text{ for } x \in \mathbf{R} \text{ with } a_i \leq x < a_{i+1} \quad (i = 1, 2, \dots).$$

In the following let m, r and n always be integers. We shall show

1.3. Theorem. *Assume $m \geq 2$. Then*

a) *For* $x \in \mathbf{R}$ *with* $(q^m - 1)/(q-1) \le x \le 2 \cdot (q^m - 1)/(q-1)$, *we have*

$$s(x, q) = (m-1)(q-1) + 1.$$

b) *For all* r *with* $2 \le r \le q-1$ *and* $x \in \mathbf{R}$ *with* $(r+1)q^{m-1} \le x \le (r+1)(q^m - 1)/(q-1)$, *we have*

$$s(x, q) = (m-1)(q-1) + r.$$

1.4. Corollary. *The following numbers are jump discontinuities of* s:

a) $a_i = i + 1$ *for* $i = 1, \dots, q-1$.

b) $a_{(m-1)(q-1)+1} = (q^m - 1)/(q-1)$ *for* $m = 2, 3, \dots$.

c) *For* $q = 2$, *all jump discontinuities of* s *are given by* a) *and* b). *If* $q \ge 3$, *then in each of the semiopen intervals* $(r(q^m - 1)/(q-1), (r+1)q^{m-1}]$ $(r = 2, \dots, q-1)$ *there is exactly one more jump discontinuity* $a_{(m-1)(q-1)+r}$ $(m = 2, 3, \dots)$. *For* $r = q-1$, *this is* $a_{m(q-1)} = q^m$; *for* $r < q-1$, *its precise locus is unfortunately not known to us.*

Here, statement a) of the corollary follows from Lemma 1.2 and part a) of the theorem for $m = 2$, since $s(q+1, q) = q$. Also observe that, for $r = q-1$, the semi-open interval contains exactly one natural number.

For $q = 2$ and $q = 3$, the corollary catches all jump discontinuities, and s is completely known: If $x \ge 2$, then

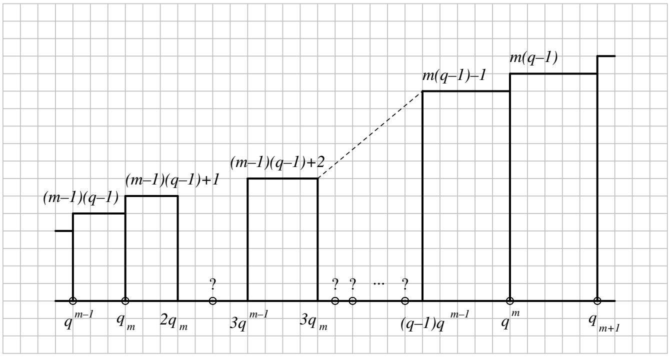
$$s(x, 2) = m \quad \text{for } 2^m - 1 \le x \le 2(2^m - 1) \ [\mathbf{2}]$$

$$s(x, 3) = \begin{cases} 2m - 1 & \text{if } (3^m - 1)/2 \le x < 3^m \\ 2m & \text{if } 3^m \le x < (3^{m+1} - 1)/2. \end{cases}$$

In the general case, from $(m-1)(q-1) + 1 \le s(x, q) \le m(q-1)$ for $(q^m - 1)/(q-1) \le x \le q(q^m - 1)/(q-1)$, it follows that

$$\lim_{x \to \infty} \frac{s(x, q)}{\log_q x} = q - 1,$$

which was conjectured in [**2**].



Set $q_m := (q^m - 1)/(q-1)$. The (symbolic) sketch above illustrates the theorem and its corollary for $q \geq 4$ in the closed interval $[q^{m-1} - 1, q_{m+1} + 1]$. Notice that $(q-1)q_m = q^m - 1$ and $q \cdot q_m = q_{m+1} - 1$. The jump discontinuities are marked by circles.

From Lemma 1.2, it follows that Theorem 1.3 is implied by the following two propositions:

1.5. Proposition. *Let* $m \geq 1$ *and* $1 \leq r \leq q-1$. *Then, for* n *with* $2 \leq n \leq (r+1)(q^m - 1)/(q-1)$,

$$s(n,q) \leq (m-1)(q-1) + r.$$

1.6. Proposition. *Let* $m \geq 1$ *and* $1 \leq r \leq q-1$. *Then*:

a) $s((r+1)q^{m-1}, q) \geq (m-1)(q-1) + r$.

b) $s((q^{m+1} - 1)/(q-1), q) \geq m(q-1) + 1$.

The proof of the propositions will be given in the next three sections.

2. A combinatorial lemma. We need a lemma about the number of certain divisors of monomials. For $X^\alpha = X_0^{\alpha_0} \cdots X_k^{\alpha_k}$, let $|\alpha| := \sum_{i=0}^{k} \alpha_i$ be the degree of X^α and $T_\alpha(d)$ the number of divisors X^β of X^α of degree d.

2.1. Rule. *Let* $\alpha, \gamma, \gamma' \in \mathbf{N}^{k+1}$ *with* $\gcd(X^\alpha, X^\gamma) = \gcd(X^\alpha, X^{\gamma'}) = 1$ *be given. If* $T_{\gamma'}(d) \geq T_\gamma(d)$ *for all* $d \in \mathbf{N}$, *then* $T_{\alpha+\gamma'}(d) \geq T_{\alpha+\gamma}(d)$ *for all* $d \in \mathbf{N}$.

This is a consequence of the formula

$$T_{\alpha+\gamma}(d) = \sum_{\nu=0}^{d} T_\alpha(\nu) T_\gamma(d-\nu) \quad (d \in \mathbf{N}).$$

2.2. Lemma. *Let* $q \geq 2$, $k \geq m \geq 0$, $q-1 \geq r \geq s \geq 1$ *and* $X^\rho := X_0^{q-1} \cdots X_{m-1}^{q-1} X_m^r$ $(\rho = (q-1, \ldots, q-1, r, 0, \ldots, 0) \in \mathbf{N}^{k+1})$. *Then*:

a) *The monomial* X^ρ *has exactly* $(r+1)(q^m-1)/(q-1)+1$ *divisors* X^β *with* $\beta \neq 0$ *and* $|\beta| \equiv s \bmod (q-1)$.

b) *For all* $\alpha \in \mathbf{N}^{k+1}$ *with* $|\alpha| = |\rho|$, $0 \leq \alpha_j \leq q-1$ $(j = 0, \ldots, k)$ *and all* $d \in \mathbf{N}$, *we have* $T_\alpha(d) \geq T_\rho(d)$.

In particular, X^α *has at least* $(r+1)(q^m-1)/(q-1)+1$ *divisors* X^β *with* $\beta \neq 0$ *and* $|\beta| \equiv s \bmod (q-1)$.

Proof. a) Use induction on m. Since, for $m = 0$, the statement is trivial, let $m \geq 1$. By induction hypothesis X^ρ has exactly $(r+1)(q^{m-1}-1)/(q-1)+1$ divisors $X^\beta \neq 1$ with $\beta_0 = 0$ and $|\beta| \equiv s \bmod (q-1)$. The monomial $X_1^{q-1} \cdots X_{m-1}^{q-1} X_m^r$ has $(r+1)q^{m-1}$ divisors $X^{\beta'} = X_1^{\beta_1} \cdots X_m^{\beta_m}$. Associate to each such divisor the divisor $X^\beta = X_0^{\beta_0} X^{\beta'}$ of X^ρ with $\beta_0 \equiv (s - |\beta'|) \bmod (q-1)$, $1 \leq \beta_0 \leq q-1$. Then all divisors $X^\beta \neq 1$ of X^ρ with $|\beta| \equiv s \bmod (q-1)$ and $\beta_0 \neq 0$ are obtained. Altogether, X^ρ has

$$(r+1)\frac{q^{m-1}-1}{q-1} + 1 + (r+1)q^{m-1} = (r+1)\frac{q^m-1}{q-1} + 1$$

such divisors.

b) Let $\alpha \in \mathbf{N}^{k+1}$ with $|\alpha| = m(q-1)+r$, $0 \leq \alpha_j \leq q-1$ for $j = 0, \ldots, k$ be given. Since T_α does not change, if we change the order of entries of α, it suffices to prove assertion b) for the $(k+1)$-tuples of the set

$$N := \{\alpha \in \mathbf{N}^{k+1} \mid q-1 \geq \alpha_0 \geq \cdots \geq \alpha_k \geq 0,\ |\alpha| = m(q-1)+r\}.$$

We use descending induction with respect to the lexicographic ordering $\geq_{\text{lex}}$ of N. We have $\rho = \text{Max}_{\text{lex}}(N)$, and b) trivially holds for $X^\alpha = X^\rho$.

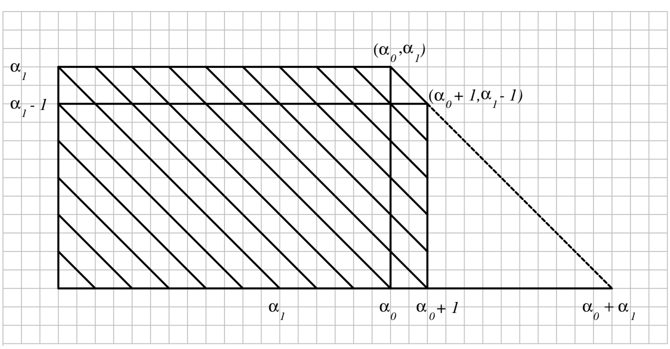
Now consider $\alpha \in N$ with $X^\alpha \neq X^\rho$, and let b) be proved for all $\beta \in N$ with $\rho \geq_{\text{lex}} \beta >_{\text{lex}} \alpha$. Since $\alpha <_{\text{lex}} \rho$, we have $k \geq 1$. In order to show $T_\alpha \geq T_\rho$, it suffices, thanks to Rule 2.1, to take only those exponents in X^α and X^ρ into regard which are distinct. Therefore, we may assume that $\alpha_j \neq \rho_j$ for $j = 0, \ldots, k$. Then $q-1 > \alpha_0 \geq \cdots \geq \alpha_k \geq 1$, $k \geq 1$, since from $\alpha_k = 0$ it would follow that $m(q-1) + r = \alpha_0 + \cdots + \alpha_{k-1} \leq k(q-1)$, $k > m$ and $\rho_k = \alpha_k$.

Replacing α by $\alpha' := (\alpha_0 + 1, \alpha_1, \ldots, \alpha_{k-1}, \alpha_k - 1)$, we also have $\alpha' \in N$. Further, $\rho \geq_{\text{lex}} \alpha' >_{\text{lex}} \alpha$. By induction hypothesis $T_{\alpha'} \geq T_\rho$. Thus, we have to show $T_\alpha \geq T_{\alpha'}$. Since α' and α differ only at their first and last entry, we can assume by Rule 2.1 that $k = 1$, i.e., that $\alpha' = (\alpha_0 + 1, \alpha_1 - 1)$, $\alpha = (\alpha_0, \alpha_1)$ with $q - 1 > \alpha_0 \geq \alpha_1 \geq 1$ and $q - 1 \geq \alpha_0 + 1 > \alpha_1 - 1 \geq 0$.

Quite generally, for $\beta = (\beta_0, \beta_1) \in \mathbf{N}^2$ with $\beta_0 \geq \beta_1$, we have

$$T_\beta(d) = \begin{cases} d+1 & \text{for } 0 \leq d \leq \beta_1 \\ \beta_1 + 1 & \text{for } \beta_1 \leq d \leq \beta_0 \\ \beta_0 + \beta_1 - d + 1 & \text{for } \beta_0 \leq d \leq \beta_0 + \beta_1 \\ 0 & \text{for } d > \beta_0 + \beta_1 \end{cases}$$

and $T_\alpha(d) \geq T_{\alpha'}(d)$ follows. □



3. Proof of Proposition 1.5. For $\mathfrak{X} \subset \mathbf{P}^k(\mathbf{F}_q)$ with $\deg \mathfrak{X} = n$, $2 \leq n \leq (r+1)(q^m-1)/(q-1)$ and $1 \leq r \leq q-1$, we have to show the estimation $r_{\mathfrak{X}} \leq (m-1)(q-1)+r$.

In $R = \mathbf{F}_q[X_0, \dots, X_k]$, we consider the homogenous vanishing ideal $I_{\mathfrak{X}}$ of $\mathfrak{X}$ and the ideal $I := (\{F^q - F\}_{F \in R}) = (\{X_i^q - X_i\}_{i=0,\dots,k})$. Set $T_{\mathfrak{X}} := R/I_{\mathfrak{X}} + I$, and let x_i denote the image of X_i in $T_{\mathfrak{X}}$ ($i = 0, \dots, k$).

3.1. *Remark.* For $d \in \mathbf{N}$, the canonical map $\varphi : R_d \to T_{\mathfrak{X}}$ has kernel $(I_{\mathfrak{X}})_d$, i.e.,

$$H_{\mathfrak{X}}(d) = \dim_{\mathbf{F_q}} \varphi(R_d).$$

Moreover, $\varphi(R_{d+1}) \subset \varphi(R_{d+q})$.

Proof. Clearly, $(I_{\mathfrak{X}})_d \subset \ker \varphi$. Let $F \in R_d$ and $\varphi(F) = 0$; hence, $F = G + H$ ($G \in I_{\mathfrak{X}}, H \in I$). Then, for all $P \in \mathfrak{X}$, it follows that $F(P) = G(P) + H(P) = 0$; hence, $F \in (I_{\mathfrak{X}})_d$. The second assertion follows from $x_i^q = x_i$ for $i = 0, \dots, k$.

Set $V_d := \varphi(R_d)$. We have $V_{(m-1)(q-1)+r} \subset V_{m(q-1)+r}$. As $H_{\mathfrak{X}}$ is increasing, it suffices to show the equality of the vector spaces, that is, $x^\alpha \in V_{(m-1)(q-1)+r}$, if $|\alpha| = m(q-1)+r$ and $0 \leq \alpha_j \leq q-1$ ($j = 0, \dots, k$).

Assume there exists such an α with $x^\alpha \notin V_{(m-1)(q-1)+r}$. Choose α maximal with respect to $\geq_{\text{lex}}$. Since $|\alpha| = m(q-1)+r$, $0 \leq \alpha_j \leq q-1$, we have $k \geq m$, and we can apply Lemma 2.2 b).

Set $\mathfrak{T}(\alpha) := \{\beta \in \mathbf{N}^{k+1} | \beta \neq 0,\ X^\beta | X^\alpha \text{ and } |\beta| \equiv |\alpha| \bmod (q-1)\}$. By the lemma, we have $|\mathfrak{T}(\alpha)| \geq (r+1)(q^m-1)/(q-1) + 1 > |\mathfrak{X}|$, but $\dim_{\mathbf{F}_q} V_{m(q-1)+r} = H_{\mathfrak{X}}(m(q-1)+r) \leq |\mathfrak{X}|$. Thus, the family x^β, $\beta \in \mathfrak{T}(\alpha)$ of vectors from $V_{m(q-1)+r}$ is linearly dependent over $\mathbf{F}_q$. Hence, there is a relation

$$x^\beta = \sum_\gamma \lambda_\gamma x^\gamma \quad \text{with } \lambda_\gamma \in \mathbf{F}_q \quad (\beta \in \mathfrak{T}(\alpha)),$$

where the summation is over the $\gamma \in \mathfrak{T}(\alpha)$ with $|\gamma| \leq |\beta|$ and $\gamma >_{\text{lex}} \beta$, if $|\gamma| = |\beta|$. Multiplying both sides by $x^{\alpha-\beta}$ gives a relation

$$x^\alpha = \sum_{\gamma'} \mu_{\gamma'} x^{\gamma'} \quad (\mu_{\gamma'} \in \mathbf{F}_q) \tag{3.2}$$

with summation over the γ' with $\gamma' \in \mathbf{N}^{k+1} \setminus \{0\}$, $|\gamma'| \le |\alpha|$, $|\gamma'| \equiv |\alpha| \bmod (q-1)$ and $\gamma' >_{\text{lex}} \alpha$, if $|\gamma'| = |\alpha|$.

a) If $|\gamma'| < |\alpha|$, $|\gamma'| \equiv |\alpha| \bmod (q-1)$, then $x^{\gamma'} \in V_{(m-1)(q-1)+r}$.

b) If $|\gamma'| = |\alpha|$ and $\gamma'_j \ge q$ for some $j \in \{0, \dots, k\}$, then since $x_j^q = x_j$, we likewise have $x^{\gamma'} \in V_{(m-1)(q-1)+r}$.

c) If $|\gamma'| = |\alpha|$, $\gamma' >_{\text{lex}} \alpha$ and $0 \le \gamma'_j \le q-1$ $(j = 0, \dots, k)$, then by the choice of α again $x^{\gamma'} \in V_{(m-1)(q-1)+r}$.

From (3.2), it follows that $x^\alpha \in V_{(m-1)(q-1)+r}$, a contradiction. □

4. Regularity in special cases and proof of Proposition 1.6. Let $S_i \subset \mathbf{F}_q$ with $|S_i| = r_i + 1$ be given $(i = 1, \dots, k)$, and let

$$\mathfrak{X} := S_1 \times \cdots \times S_k \subset \mathbf{F}_q^k \subset \mathbf{P}^k(\mathbf{F}_q),$$

where we identify $(a_1, \dots, a_k) \in \mathfrak{X}$ with $\langle 1, a_1, \dots, a_k \rangle \in \mathbf{P}^k(\mathbf{F}_q)$. Then $I_{\mathfrak{X}} = (\{F_1, \dots, F_k\})$ with $F_i := \prod_{a \in S_i}(X_i - aX_0)$ $(i = 1, \dots, k)$ is the homogenous vanishing ideal of $\mathfrak{X}$ and

$$\begin{aligned} R/I_{\mathfrak{X}} + (X_0) &= \mathbf{F}_q[X_1, \dots, X_k]/(X_1^{r_1+1}, \dots, X_k^{r_k+1}) \\ &= \bigotimes_{i=1}^{k} \mathbf{F}_q[X_i]/(X_i^{r_i+1}). \end{aligned}$$

Therefore,

$$r_{\mathfrak{X}} = \sum_{i=1}^{k} r_i.$$

In particular, for $\mathfrak{X} = \mathbf{F}_q^k$, we have $r_{\mathfrak{X}} = k(q-1)$.

For the proof of Proposition 1.6 a), we choose for $S_1 \subset \mathbf{F}_q$ a set of $r+1$ elements, and set $S_i = \mathbf{F}_q$ $(i = 2, \dots, k)$. Then $\deg \mathfrak{X} = (r+1)q^{k-1}, r_{\mathfrak{X}} = (k-1)(q-1) + r$, and it follows that

$$s((r+1)q^{k-1}, q) \ge (k-1)(q-1) + r \quad (r = 1, \dots, q-1).$$

For the proof of Proposition 1.6 b), consider $\mathfrak{X} := \mathbf{P}^k(\mathbf{F}_q)$ and, as in Section 3, the ring $T_{\mathfrak{X}} = R/I_{\mathfrak{X}} + I$ with $I = (\{X_i^q - X_i\}_{i=0,\dots,k})$.

Further, let

$$T := R/I = \bigotimes_{i=0}^{k} \mathbf{F}_q[X_i]/(X_i^q - X_i) = \bigoplus_{0\le\alpha_i\le q-1} \mathbf{F}_q x_0^{\alpha_0}\cdots x_k^{\alpha_k}$$

with $x_i := X_i + I$ $(i = 0, \dots, k)$. As an $\mathbf{F}_q$-vector space, we identify T with the subspace of R generated by the monomials $X^\alpha = X_0^{\alpha_0}\cdots X_k^{\alpha_k}$ $(0 \le \alpha_i \le q-1)$.

4.1. Lemma. *We have $I_{\mathfrak{X}} \subset I$; hence, $T_{\mathfrak{X}} = T$.*

Proof. Obviously, I is the vanishing ideal of $\mathbf{F}_q^{k+1}$, and hence $I_{\mathfrak{X}} \subset I$.

Write $T = \mathbf{F}_q \oplus \bigoplus_{s=1}^{q-1} T_s$, with

$$T_s := \langle\{X^\alpha \mid \alpha \ne 0,\ 0 \le \alpha_i \le q-1 \text{ and } |\alpha| \equiv s \bmod (q-1)\}\rangle.$$

Lemma 2.2 a) with $r = q-1$ implies

$$\dim_{\mathbf{F}_q} T_s = q\frac{q^k-1}{q-1} + 1 = \frac{q^{k+1}-1}{q-1} \quad (s = 1, \dots, q-1).$$

The canonical epimorphism $\varphi : R \to T$ maps $R_{k(q-1)+s}$ into T_s. By Remark 3.1, we have $H_{\mathfrak{X}}(k(q-1)+s) = \dim_{\mathbf{F}_q}\varphi(R_{k(q-1)+s})$ and $\varphi(R_{k(q-1)+s}) = \sum_{j=0}^{k}\varphi(R_{j(q-1)+s}) = T_s$. Thus,

$$H_{\mathfrak{X}}(k(q-1)+s) = \frac{q^{k+1}-1}{q-1} \quad (s = 1, \dots, q-1).$$

On the other hand, $\varphi : R_{k(q-1)} \to T_{q-1}$ is not surjective, since the monomial $X_0^{q-1}\cdots X_k^{q-1} \in T_{q-1}$ has no preimage in $R_{k(q-1)}$. Therefore,

$$H_{\mathfrak{X}}(k(q-1)) < \frac{q^{k+1}-1}{q-1},$$

and consequently $r_{\mathfrak{X}} = k(q-1)+1$. Since $\deg\mathfrak{X} = (q^{k+1}-1)/(q-1)$, we obtain

$$s\left(\frac{q^{k+1}-1}{q-1}, q\right) \ge k(q-1)+1,$$

what we had to show in Proposition 1.6 b). ◻

Remark. For a complete description of $H_{\mathfrak{X}}$, see, for example, [**3**].

5. Alternate views. Now let K be an arbitrary field. In this section we use the following description of the Hilbert function $H_{\mathfrak{X}}$ for $\mathfrak{X} \subset \mathbf{P}^k(K), \mathfrak{X} = \{P_1, \dots, P_n\}$ where the P_i are not necessarily distinct.

5.1. *Remark.* Let $P_i = \langle a_{i0}, \dots, a_{ik} \rangle$ $(i = 1, \dots, n)$. For $l \in \mathbf{N}$, let $\varphi_l : R_l \to K^n$ be the linear map given by $F \mapsto (F(P_1), \dots, F(P_n))$, where $F(P_i) = F(a_{i0}, \dots, a_{ik})$ $(i = 1, \dots, n)$. Then $H_{\mathfrak{X}}(l) = \dim_K(\operatorname{im} \varphi_l)$. Here $\operatorname{im} \varphi_1 =: V$ is the vector space spanned by columns of the matrix $(a_{ij})_{i=1,\dots,n, j=0,\dots,k}$, and $V^{(l)} := \operatorname{im} \varphi_l$ is generated by all vectors $v_1 \cdots v_l$ with $v_j \in V$ $(j = 1, \dots, l)$, where the multiplication of the vectors is performed componentwise.

These assertions are clear since $(I_{\mathfrak{X}})_l = \ker \varphi_l$.

Remark 1.1 asks for which s to given distinct points $P_1, \dots, P_n \in \mathbf{P}^{n-1}(\mathbf{F}_q)$ there exist polynomials $F_1, \dots, F_n \in R_s$ with $F_i(P_j) = \delta_{ij}$ $(i, j = 1, \dots, n)$? By Remark 5.1, this is equivalent to $H_{\mathfrak{X}}(s) = n$, which is correct for $s \geq s(n, q)$ and false for $s < s(n, q)$.

There is also a connection between the function $s(n, q)$ and the index of stability of certain one-dimensional local Cohen-Macaulay rings. If $(R, \mathfrak{m})$ is such a ring and $\mathfrak{a}$ an $\mathfrak{m}$-primary ideal, then $\mathfrak{a}$ is called *stable* if there exists an $x \in \mathfrak{a}$ such that $\mathfrak{a}^2 = x\mathfrak{a}$. For large l, the ideal $\mathfrak{a}^l$ is always stable, and one defines $r(\mathfrak{a}) := \operatorname{Min}\{l \in \mathbf{N}_+ | \mathfrak{a}^l \text{ is stable}\}$. The number $s(R) := \sup\{r(\mathfrak{a}) \mid \mathfrak{a}\ \mathfrak{m}\text{-primary}\}$ is called the *index of stability* of R. See [**1**] for numerous references, assertions and examples about the index of stability.

In the following, let $P = K[[Y_1, \dots, Y_n]]$ be the formal power series ring over K in the variables $Y_1, \dots, Y_n$. Set $I := (\{Y_i Y_j\}_{1 \leq i < j \leq n})$, and let $Q_n := P/I$ be the completion of the local ring at the origin of the curve in $\mathbf{A}^n$ which is the union of the coordinate axes. Let y_i be the image of Y_i in Q_n and $\mathfrak{m}$ the maximal ideal of Q_n.

5.2. Proposition. *Let $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbf{N}_+^n$ and $V \subset K^n$ be a linear subspace of dimension d which is not contained in a coordinate hyperplane. Then*:

a) *The elements $a_1 y_1^{\alpha_1} + \dots + a_n y_n^{\alpha_n}$ with $(a_1, \dots, a_n) \in V$ generate an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ in Q_n.*

b) *If the vectors* $(a_{1j}, \dots, a_{nj})$ $(j = 1, \dots, d)$ *form a basis of* V, *then the elements* $a_{1j}y_1^{\alpha_1} + \cdots + a_{nj}y_n^{\alpha_n}$ $(j = 1, \dots, d)$ *form a minimal system of generators of* $\mathfrak{a}$. *In particular,* $\mu(\mathfrak{a}) = \dim_K \mathfrak{a}/\mathfrak{m}\alpha = \dim_K V$.

c) *If* V *contains the* i*th unit vector* $e_i = (0, \dots, 1, \dots, 0)$ *for some* $i \in \{1, \dots, n\}$, *then* α_i *is the smallest number such that* $y_i^{\alpha_i} \in \mathfrak{a}$. *Otherwise* $\alpha_i + 1$ *is the smallest such number.*

Proof. Choose for $i \in \{1, \dots, n\}$ a vector $(a_1, \dots, a_n) \in V$ with $a_i \neq 0$. Then $y_i \sum_{j=1}^n a_j y_j^{\alpha_j} = a_i y_i^{\alpha_i+1} \in \mathfrak{a}$, and hence $\mathfrak{a}$ is $\mathfrak{m}$-primary. If $e_i \in V$, then already $y_i^{\alpha_i} \in \mathfrak{a}$. The assertions of the proposition follow easily. □

Write $\mathfrak{a} =: \mathfrak{a}(\alpha, V)$ if $\mathfrak{a}$ is an ideal as in Proposition 5.2 a).

5.3. Proposition. *Any* $\mathfrak{m}$*-primary ideal* $\mathfrak{a}$ *of* Q_n *has the form* $\mathfrak{a} = \mathfrak{a}(\alpha, V)$ *with* α *and* V *as in Proposition* 5.2.

Proof. For an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ of Q_n and any $i \in \{1, \dots, n\}$, there is a smallest number $\alpha_i \in \mathbf{N}_+$ such that $y_i^{\alpha_i} \in \mathfrak{a} + (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$. Therefore, $y_i^{\alpha_i+1} \in \mathfrak{m}a$, and any $f \in \mathfrak{a}$ can be written $f = \sum_{i=1}^n f_i$ with $f_i \in y_i^{\alpha_i} K[[y_i]]$, i.e., $f \equiv \sum_{i=1}^n a_i y_i^{\alpha_i} \bmod \mathfrak{m}a$ with $a_1, \dots, a_n \in K$. By Nakayama's lemma, the assertion follows. □

If $\mathfrak{a}$ is a principal ideal, then $\mathfrak{a}$ is stable and $r(\mathfrak{a}) = 1$. Therefore, we assume in the following that $\dim V := k+1 \geq 2$. Then we also have only to consider the case $n \geq 2$. For $\mathfrak{a} = \mathfrak{a}(\alpha, V)$ we have $\mathfrak{a}^l = \mathfrak{a}(l\alpha, V^{(l)})$ where, as above, $V^{(l)}$ is the vector space spanned by all $v_1 \cdots v_l$ with $v_i \in V$ and componentwise multiplication.

Now choose a basis $v_j = (a_{1j}, \dots, a_{nj})$ $(j = 0, \dots, k)$ of V, and consider the points $P_i = \langle a_{i0}, \dots, a_{ik} \rangle \in \mathbf{P}^k(K)$ $(i = 1, \dots, n)$ corresponding to the rows of the matrix $(a_{ij})_{i=1,\dots,n, j=0,\dots,k}$. For $\mathfrak{X} := \{P_1, \dots, P_n\}$ and $\mathfrak{a} = \mathfrak{a}(\alpha, V)$, Remark 5.1 implies that $\dim V^{(l)} = H_{\mathfrak{X}}(l) = \mu(\mathfrak{a}^l)$. In particular,

$$\dim V = \dim V^{(1)} < \dim V^{(2)} < \cdots < \dim V^{(r_{\mathfrak{X}})} = \dim V^{(l)} = \deg \mathfrak{X} \leq n$$

for $l \geq r_{\mathfrak{X}}$.

5.4. Proposition. $r(\mathfrak{a}) = r_{\mathfrak{X}}$.

Proof. For $1 \leq l < r_{\mathfrak{X}}$, we have $\mu(\mathfrak{a}^l) = \dim V^{(l)} < \dim V^{(2l)} = \mu(\mathfrak{a}^{2l})$; hence, $\mathfrak{a}^l$ is not stable.

Conversely, we can assume that $\mathfrak{X} = \{P_1, \dots, P_{n'}\}$, $\deg \mathfrak{X} = n' \le n$. By Remark 5.1, there is an $F \in R_{r_\mathfrak{X}}$ with $F(P_i) = 1$ for $i = 1, \dots, n'$. Then $v := (F(P_1), \dots, F(P_n)) \in V^{(r_\mathfrak{X})}$ and $F(P_i) \neq 0$ for $i = 1, \dots, n$. Since $\dim V^{(r_\mathfrak{X})} = \dim V^{(2r_\mathfrak{X})}$, we obtain $V^{(2r_\mathfrak{X})} = v \cdot V^{(r_\mathfrak{X})}$, and with $x := \sum_{i=1}^n F(P_i) y_i^{r_\mathfrak{X}\alpha_i}$, it follows that $(\mathfrak{a}^{r_\mathfrak{X}})^2 = x \cdot \mathfrak{a}^{r_\mathfrak{X}}$, i.e., $\mathfrak{a}^{r_\mathfrak{X}}$ is stable, $r(\mathfrak{a}) = r_\mathfrak{X}$.

If K is an infinite field one can choose pairwise distinct $a_1, \dots, a_n \in K$. For the vector space V generated by $(1, \dots, 1)$ and $(a_1, \dots, a_n)$, we have by van der Monde that $\dim_K V^{(l)} = l + 1$ for $l = 1, \dots, n-1$ and $\dim_K V^{(l)} = n$ for $l \ge n$. For $\mathfrak{a} = \mathfrak{a}(\alpha, V)$ with any $\alpha \in \mathbf{N}_+^n$, we obtain $r(\mathfrak{a}) = n - 1$; hence, $s(Q_n) = n - 1 = m(Q_n) - 1$, where $m(Q_n)$ denotes the multiplicity of Q_n.

For $K = \mathbf{F}_q$ the situation is quite different. Proposition 5.4 implies that, in this case, $s(Q_n) = s(n, q)$, so that Theorem 1.3 and its conclusions can also be applied to the index of stability $s(Q_n)$. In particular, we have

$$\lim_{n\to\infty} \frac{s(Q_n)}{\log_q n} = q - 1. \qquad \square$$

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REFERENCES

1. J. Herzog and R. Waldi, *A note on the Hilbert function of a one-dimensional Cohen-Macaulay ring*, Manuscr. Math. **16** (1975), 251–260.

2. M. Kreuzer and R. Waldi, *On the Castelnuovo-Mumford regularity of a projective system*, Comm. Algebra **25** (1997), 2919–2929.

3. A.B. Sorensen, *Projective Reed-Muller codes*, IEEE Trans. Inf. Theor. **37** (1991), 1567–1576.

4. M. Tsfasman, S. Vladut and D. Nogin, *Algebraic-geometric codes*: *Basic notions*, Math. Surv. Mono. **139**, American Mathematical Society, Providence, R.I., 2007.

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