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COMULTIPLICATION MODULES OVER COMMUTATIVE RINGS

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ABSTRACT. Let R be a commutative ring with identity. A unital R-module M is a comultiplication module provided for each submodule N of M there exists an ideal A of R such that N is the set of elements m in M such that $Am = 0$. It is proved that if M is a finitely generated comultiplication R-module with annihilator B in R then the ring R/B is semilocal and in certain cases M is quotient finite dimensional. Moreover, certain comultiplication modules satisfy the $AB5^*$-condition. If an R-module $X = \oplus_{i \in I} U_i$ is a direct sum of simple submodules U_i $(i \in I)$ and if P_i is the annihilator of U_i in R for each i in I then X is a comultiplication module if and only if $\cap_{j \neq i} P_j \nsubseteq P_i$ for all $i \in I$. A Noetherian comultiplication module is Artinian and a finitely generated Artinian module M is a comultiplication module if and only if the socle of M is a (finite) direct sum of pairwise non-isomorphic simple submodules. In case R is a Dedekind domain, an R-module M is a comultiplication module if and only if M is cocyclic or $M \cong (R/P_1^{k(1)}) \oplus \cdots \oplus (R/P_n^{k(n)})$ for some positive integers n, $k(i)$ $(1 \leq i \leq n)$ and distinct maximal ideals P_i $(1 \leq i \leq n)$ of R. For a general ring R a Noetherian R-module M is comultiplication if and only if the R_P-module M_P is comultiplication for every maximal ideal P of R, but it is shown that this is not true in general. It is shown that comultiplication modules and quasi-injective modules are related in certain circumstances.

1. Comultiplication modules. All rings are commutative with identity and all modules are unital. Let R be a ring, and let M be any R-module. Given submodules N and L of M we denote by $(N :_R L)$ the set of elements r in R such that $rL \subseteq N$. Note that $(N :_R L)$ is the annihilator in R of the R-module $(L+N)/N$ and is an ideal of R. In particular, if N is a submodule of M and $m \in M$ then $(N :_R Rm)$ will be denoted simply by $(N :_R m)$, so that $(N :_R m) = \{r \in R : rm \in N\}$. On the other hand, if N is again a submodule of M and A is an ideal of R then $(N :_M A)$ is the set of elements m in M such that $Am \subseteq N$

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and it is clear that $(N :_M A)$ is a submodule of M. In particular, if N is a submodule of M we shall be interested in the submodule $(0 :_M (0 :_R N))$ of M consisting of all elements $m \in M$ such that $(0 :_R N)m = 0$. In other words, an element $m \in (0 :_M (0 :_R N))$ if and only if $rm = 0$ for every element r in R such that $rN = 0$. Next the injective envelope of M will be denoted by $E(M)$ and if $\varphi : M \to M'$ is any homomorphism of R-modules then the kernel of φ will be denoted by $\ker \varphi$. For any unexplained notions or terminology please see [**4, 19, 28**].

Let R be a ring. Recall that an R-module M is called a *multiplication module* provided, for each submodule N of M, there exists an ideal A of R such that $N = AM$. Multiplication modules have been extensively studied (see, for example, [**1, 2, 10, 11, 24**–**27**]). Note the following characterization of multiplication modules which we think may be new.

Lemma 1.1. *Let R be any ring. Then an R-module M is a multiplication module if and only if for any $m \in M$ and submodule N of M, $(Rm :_R M) \subseteq (N :_R M)$ implies that $m \in N$.*

Proof. Suppose first that M is a multiplication module. Then $L = (L :_R M)M$ for every submodule L of M (see, for example, [**27**, page 223]). Suppose that $(Rm :_R M) \subseteq (N :_R M)$ for some $m \in M$ and submodule N of M. Then

$$Rm = (Rm :_R M)M \subseteq (N :_R M)M = N,$$

as required. Conversely, suppose that M has the stated condition. Let $x \in M$. Then $(Rx :_R M)M$ is a submodule of the cyclic module Rx. Because Rx is a multiplication module, $(Rx :_R M)M = ((Rx :_R M)M :_R Rx)Rx$. It follows that

$$(Rx :_R M) \subseteq (((Rx :_R M)M :_R Rx)Rx :_R M)$$

and hence, by hypothesis, $x \in ((Rx :_R M)M :_R Rx)Rx \subseteq (Rx :_R M)M$. Thus $Rx = (Rx :_R M)M$. It follows that $Rx = (Rx :_R M)M$ for every element x of M and hence M is a multiplication module by [**11**, Proposition 1.1]. $\square$

We now dualize the notion of a multiplication module to obtain comultiplication modules. An R-module M is called a *comultiplication* *module* provided for each submodule N of M there exists an ideal A of R such that $N = (0 :_M A)$. Comultiplication modules have been studied in a number of papers; see, for example, [**5–9**]. Rings R such that ${}_RR$ is a comultiplication module are called *dual rings* or *D-rings* and have also been studied by different authors; see, for example, [**14, 15, 17, 18, 20, 21**] and, for a survey, see [**13**]. Note the following elementary fact.

Lemma 1.2. *Let R be a ring, and let M be an R-module such that $AM = 0$ for some ideal A of R. Then the R-module M is a comultiplication module if and only if the (R/A)-module M is a comultiplication module.*

Proof. We make the set M into an (R/A)-module in the usual way by defining $(r + A)m = rm$ for all $r \in R$ and $m \in M$. Clearly a subset N of the set M is a submodule of the (R/A)-module M if and only if N is a submodule of the R-module M. Suppose that the R-module M is a comultiplication module. Let L be a submodule of the (R/A)-module M. There exists an ideal B of R such that $L = (0 :_M B)$ and hence $L = (0 :_M (B + A)/A)$. It follows that the (R/A)-module M is comultiplication. The converse is equally easy. $\square$

As we have already seen, an R-module M is a multiplication module if and only if for each $m \in M$ there exists an ideal A of R such that $Rm = AM$ (see [**11**, Proposition 1.1]). The analogue of this fact is given in the next result. Recall that a module M over an arbitrary ring R is called *cocyclic* provided M has a simple essential socle. In other words, M is cocyclic if and only if M contains a simple submodule U such that $U \subseteq N$ for every non-zero submodule N of M. In [**4**, page 94, Example 20] cocyclic modules are called *subdirectly irreducible*.

Proposition 1.3. *Let R be any ring. Then an R-module M is a comultiplication module if and only if for each submodule N of M such that M/N is cocyclic there exists an ideal A of R such that $N = (0 :_M A)$.*

Proof. The necessity is clear. Conversely, suppose that M has the stated property. Let L be any proper submodule of M. It is well known that there exist an index set I and submodules L_i ($i \in I$) of M such that $L = \cap_{i \in I} L_i$ and the module M/L_i is cocyclic for each $i \in I$ (see, for example, [**4**, page 94, Example 20]). For each $i \in I$ there exists an ideal A_i of R such that $L_i = (0 :_M A_i)$. Then $L = (0 :_M \sum_{i \in I} A_i)$. It follows that M is a comultiplication module. □

The next result is perhaps well known but is given for completeness.

Lemma 1.4. *Let R be a ring, and let M be an R-module such that $N \neq (0 :_M (0 :_R N))$ for some submodule N. Then there exists a submodule K of M such that $N \subseteq K$ and the module $(0 :_M (0 :_R K))/K$ has non-zero socle.*

Proof. By hypothesis there exists an element x of M such that $x \in (0 :_M (0 :_R N))$ but $x \notin N$ and in this case $(0 :_R N) \subseteq (0 :_R x)$. Let $\mathcal{S}$ denote the collection of submodules L of M such that $N \subseteq L$ and $(0 :_R L) \subseteq (0 :_R x)$ but $x \notin L$. Clearly $\mathcal{S}$ is non-empty because N belongs to $\mathcal{S}$. Let L_i ($i \in I$) be any chain in $\mathcal{S}$, and let $L = \cup_{i \in I} L_i$. Then L is a submodule of M, $N \subseteq L$ and $x \notin L$. Moreover $(0 :_R L) \subseteq (0 :_R L_i) \subseteq (0 :_R x)$ for all $i \in I$. It follows that $L \in \mathcal{S}$. By Zorn's lemma, $\mathcal{S}$ contains a maximal member K (say). Note that $N \subseteq K$ and $(0 :_R K) \subseteq (0 :_R x)$ so that $x \in (0 :_M (0 :_R K))$ but $x \notin K$. Let a be any element of R such that a does not belong to the proper ideal $(K :_R x)$. Then $ax \notin K$. It follows that K is a proper submodule of $K + Rax$ and hence $K + Rax \notin \mathcal{S}$. Note that $(0 :_R K + Rax) \subseteq (0 :_R K) \subseteq (0 :_R x)$ so that $x \in K + Rax$. Therefore there exist $b \in R$ and $u \in K$ such that $x = u + bax$ and $1 - ba \in (K :_R x)$. It follows that $(K :_R x)$ is a maximal ideal of R and hence $x + K$ is a non-zero element of the socle of the module $(0 :_M (0 :_R K))/K$. □

Let R be a ring and M an R-module. Given a submodule L of M, a homomorphism $\beta : L \to M$ will be called *trivial* provided there exists an $r \in R$ such that $\beta(x) = rx$ ($x \in L$). Clearly every trivial homomorphism $\beta : L \to M$ lifts to an endomorphism of M. In particular, an endomorphism φ of the module M will be called *trivial* if $\varphi : M \to M$ is trivial in the above sense. For example, if M is a cyclic module then every endomorphism of M is trivial. Recall that a module

M is a *self-cogenerator* provided for each submodule N of M, the factor module M/N embeds in the direct product M^I of copies of M, for some index set I. It is easy to check that a module M is a self-cogenerator if and only if for each submodule N of M there exist an index set I and endomorphisms φ_i $(i \in I)$ of M such that $N = \cap_{i\in I} \ker \varphi_i$. We shall call a module M *strongly self-cogenerated* provided for each submodule N of M there exists a family φ_i $(i \in I)$ of trivial endomorphisms of M, for some index set I, such that $N = \cap_{i\in I} \ker \varphi_i$. The following result gives some characterizations of comultiplication modules. Compare part (v) with Lemma 1.1.

Theorem 1.5. *Let R be any ring. Then the following statements are equivalent for an R-module M.*

(i) *M is a comultiplication module.*

(ii) $N = (0 :_M (0 :_R N))$ *for every submodule N of M.*

(iii) *The module $(0 :_M (0 :_R N))/N$ has zero socle for every submodule N of M.*

(iv) *Given submodules K, L of M, $(0 :_R K) \subseteq (0 :_R L)$ implies that $L \subseteq K$.*

(v) *Given any submodule N of M and $m \in M$, $(0 :_R N) \subseteq (0 :_R m)$ implies that $m \in N$.*

(vi) *Given any submodule N of M and $m \in M$, $(0 :_R N) \subseteq (0 :_R m)$ implies that $(N :_R m)$ is not a maximal ideal of R.*

(vii) $(L :_R N) = ((0 :_R N) :_R (0 :_R L))$ *for all submodules L and N of M.*

(viii) *M is strongly self-cogenerated.*

Proof. (i) $\Leftrightarrow$ (ii). By [**5**, Lemma 3.7].

(ii) $\Leftrightarrow$ (iii). By Lemma 1.4.

(ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi). Clear.

(vi) $\Rightarrow$ (ii). By Lemma 1.4.

(ii) $\Rightarrow$ (vii). Let A denote the ideal $(0 :_R L)$ of R. By (ii),

$L = (0 :_M A)$. Let $r \in R$. Then

$$\begin{aligned} r \in (L :_R N) &\Longleftrightarrow rN \subseteq L \Longleftrightarrow rAN = 0 \Longleftrightarrow rA \subseteq (0 :_R N) \\ &\Longleftrightarrow r \in ((0 :_R N) :_R A) = ((0 :_R N) :_R (0 :_R L)). \end{aligned}$$

(vii) $\Rightarrow$ (iv). Let K and L be submodules of M such that $(0 :_R K) \subseteq (0 :_R L)$. Then by (vii),

$$(K :_R L) = ((0 :_R L) :_R (0 :_R K)) = R,$$

and hence $L \subseteq K$.

(i) $\Rightarrow$ (viii). Let N be any submodule of M. There exists an ideal A of R such that $N = (0 :_M A)$. For each $a \in A$, let φ_a denote the trivial endomorphism of M defined by $\varphi_a(m) = am$ $(m \in M)$. Clearly $N = \cap_{a \in A} \ker \varphi_a$.

(viii) $\Rightarrow$ (i). Let L be any submodule of M. By hypothesis there exist an index set J and trivial homomorphisms θ_j $(j \in J)$ such that $L = \cap_{j \in J} \ker \theta_j$. For each $j \in J$ there exists $b_j \in R$ such that $\theta_j(m) = b_j m$ $(m \in M)$. Let B denote the ideal $\sum_{j \in J} Rb_j$. It is easy to check that $L = (0 :_M B)$. Now (i) follows. □

Corollary 1.6. *Let R be a ring, and let K and L be submodules of a comultiplication R-module M. Then $K \subseteq L$ if and only if there exists a monomorphism $\varphi : K \to L$. Moreover, $K = L$ if and only if $(0 :_R K) = (0 :_R L)$ and this happens if and only if $K \cong L$.*

Proof. Suppose there exists a monomorphism $\varphi : K \to L$. Then $K \cong \varphi(K)$, and hence $(0 :_R K) = (0 :_R \varphi(K))$. By Theorem 1.5, $K = \varphi(K) \subseteq L$. The rest follows by Theorem 1.5. □

Recall that if R is any ring then an R-module M is called *nonsingular* provided $Am \neq 0$ for every essential ideal A of R and non-zero element m of M.

Corollary 1.7. *For any ring R, every nonsingular comultiplication module is semisimple and projective.*

Proof. Let M be a nonsingular comultiplication R-module. Let N be any essential submodule of M. By Theorem 1.5 (viii), the module M/N embeds in a module M^I, for some index set I. However, M^I is a nonsingular module and hence $M = N$. Thus M is the only essential submodule of M. By [**4**, Proposition 9.7], M is semisimple. It follows easily that M is projective because M is nonsingular. □

We have already seen that strongly self-cogenerated modules are precisely comultiplication modules (see Theorem 1.5). We end this section by considering self-cogenerated modules.

Theorem 1.8. *Let R be a ring and M a self-cogenerated R-module such that, for each finitely generated submodule N of M, every homomorphism $\varphi : N \to M$ is trivial. Then $L = (0 :_M (0 :_R L))$ for every finitely generated submodule L of M.*

Proof. Let L be any finitely generated submodule of M. Since M is self-cogenerated it follows that there exists a non-empty index set I and endomorphisms θ_i $(i \in I)$ of M such that $L = \cap_{i \in I} \ker \theta_i$. Let $m \in M$ such that $m \in (0 :_M (0 :_R L))$, so that $(0 :_R L) \subseteq (0 :_R m)$. Let $i \in I$, and let $\theta = \theta_i$. By hypothesis, because the submodule $L + Rm$ is finitely generated, there exists $r \in R$ such that $\theta(x + am) = r(x + am)$ for all $x \in L$ and $a \in R$. It follows that $rx = \theta(x) = 0$ for all $x \in L$, and hence $r \in (0 :_R L) \subseteq (0 :_R m)$. Thus $rm = 0$ and hence $\theta(m) = rm = 0$. It follows that $\theta_i(m) = 0$ for all $i \in I$ and hence $m \in \cap_{i \in I} \ker \theta_i = L$. Hence $L = (0 :_M (0 :_R L))$, as required. □

2. Properties of comultiplication modules.

In this section we shall show that comultiplication modules have some interesting properties. We begin with a result that lists some useful properties of comultiplication modules.

Lemma 2.1. *Let R be a ring, and let M be a comultiplication R-module. Then*

(i) *Every submodule of M is a comultiplication module.*

(ii) *$\varphi(N) \subseteq N$ for every submodule N of M and homomorphism $\varphi : N \to M$.*

(iii) *For each ideal* A *of* R *either* $(0:_M A) \neq 0$ *or for each* $m \in M$ *there exists* $a \in A$ *such that* $(1-a)m = 0$.

(iv) M *has essential socle.*

(v) $(0:_M P)$ *is zero or simple for every maximal ideal* P *of* R.

(vi) *Every finitely generated submodule of* M *is finitely cogenerated.*

Proof. (i) By [**5**, Theorem 3.17 (d)].

(ii) Clear.

(iii) By [**8**, Proposition 3.1 (c)].

(iv) By [**8**, Theorem 3.2 (a)].

(v) By [**8**, Proposition 3.1 (a)].

(vi) By [**8**, Theorem 3.5 (b)]. □

Corollary 2.2. *Let* R *be a ring, and let* M *be a non-zero finitely generated comultiplication* R*-module. Then the ring* $R/(0:_R M)$ *is semilocal. Moreover, for each prime ideal* P *of* R, $M \neq PM$ *if and only if* $(0:_M P) \neq 0$.

Proof. By Lemma 2.1 (vi), there exist a positive integer n and distinct maximal ideals P_i $(1 \leq i \leq n)$ such that every simple submodule of M is isomorphic to one of the simple submodules $(0:_M P_i)$ $(1 \leq i \leq n)$. Let P be any maximal ideal of R such that $(0:_R M) \subseteq P$. Suppose that $M = PM$. Because M is finitely generated, the usual determinant argument gives $(1-p)M = 0$ for some $p \in P$ and hence $1-p \in A \subseteq P$, a contradiction. Thus $M \neq PM$. By Lemma 2.1 (iii) and (v), $(0:_M P)$ is a simple submodule of M and hence $P = P_i$ for some $1 \leq i \leq n$. Let Q be any maximal ideal of R. Suppose that $M \neq QM$. By Lemma 2.1 (iii), $(0:_M Q) \neq 0$. Now suppose that $M = QM$. Again by the usual determinant argument, there exists $q \in Q$ such that $(1-q)M = 0$. In particular, this implies that $(1-q)(0:_M Q) = 0$, and hence $(0:_M Q) = 0$. □

Let $\mathbf{Z}$ denote the ring of integers, let p be any prime in $\mathbf{Z}$ and let M denote the $\mathbf{Z}$-module $(\mathbf{Z}/\mathbf{Z}p) \oplus (\mathbf{Z}/\mathbf{Z}p)$. Then M satisfies (iii), (iv) and

(vi) of Lemma 2.1 but M is not a comultiplication module by (v). Later we shall give an example of a $\mathbf{Z}$-module M which satisfies (ii)–(vi) of Lemma 2.1 but which is not a comultiplication module.

Lemma 2.3. *Let R be a ring, and let M be a comultiplication R-module such that there exists a family of submodules L_i $(i \in I)$ such that $\cap_{i\in I} L_i = 0$. Then*

(i) *$N = \cap_{i\in I}(N + L_i)$ for every submodule N of M.*

(ii) *For each finitely generated submodule K of M there exists a finite subset J of I such that $R = \sum_{j\in J}(0 :_R L_j) + (0 :_R K)$, and hence $K \cap (\cap_{j\in J} L_j) = 0$.*

Proof. (i) By [**5**, Proposition 3.14 (a)].

(ii) By [**8**, Theorem 3.5 (a)]. □

Elements m_i $(i \in I)$ of an R-module M are called *independent* provided the sum $\sum_{i\in I} Rm_i$ is direct.

Proposition 2.4. *Let R be a ring, and let M be a comultiplication R-module. Let n be a positive integer, and let m_i $(1 \le i \le n)$ be independent elements of M. Then the submodule $Rm_1 \oplus \cdots \oplus Rm_n$ is cyclic.*

Proof. We prove the result by induction on n. If $n = 1$, then there is clearly nothing to prove. Suppose that $n \ge 2$. Let $L = Rm_2 \oplus \cdots \oplus Rm_n$. By induction on n, L is cyclic. Thus it is sufficient to prove the result when $n = 2$. Now $Rm_1 \cap Rm_2 = 0$ gives that

$$R = (0 :_R Rm_1) + (0 :_R Rm_2) + (0 :_R Rm_1) = (0 :_R Rm_1) + (0 :_R Rm_2),$$

by Lemma 2.3. If $A_1 = (0 :_R Rm_1)$ and $A_2 = (0 :_R Rm_2)$, then

$$\begin{aligned} Rm_1 \oplus Rm_2 &\cong (R/A_1) \oplus (R/A_2) \cong (A_2/(A_1 \cap A_2)) \oplus (A_1/(A_1 \cap A_2)) \\ &= R/(A_1 \cap A_2), \end{aligned}$$

and thus $Rm_1 \oplus Rm_2$ is cyclic. □

We now consider how comultiplication modules behave under localization. Let R be a ring, and let M be an R-module. For any prime ideal P of R we set

$$I_P = \{r \in R : rc = 0 \text{ for some } c \in R \setminus P\},$$

and

$$T_P = \{m \in M : cm = 0 \text{ for some } c \in R \setminus P\}.$$

Note that I_P is an ideal of R, T_P is a submodule of R and $I_P M \subseteq T_P$. We shall call the prime ideal P *good for* M if there exists $d \in R \setminus P$ such that $dT_P = 0$. For example, P is good for M provided T_P is finitely generated. Let $\overline{R}$ denote the ring R/I_P and $\overline{M}$ the R- (and also $\overline{R}$-)module M/T_P. Given $r \in R$ and $m \in M$ denote the element $r + I_P$ of $\overline{R}$ by $\overline{r}$ and the element $m + T_P$ of $\overline{M}$ by $\overline{m}$. Form the local ring R_P in the usual way and note that it consists of elements of the form $\overline{r}/\overline{a}$ where $r \in R$ and $a \in R \setminus P$. Similarly the R_P-module M_P consists of elements $\overline{m}/\overline{a}$, where $m \in M$ and $a \in R \setminus P$.

Theorem 2.5. *Let R be a ring, and let M be an R-module such that every maximal ideal of R is good for M. Then M is a comultiplication R-module if and only if M_P is a comultiplication R_P-module for every maximal ideal P of R.*

Proof. Suppose that M is a comultiplication R-module. Let P be any maximal ideal of R. Then $dT_P = 0$ for some $d \in R \setminus P$. Let X be any submoduile of M_P, and let $N = \{m \in M : \overline{m}/\overline{1} \in X\}$. Then N is a submodule of the R-module M. Now let $y \in M_P$ such that $(0 :_{R_P} X) \subseteq (0 :_{R_P} y)$. Then $y = \overline{m}/\overline{c}$ for some $m \in M$, $c \in R \setminus P$. Let $r \in (0 :_R N)$. Given $x \in X$ there exist $u \in M$, $a \in R \setminus P$ such that $x = \overline{u}/\overline{a}$ and in this case $u \in N$. Then $ru = 0$ implies that $(\overline{r}/\overline{1})x = 0$. Thus $\overline{r}/\overline{1} \in (0 :_{R_P} X) \subseteq (0 :_{R_P} y)$, and hence $(\overline{r}/\overline{1})(\overline{m}/\overline{c}) = 0$. It follows that $drm = 0$. We have proved that $(0 :_R N) \subseteq (0 :_R dm)$. By Theorem 1.5, $dm \in N$ and hence $y \in X$. It follows that M_P is a comultiplication module by Theorem 1.5 again.

Conversely, suppose that the R_P-module M_P is a comultiplication module for every maximal ideal P of R. Let L be a submodule of

M, and let $v \in M$ be such that $(0 :_R L) \subseteq (0 :_R v)$. Suppose that $(L :_R v) \neq R$. Then there exists a maximal ideal Q of R such that $(L :_R v) \subseteq Q$. By hypothesis, there exists a $b \in R \setminus Q$ such that $bT_Q = 0$. Let Y denote the submodule of M_Q consisting of all elements $\overline{w}/\overline{e}$, where $w \in L$, $e \in R \setminus Q$. Let $s \in R$, $f \in R \setminus Q$ such that $\overline{s}/\overline{f} \in (0 :_{R_Q} Y)$. Then $bsL = 0$, and hence $bsv = 0$. It follows that $(\overline{s}/\overline{f})(\overline{v}/\overline{1}) = 0$. Thus $(0 :_{R_Q} Y) \subseteq (0 :_{R_Q} \overline{v}/\overline{1})$. By hypothesis, $\overline{v}/\overline{1} \in Y$, and hence $gv \in L$ for some $g \in R \setminus Q$. But this implies that $g \in (L :_R v) \subseteq Q$, a contradiction. Thus $(L :_R v) = R$ and $v \in L$. It follows that the R-module M is a comultiplication module. □

In particular, Theorem 2.5 applies to Noetherian modules M. Note also that the proof of Theorem 2.5 shows that if M is a comultiplication R-module and P is a prime (but not necessarily maximal) ideal of R which is good for M then the R_P-module M_P is a comultiplication module. The proof of Theorem 2.5 also gives the following fact.

Corollary 2.6. *Let R be a Noetherian ring, and let M be a comultiplication R-module. Then the R_P-module M_P is a comultiplication module for every prime ideal P of R.*

Proof. Suppose that M is a comultiplication R-module. Let P be any prime ideal of R. Following the necessity part of the proof of Theorem 2.5, we obtain $(0 :_R N)m \subseteq T_P$. Because R is a Noetherian ring it follows that $(0 :_R N)m$ is a finitely generated submodule of T_P, and hence $h(0 :_R N)m = 0$ for some $h \in R \setminus P$. As before, it follows that M_P is a comultiplication R_P-module. □

In the next section we shall give an example to show that the converse of Corollary 2.6 is false in general. Let R be a ring and M an R-module. Clearly

$$\left(0 :_M \sum_{i \in I} A_i\right) = \bigcap_{i \in I} (0 :_M A_i),$$

for any collection of ideals A_i $(i \in I)$ of R. Moreover, if A and B are ideals of R, then $(0 :_M A) + (0 :_M B) \subseteq (0 :_M A \cap B)$. Note that if A

and B are ideals of R such that $R = A + B$ then

$$(0 :_M A \cap B) = (0 :_M A \cap B)B + (0 :_M A \cap B)A \subseteq (0 :_M A) + (0 :_M B),$$

and hence $(0 :_M A \cap B) = (0 :_M A) + (0 :_M B)$. Moreover, if U is a simple module and A and B are any ideals of R such that $(A \cap B)U = 0$, then $ABU = 0$ and hence $AU = 0$ or $BU = 0$, so that $(0 :_U A \cap B) = (0 :_U A) + (0 :_U B)$. This can easily be extended to semisimple modules, so that $(0 :_X A \cap B) = (0 :_X A) + (0 :_X B)$ for every semisimple R-module X and arbitrary ideals A and B of R. However, in general, $(0 :_M A) + (0 :_M B) \neq (0 :_M A \cap B)$, as the following example shows (we shall need this example later).

Example 2.7. Let K be any field, let $K_i = K$ $(i \in \mathbf{N})$, where $\mathbf{N}$ denotes the set of positive integers, and let T denote the direct product $\prod_{i \in \mathbf{N}} K_i$. Then T is a commutative ring. Let R denote the subring of T consisting of all elements $(k_1, k_2, k_3, \dots)$, where $k_i \in K_i$ $(i \in \mathbf{N})$, such that there exists a positive integer n with $k_n = k_{n+1} = k_{n+2} = \cdots$. Then R is a commutative von Neumann regular ring with socle $S = \oplus_{i \in \mathbf{N}} K_i$ such that $(0 :_R A \cap B) \neq (0 :_R A) + (0 :_R B)$ for some ideals A and B of R.

Proof. It is easy to check that S is the socle of R. Let I and J be disjoint infinite subsets of $\mathbf{N}$. Let $A = \oplus_{i \in I} K_i$ and $B = \oplus_{j \in J} K_j$. Then A and B are ideals of R. Moreover, $(0 :_R A) \subseteq S$ and $(0 :_R B) \subseteq S$ because I and J are infinite sets. Thus

$$(0 :_R A) + (0 :_R B) \subseteq S \neq R = (0 :_R A \cap B),$$

because $A \cap B = 0$. $\square$

Let R be any ring, and let N and L be submodules of an R-module M. Then

$$(0 :_R N) + (0 :_R L) \subseteq (0 :_R N \cap L).$$

However, as we shall see in the next section, if R is the ring and S the ideal of R in Example 2.7, then S is a comultiplication R-module such that

$$(0 :_R N \cap L) \neq (0 :_R N) + (0 :_R L)$$

for some submodules N and L of the R-module S.

Let R be a ring, and let M be any R-module. A family of submodules L_i $(i \in I)$ is called *direct* provided for each $i, j \in I$ there exists a $k \in I$ such that $L_i + L_j \subseteq L_k$ and in this case

$$N \cap \left(\sum_{i \in I} L_i \right) = \sum_{i \in I} (N \cap L_i),$$

for every submodule N of M. On the other hand, a family of submodules H_i $(i \in I)$ of M is called *inverse* if for each $i, j \in I$ there exists a $k \in I$ such that $H_k \subseteq H_i \cap H_j$. Note the following simple fact.

Lemma 2.8. *Let R be a ring, and let H_i $(i \in I)$ be any inverse family of submodules of M. Then $(0 :_R H_i)$ $(i \in I)$ is a direct family of ideals of R.*

Proof. Let $i, j \in I$. By hypothesis there exists a $k \in I$ such that $H_k \subseteq H_i \cap H_j$, and in this case

$$(0 :_R H_i) + (0 :_R H_j) \subseteq (0 :_R H_i \cap H_j) \subseteq (0 :_R H_k). \qquad \square$$

Lemma 2.3 (i) shows that if M is a comultiplication module then

$$N + \cap_{i \in I} L_i = \cap_{i \in I} (N + L_i),$$

for all submodules N and L_i $(i \in I)$ of M such that $\cap_{i \in I} L_i = 0$. Recall that a module M is said to satisfy the *$AB5^*$-condition* and is called an *$AB5^*$ module* provided for every inverse family H_i $(i \in I)$ of submodules,

$$N + \cap_{i \in I} H_i = \cap_{i \in I} (N + H_i),$$

for every submodule N of M. For more information about $AB5^*$-modules see, for example, [**28**]. An arbitrary module M is called (*Goldie*) *finite dimensional* if M does not contain an infinite direct sum of non-zero submodules. In addition, M is called *quotient finite dimensional* if every homomorphic image of M is finite dimensional. For example, Noetherian modules are quotient finite dimensional. More generally, modules with Krull dimension are quotient finite dimensional (see, for example, [**19**, Lemma 6.2.6]). The next result is proved by adapting the proof of [**21**, Lemma 6.28].

Theorem 2.9. *Let R be any ring, and let M be a comultiplication R-module such that $(0 :_M A \cap B) = (0 :_M A) + (0 :_M B)$ for all ideals A and B of R. Then M is an $AB5^*$ module. Moreover, if in addition M is finitely generated then M is quotient finite dimensional.*

Proof. Let H_i $(i \in I)$ be any inverse family of submodules of M, and let N be any submodule of M. Because M is a comultiplication module, Theorem 1.5 (ii) gives that

$$\cap_{i\in I}(N + H_i) = \cap_{i\in I}[(0 :_M (0 :_R N)) + (0 :_M (0 :_R H_i))]$$

and by hypothesis,

$$\cap_{i\in I}\,[(0 :_M (0 :_R N)) + (0 :_M (0 :_R H_i))] \\ = \cap_{i\in I}(0 :_M [(0 :_R N) \cap (0 :_R H_i)]).$$

Next it is clear that

$$\cap_{i\in I}(0 :_M [(0 :_R N) \cap (0 :_R H_i)]) = \left(0 :_M \sum_{i\in I}[(0 :_R N) \cap (0 :_R H_i)]\right).$$

But Lemma 2.8 and our above remarks give

$$\sum_{i\in I}[(0 :_R N) \cap (0 :_R H_i)] = (0 :_R N) \cap \sum_{i\in I}(0 :_R H_i),$$

and by hypothesis,

$$\left(0 :_M \left[\left(0 :_R N\right) \cap \sum_{i\in I}(0 :_R H_i)\right]\right) \\ = (0 :_M : (0 :_R N) + \left(0 :_M \sum_{i\in I}(0 :_R H_i)\right).$$

Next it is clear that

$$\left(0 :_M \sum_{i\in I}(0 :_R H_i)\right) = \cap_{i\in I}(0 :_M (0 :_R H_i)).$$

Finally by Theorem 1.5 (ii) again, we can conclude that

$$\cap_{i\in I}(N + H_i) = N + \cap_{i\in I}H_i.$$

It follows that M is an $AB5^*$ module.

Now suppose that in addition M is finitely generated. Without loss of generality M is non-zero. Let $A = (0 :_R M)$. By Lemma 1.2 the (R/A)-module M is a comultiplication module. It is easy to check that as an (R/A)-module M still satisfies the condition on annihilators stated in the theorem and thus the (R/A)-module M is an $AB5^*$ module. But the ring R/A is semilocal by Corollary 2.2 so that M is a quotient finite dimensional (R/A)-module by [**21**, Proposition 6.25 (1) and Corollary 6.26]. Now it is clear that the R-module M is quotient finite dimensional. □

We do not know an example of a ring R and a comultiplication R-module M such that $(0 :_M A\cap B) \neq (0 :_M A)+(0 :_M B)$ for some ideals A, B of R. It is proved in [**16**, Corollary 3.3] that every Noetherian injective module over a commutative ring is Artinian. Now we have the following result.

Lemma 2.10. *Let R be a Noetherian ring. Then every comultiplication R-module is Artinian.*

Proof. Let $N_1 \supseteq N_2 \supseteq \cdots$ be any descending chain of submodules of a comultiplication R-module M. Then $(0 :_R N_1) \subseteq (0 :_R N_2) \subseteq \cdots$ is an ascending chain of ideals of R. There exists a positive integer k such that $(0 :_R N_k) = (0 :_R N_{k+1}) = \cdots$, and hence $N_k = N_{k+1} = \cdots$ by Theorem 1.5. □

Corollary 2.11. *Let R be any ring. Then every Noetherian comultiplication R-module is Artinian.*

Proof. Let M be a Noetherian comultiplication module over the ring R. Let $A = (0 :_R M)$. Then M is finitely generated and in the usual way the R-module R/A embeds in the R-module $M^{(n)}$, for some positive integer n. It follows that R/A is a Noetherian ring. By Lemma 1.2 M is a comultiplication (R/A)-module, and hence M is an Artinian (R/A)-module by Lemma 2.10. Thus the R-module M is Artinian. □

3. Some examples of comultiplication modules. Let R be any ring. We already know that every submodule of a comultiplication module over R is also a comultiplication module (see Lemma 2.1). However homomorphic images of comultiplication modules need not be comultiplication modules. For an example see [**12**, page 217, Example 4]. Moreover, every simple R-module U is clearly a comultiplication module, but the module $U \oplus U$ is not by Corollary 1.6. Thus the class of comultiplication R-modules is not closed under either direct sums or extensions.

Let R be a ring, and let an R-module M be a self-cogenerator. Let S denote the endomorphism ring of the R-module M, and suppose that S is a commutative ring (this happens, for example, when every submodule of M is fully invariant). Then M is an S-module when we define (as usual) $\varphi m = \varphi(m)$ for all $\varphi \in S$ and $m \in M$. For each submodule N of M, M/N embeds in M^I, for some index set I, and hence there exists an ideal A in S such that $N = \{m \in M : Am = 0\} = (0 :_M A)$. It follows that, in this case, the S-module M is a comultiplication module. In particular, if R is a Noetherian local ring with unique maximal ideal P and E is the injective hull $E(R/P)$ then E is a self-cogenerator and the endomorphism ring S of E is commutative (see, for example, [**34**, Proposition 5.13]. In this case, S is the completion of the local ring R (see [**23**, Theorem 5.14, Corollary 1]. Thus, if R is a complete local ring, then $R = S$ and E is a comultiplication R-module (see [**5**, Theorem 3.17] where a different proof is given). Now we consider certain direct sums.

Theorem 3.1. *Let R be a ring, let P_i $(i \in I)$ be any non-empty collection of distinct maximal ideals of R, let $k(i)$ $(i \in I)$ be any collection of positive integers and let M_i be any non-zero R-module such that $P_i^{k(i)} M_i = 0$ for all $i \in I$. Then the R-module $M = \oplus_{i \in I} M_i$ is a* *comultiplication module if and only if M_i is a comultiplication module and $\cap_{j\neq i} P_j^{k(j)} \nsubseteq P_i$ for all $i \in I$.*

Proof. Suppose first that M is a comultiplication module. By Lemma 2.1 (i), M_i is a comultiplication module for each $i \in I$. Let $i \in I$. We set $A_i = \cap_{j\neq i} P_j^{k(j)}$. Suppose that $A_i \subseteq P_i$. Because $A_i(\oplus_{j\neq i} M_j) = 0$, we see that $A_i^{k(i)}M = 0$. Suppose that $A_iM = 0$. Because M is a comultiplication module we have:

$$\oplus_{j\neq i} M_j = (0 :_M (0 :_R \oplus_{j\neq i} M_j)) = (0 :_M \cap_{j\neq i} P_j^{k(j)}) = (0 :_M A_i) \supseteq M,$$

and hence $M_i = 0$, a contradiction. Thus $A_iM \neq 0$. There exists an integer t with $1 \leq t \leq k(i) - 1$ such that $A_i^tM \neq 0$ but $A_i^{t+1}M = 0$. It follows that $A_i^tM \subseteq (0 :_M A_i) = \oplus_{j\neq i} M_j$ and hence $A_i^tM_i = 0$. This implies that $A_i^tM = 0$, a contradiction. We conclude that $A_i \nsubseteq P_i$.

Conversely, suppose that M_i is a comultiplication module and $\cap_{j\neq i} P_j^{k(j)} \nsubseteq P_i$ for all $i \in I$. Let N be any submodule of M. Let $x \in N$. There exist a positive integer s and distinct elements $i_j \in I$ $(1 \leq j \leq s)$ such that $x_{i_j} \in M_{i_j}$ $(1 \leq j \leq s)$ and $x = x_{i_1} + \cdots + x_{i_s}$. By hypothesis, $R = P_{i_1}^{k(i_1)} + (P_{i_2}^{k(i_2)} \cap \cdots \cap P_{i_s}^{k(i_s)})$ so that $(1-a)(x - x_{i_1}) = 0$ for some $a \in P_{i_1}^{k(i_1)}$. Hence $x_{i_1} = (1-a)x \in N$. Similarly $x_{i_j} \in N$ $(2 \leq j \leq s)$. We have proved that $N = \oplus_{i\in I} (N \cap M_i)$. By hypothesis, for each $i \in I$, there exists an ideal B_i in R such that $N \cap M_i = (0 :_{M_i} B_i)$. Clearly $(0 :_R N) = \cap_{i\in I} B_i$. Let $m \in M$ such that $(\cap_{i\in I} B_i)m = 0$. For each j in I let $\pi_j : M \to M_j$ denote the canonical projection. Let $j \in I$. Then $\pi_j(m) \in M_j$ and $(\cap_{i\in I} B_i)\pi_j(m) = 0$. Note that, for each $i \in I$, $P_i^{k(i)} \subseteq B_i$. Hence $\cap_{i\neq j} B_i \nsubseteq P_j$ and from this we deduce that $R = P_j^{k(j)} + (\cap_{i\neq j} B_i)$. Now $B_j\pi_j(m) = 0$ and $\pi_j(m) \in N \cap M_j$. It follows that $m \in \oplus_{i\in I} (N \cap M_i) = N$. By Theorem 1.5, we have proved that M is a comultiplication module. $\square$

Corollary 3.2. *Let R be a ring, and let an R-module $M = \oplus_{i\in I} U_i$ be a direct sum of simple submodules U_i $(i \in I)$, for some index set I. Then M is a comultiplication module if and only if $\cap_{j\neq i}(0 :_R U_j) \nsubseteq (0 :_R U_i)$ for all $i \in I$.*

Proof. By Theorem 3.1. $\square$

Corollary 3.3. *Let R be a semiprime ring with socle S. Then the R-module S is a comultiplication module.*

Proof. Suppose that $S \neq 0$. There exist an index set I and minimal ideals U_i $(i \in I)$ of R such that $S = \oplus_{i \in I} U_i$. For each $i \in I$, let $P_i = (0 :_R U_i)$. Let $i \in I$. For each $j \in I \setminus \{i\}$, $U_i U_j = 0$ and hence $U_i \subseteq P_j$. Thus $U_i \subseteq \cap_{j \neq i} P_j$ and $U_i \nsubseteq P_i$ because $U_i^2 \neq 0$. It follows that $\cap_{j \neq i} P_j \nsubseteq P_i$ for all $i \in I$. By Corollary 3.2, S is a comultiplication R-module. □

Using Corollary 3.3 we show that there exist a ring R and a comultiplication R-module M such that $(0 :_R N \cap L) \neq (0 :_R N) + (0 :_R L)$ for some submodules N and L of M.

Example 3.4. Let R and S be as in Example 2.7. Then S is a comultiplication R-module such that $(0 :_R N \cap L) \neq (0 :_R N) + (0 :_R L)$ for some submodules N and L of S.

Proof. By Corollary 3.3 S is a comultiplication R-module. Let A and B be as in Example 2.7. Then A and B are submodules of the R-module S and, as before, $(0 :_R A \cap B) \neq (0 :_R A) + (0 :_R B)$. □

Here is another example. Let K be any field, and let $K_i = K$ $(i \in I)$ for any index set I. Let $R = \prod_{i \in I} K_i$ be the direct product of the fields K_i $(i \in I)$. Then R is a commutative ring. For each $i \in I$ let $\pi_i : R \to K_i$ denote the canonical projection, and let P_i denote the kernel of π_i. Then P_i is a maximal ideal of R for each $i \in I$, and it is easy to check that if I has at least two elements then $\cap_{j \neq i} P_j \nsubseteq P_i$ for all $i \in I$. By Corollary 3.2 the semisimple R-module $\oplus_{i \in I}(R/P_i)$ is a comultiplication module.

We now give an example to show that in Theorem 2.5 some condition is required on the maximal ideals of R. If p is any prime in the ring $\mathbf{Z}$ of integers, then $\mathbf{Z}_p$ will denote the localization of $\mathbf{Z}$ at the maximal ideal $\mathbf{Z}p$ and M_p the localization of any $\mathbf{Z}$-module M at $\mathbf{Z}p$.

Example 3.5. Let I denote an infinite set of primes in $\mathbf{Z}$, and let M denote the $\mathbf{Z}$-module $\oplus_{p \in I}(\mathbf{Z}/\mathbf{Z}p)$. Then M_p is a simple (and hence comultiplication) $\mathbf{Z}_p$-module for every $p \in I$, but the $\mathbf{Z}$-module M is not a comultiplication module. Moreover, M satisfies (ii)–(vi) of Lemma 2.1.

Proof. For each prime $q \in I$, $T_q = \{m \in M : cm = 0 \text{ for some } c \in \mathbf{Z} \setminus \mathbf{Z}q\} = \oplus_{p \neq q}(\mathbf{Z}/\mathbf{Z}p)$ and hence $M/T_q \cong \mathbf{Z}/\mathbf{Z}q$. It follows that M_q is simple for each prime q in I. By Corollary 3.2, M is not a comultiplication $\mathbf{Z}$-module. To check that M satisfies (ii)–(vi) of Lemma 2.1 is elementary. $\square$

Before proceeding, we give a simple example to show that, in general, the condition that $\cap_{j \neq i} P_j^{k(j)} \nsubseteq P_i$ in Theorem 3.1 cannot be replaced by the simpler condition that $\cap_{j \neq i} P_j \nsubseteq P_i$.

Example 3.6. Let R denote the polynomial ring $\mathbf{Z}[x]$ in an indeterminate x, let I be an infinite set of primes in $\mathbf{Z}$ and let M_p denote the maximal ideal $\mathbf{Z}p + Rx$ of R for each $p \in I$. Let $\{n(p) : p \in I\}$ be any unbounded collection of positive integers. Let Q be any maximal ideal of R such that $x \notin Q$. Then $\cap_{p \in I} M_p \nsubseteq Q$, but $\cap_{p \in I} M_p^{n(p)} \subseteq Q$.

Proof. Because R has zero Jacobson radical, there exists a maximal ideal Q of R such that $x \notin Q$. Clearly $x \in \cap_{p \in I} M_p$, and hence $\cap_{p \in I} M_p \nsubseteq Q$. Let $r \in \cap_{p \in I} M_p^{n(p)}$. Suppose that $r \neq 0$. Then $r = a_0 + a_1 x + \cdots + a_k x^k$ for some integer $k \geq 0$ and elements $a_i \in \mathbf{Z}$ $(0 \leq i \leq k)$ with $a_k \neq 0$. There exists an infinite subset J of I such that $n(p) \geq k + 1$ for all $p \in J$. Then $r \in \cap_{p \in J} M_p^{n(p)}$ implies that $a_k \in \cap_{p \in J} \mathbf{Z}p = 0$, a contradiction. Thus $r = 0$. It follows that $\cap_{p \in I} M_p^{n(p)} = 0 \subseteq Q$. $\square$

Now we consider trivial extensions. For a good account of trivial extensions see [**3**]. Let S be a ring, and let V be an S-module. Then the *trivial extension* of V by S, denoted by $V \rtimes S$, is the set of ordered pairs (s, v) with $s \in S$ and $v \in V$. The set $V \rtimes S$ can be made into a ring by defining

$$(s, v) + (s', v') = (s + s', v + v'),$$

and

$$(s, v)(s', v') = (ss', sv' + s'v),$$

for all $s, s' \in S$ and $v, v' \in V$. It is well known, and easy to check, that $R = V \rtimes S$ is a commutative ring with identity $(1, 0)$, where 1 is the identity of S. Moreover $A = V \rtimes 0 = \{(0, v) : v \in V\}$ is an ideal of R such that $A^2 = 0$ and $R/A \cong S$.

In [**5**, Question 3.25] it is asked whether every cocyclic module over a commutative ring is a comultiplication module. It is easy to give examples to show that this is not the case.

Example 3.7. Let S be any non-zero integral domain which is not a local ring (e.g., S could be the ring $\mathbf{Z}$ of rational integers). Let U be any simple S-module, and let E be the injective envelope of U. Let $R = E \rtimes S$ be the trivial extension of E by S. Then the R-module R is a cocyclic module which is not a comultiplication module.

Proof. For any S-submodule X of E we denote by $(0, X)$ the ideal of R consisting of all elements $(0, x)$ with $x \in X$. Let $0 \neq r \in R$. Then $r = (s, e)$ for some $s \in S$, $e \in E$. Suppose that $s \neq 0$. Then $E = sE$ by [**23**, Proposition 2.6]. It follows that $(0, E) = (0, E)(s, e) \subseteq Rr$. Thus $(0, U) \subseteq Rr$. Now suppose that $s = 0$. Then $Rr = (0, Se)$. But $Se \neq 0$ so that $U \subseteq Se$ and hence again we have $(0, U) \subseteq Rr$. It follows that the R-module R is cocyclic. Now let $P = (0 :_S U)$. Then P is a maximal ideal of S. By hypothesis, there exists a non-unit $a \in S$ such that $a \notin P$. Let A be the ideal of R generated by the element $(a, 0)$. It is easy to check that $A = \{(ba, f) : b \in S,\ f \in E\}$. It follows that $(0 :_R A) = 0$ and hence $(0 :_R (0 :_R A)) = R \neq A$, because a is not a unit. Thus the R-module R is not a comultiplication module. $\square$

Next we give some examples of cocyclic modules over Artinian rings that are comultiplication modules.

Example 3.8. Let K be a field, let V be a non-zero finite dimensional vector space over K, and let $R = V \rtimes K$ be the trivial extension of V by K. Then R is an Artinian local ring with unique (up to isomorphism) simple module U and the cocyclic (and injective) R-module $E(U)$ is a comultiplication module.

Proof. Let $J = \{(0, v) : v \in V\}$. Then R is an Artinian ring (because V is finite dimensional) with unique maximal ideal J and $J^2 = 0$.

Note that $R/J \cong K$ so that we can regard K as the unique simple R-module. Now we let X denote the set of elements of the form (φ, k) with $\varphi \in \mathrm{Hom}_K(V, K)$ and $k \in K$. We define

$$(\varphi, k) + (\varphi', k') = (\varphi + \varphi', k + k'),$$

and

$$(a, v)(\varphi, k) = (a\varphi, ak + \varphi(v)),$$

for all $\varphi, \varphi' \in \mathrm{Hom}_K(V, K)$ and $a, k, k' \in K$. In this way X becomes an R-module.

Let (φ, k) be any non-zero element of X. If $\varphi \neq 0$, then there exists $v \in V$ such that $\varphi(v) \neq 0$ and in this case $(0, \varphi(v)) = (0, v)(\varphi, k) \in R(\varphi, k)$. It follows that if U is the submodule $\{(0, k) : k \in K\}$ of X then U is a simple essential submodule of X and hence X is cocyclic. We claim that X is an injective R-module. Let B be a proper ideal of R, and let $\alpha : B \to X$ be an R-homomorphism. Then $B \subseteq J$, J is semisimple and there exists an ideal C of R such that $J = B \oplus C$. Thus α can be lifted to a homomorphism $\beta : J \to X$. Note that $J\beta(J) = \beta(J^2) = 0$ so that $\beta(J) \subseteq (0 :_X J) = U$. Let $\varepsilon : U \to K$ denote the natural K-isomorphism. Define $\theta : V \to K$ by $\theta(v) = \varepsilon\beta(0, v)\,(v \in V)$. Then $\theta \in \mathrm{Hom}_K(V, K)$, $(\theta, 0) \in X$, and

$$\beta(0, v) = (0, v)(\theta, 0)\,(v \in V).$$

It follows that X is an injective R-module and hence $X = E(U)$.

Let Y be any non-zero submodule of X. If $Y = U$, then $Y = (0 :_X J)$. Suppose that $Y \neq U$. There exists a non-zero subspace W of the K-vector space $\mathrm{Hom}_K(V, K)$ such that $Y = \{(\lambda, k) : \lambda \in W, k \in K\}$. It is easy to check that $(0 :_R Y) = \{(0, v) : \lambda(v) = 0 \text{ for all } \lambda \in W\}$. Let V' denote the subspace of V consisting of all elements v in V such that $\lambda(v) = 0$ for all $\lambda \in W$. Clearly

$$(0 :_X (0 :_R Y)) = \{(\mu, k) : k \in K, \mu \in \mathrm{Hom}\, K(V, K) \text{ and } \mu(v) = 0\ (v \in V')\}.$$

But if V is n-dimensional over K, for some positive integer n, then $\mathrm{Hom}_K(V, K)$ is also n-dimensional. Now Y is t-dimensional for some

$2 \leq t \leq n+1$, and hence W is $(t-1)$-dimensional, V' is $(n-t+1)$-dimensional and $(0 :_X (0 :_R Y))$ is t-dimensional, all over K. It follows that $(0 :_X (0 :_R Y)) = Y$. Thus X is a comultiplication module by Theorem 1.5. □

Let R be any ring, let P be a maximal ideal of R and let M be an R-module. Then the set of elements $m \in M$ such that $P^n m = 0$ for some positive integer n (depending on m) is a submodule of M that we shall denote by $M(P)$. We complete this section by considering modules over Dedekind domains. Let R be a Dedekind domain. Let U be a simple R-module, let $E = E(U)$ and let $P = (0 :_R U)$. It is well known that the submodules of E are totally ordered and form a chain

$$0 \subset U = (0 :_E P) \subset (0 :_E P^2) \subset \cdots \subseteq \cup_{n \geq 1}(0 :_E P^n) = E.$$

It follows that E is a comultiplication module. In addition the R-module R/P^n is a comultiplication module because $R/P^n \cong (0 :_E P^n)$ (see Lemma 2.1 (i)). Combining these facts with Theorem 3.1, we obtain the sufficiency part of the next result.

Theorem 3.9. *Let R be a Dedekind domain. Then a non-zero R-module M is a comultiplication module if and only if M is cocyclic or there exist positive integers $n, k(1), \ldots, k(n)$ and distinct maximal ideals P_i $(1 \leq i \leq n)$ of R such that $M \cong (R/P_1^{k(1)}) \oplus \cdots \oplus (R/P_n^{k(n)})$.*

Proof. For the sufficiency see the above remarks. Conversely, suppose that M is a comultiplication R-module. By Lemma 2.10 M is Artinian. By [**22**, Example 8.48] there exist a positive integer n and distinct maximal ideals P_i $(1 \leq i \leq n)$ of R such that $M = M(P_1) \oplus \cdots \oplus M(P_n)$. Next, by Lemma 2.1, for each $1 \leq i \leq n$, $(0 :_M P_i)$ is simple and hence $M(P_i)$ is cocyclic. Suppose that $(0 :_R M(P_i)) = 0$ for some $1 \leq i \leq n$. Then $M(P_i) = (0 :_M (0 :_R M(P_i)) = M$, and in this case M is cocyclic. Otherwise $(0 :_R M(P_i)) \neq 0$ and $M(P_i) \cong R/P_i^{k(i)}$ for some positive integer $k(i)$, for each $1 \leq i \leq n$. □

In view of Example 3.7 and Theorem 3.9 it is natural to ask for a Noetherian ring R whether every cocyclic R-module is comultiplication. Because R is Noetherian, every cocyclic R-module is countably generated. We now aim to show that finitely generated cocyclic modules over Noetherian rings are comultiplication.

Lemma 3.10. *Let R be a Noetherian ring, let M be a cocyclic R-module with simple socle U, and let P denote the maximal ideal $(0 :_R U)$ of R. Then for each element m in M there exists a positive integer n such that $P^n m = 0$. Moreover, $L = (0 :_M (0 :_R L))$ for every finitely generated submodule L of M. In particular, $N = (0 :_M (0 :_R N))$ for every submodule N of M such that $P^k N = 0$ for some positive integer k.*

Proof. Let $E = E(U)$. Then, without loss of generality, we can suppose that M is a submodule of E. By [**23**, Proposition 4.23], $E = \cup_{n=1}^{\infty}(0 :_E P^n)$ and hence $M = \cup_{n=1}^{\infty}(0 :_M P^n)$. Next, let L be any finitely generated submodule of M and let $\varphi : L \to M$ be any homomorphism. Then φ lifts to an endomorphism θ of E. There exists a positive integer k such that $L \subseteq (0 :_M P^k)$. By [**23**, Proposition 5.12], there exists $r \in R$ such that $\theta(m) = rm$ for all $m \in (0 :_E P^k)$. It follows that $\varphi(x) = rx$ for all $x \in L$. Thus every homomorphism $\varphi : L \to M$ is trivial. By Theorem 1.8, $L = (0 :_M (0 :_R L))$ for every finitely generated submodule L of M. Now let N be any submodule of M such that $P^k N = 0$ for some positive integer k. Because M is Artinian by [**23**, Theorem 4.30], N is finitely generated and hence $N = (0 :_M (0 :_R N))$, as required. □

We shall call a semisimple module M *completely inhomogeneous* if M is a direct sum of pairwise non-isomorphic simple submodules. We end this section with the following result.

Theorem 3.11. *A finitely generated Artinian module is a comultiplication module if and only if its socle is completely inhomogeneous.*

Proof. Let R be any ring, and let M be a finitely generated Artinian R-module. By Lemma 1.2 we can suppose without loss of generality that the module M is faithful. Recall that the R-module R embeds in a finite direct sum of copies of M and hence the ring R is Artinian. By [**23**, Theorem 3.25], R is a Noetherian ring. If M is a comultiplication module, then the socle Soc (M) of M is a comultiplication module by Lemma 2.1 (i) and is completely inhomogeneous by Lemma 2.1 (v). Conversely, suppose that Soc (M) is completely inhomogeneous. As in the proof of Theorem 3.9, there exist a positive integer n and distinct maximal ideals P_i $(1 \leq i \leq n)$ of R such that $M = M(P_1) \oplus \cdots \oplus M(P_n)$. By hypothesis, $M(P_i)$ is a cocyclic module for each $1 \leq i \leq n$. Next, for each $1 \leq i \leq n$, $M(P_i)$ is a comultiplication module by Lemma 3.10. Finally M is a comultiplication module by Theorem 3.1. □

4. Quasi-injective modules. Let R be a (commutative) ring. Given an R-module M, recall that an R-module X is called M-*injective* provided that for each submodule L of M and homomorphism $\varphi : L \to X$, φ can be lifted to M. If X is M-injective for every R-module M, then, of course, X is injective. A module M is called *quasi-injective* in case M is M-injective. The ring R is usually called *self-injective* if the R-module R is quasi-injective. Moreover, the ring R is called *quasi-Frobenius* or simply a *QF ring* if R is Artinian self-injective. The first lemma in this section is taken from [**12**, Theorems 24.4 and 24.5]. Recall that a ring R is a dual ring provided the R-module R is comultiplication.

Lemma 4.1. *The following statements are equivalent for a ring R.*

(i) *R is a QF ring.*

(ii) *R is an Artinian dual ring.*

(iii) *R is a Noetherian dual ring.*

(iv) *R is a Noetherian self-injective ring.*

Now we examine comultiplication modules and some somewhat more general modules in relation to injectivity properties. The next lemma is well known but we include its proof for completeness.

Lemma 4.2. *Let R be any ring. Consider the following statements for an R-module M.*

(i) *$Rm = (0 :_M (0 :_R Rm))$ for all $m \in M$.*

(ii) *$\beta(m) \in Rm$ for every $m \in M$ and homomorphism $\beta : Rm \to M$.*

(iii) *Rm is a fully invariant submodule of M for every $m \in M$.*

(iv) *Every submodule of* M *is fully invariant.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv).

Proof. (i) $\Rightarrow$ (ii). Let $m \in M$, and let $\beta : Rm \to M$ be a homomorphism. Then $(0 :_R Rm)\beta(m) = \beta((0 :_R Rm)m) = \beta(0) = 0$, so that $\beta(m) \in (0 :_M (0 :_R Rm)) = Rm$.

(ii) $\Rightarrow$ (i). Let $m \in M$. Clearly $Rm \subseteq (0 :_M (0 :_R Rm))$. Let $x \in (0 :_M (0 :_R Rm))$. Define a mapping $\alpha : Rm \to M$ by $\alpha(rm) = rx$ $(r \in R)$. It is easy to check that α is well-defined and a homomorphism. By hypothesis, $x = \alpha(m) \in Rm$. It follows that $(0 :_M (0 :_R Rm)) \subseteq Rm$ and hence $Rm = (0 :_M (0 :_R Rm))$.

(ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv). Clear. □

Lemma 4.3. *Let* R *be any ring. The following statements are equivalent for an* R*-module* M.

(i) *For each finitely generated submodule* L *of* M, *every homomorphism* $\beta : L \to M$ *is trivial.*

(ii) $Rm = (0 :_M (0 :_R Rm))$ *for all* $m \in M$ *and* $(0 :_R N \cap K) = (0 :_R N) + (0 :_R K)$ *for all finitely generated submodules* N *and* K *of* M.

Proof. (i) $\Rightarrow$ (ii). Suppose that (i) holds. Then $Rm = (0 :_M (0 :_R Rm))$ for all $m \in M$ by Lemma 4.2. Let N and K be finitely generated submodules of M. Clearly, $(0 :_R N) + (0 :_R K) \subseteq (0 :_R N \cap K)$. Let $r \in (0 :_R N \cap K)$. Define a mapping $\alpha : N + K \to M$ by $\alpha(u + v) = ru$ for all $u \in N$, $v \in K$. If $u \in N$ and $v \in K$ are such that $u + v = 0$, then $u = -v \in N \cap K$ and hence $ru = 0$. Thus α is well-defined. It is easy to see that α is a homomorphism. By (i) there exists an $s \in R$ such that $\alpha(u + v) = s(u + v)$ for all $u \in N$, $v \in K$. Note that, for all $u \in N$, $ru = \alpha(u) = su$ and hence $r - s \in (0 :_R N)$. Also, for all $v \in K$, $0 = \alpha(v) = sv$ so that $s \in (0 :_R K)$. Thus $r = (r - s) + s \in (0 :_R N) + (0 :_R K)$. It follows that $(0 :_R N \cap K) \subseteq (0 :_R N) + (0 :_R K)$, and hence $(0 :_R N \cap K) = (0 :_R N) + (0 :_R K)$.

(ii) $\Rightarrow$ (i). Let L be any finitely generated submodule of M, and let $\beta : L \to M$ be a homomorphism. There exist a positive integer n and elements $x_i \in L$ $(1 \leq i \leq n)$ such that $L = Rx_1 + \cdots + Rx_n$. We prove by induction on n that there exists an $r \in R$ with $\beta(x) = rx$ for all $x \in L$. Suppose first that $n = 1$. Note that $Rx_1 = (0 :_M (0 :_R Rx_1))$. By Lemma 4.2, $\beta(x_1) = ax_1$ for some $a \in R$, and hence $\beta(x) = ax$ for all $x \in Rx_1 = L$. Now suppose that $n \geq 2$. Let $L_1 = Rx_1$ and $L_2 = Rx_2 + \cdots + Rx_n$ so that $L = L_1 + L_2$. By the case $n = 1$, there exists a $b_1 \in R$ such that $\beta(w_1) = b_1w_1$ for all $w_1 \in L_1$. Next, by induction on n, there exists a $b_2 \in R$ such that $\beta(w_2) = b_2w_2$ for all $w_2 \in L_2$. Now if $w \in L_1 \cap L_2$ then $b_1w = \beta(w) = b_2w$ and hence

$$b_1 - b_2 \in (0 :_R L_1 \cap L_2) = (0 :_R L_1) + (0 :_R L_2).$$

There exist elements $c_1 \in (0 :_R L_1)$, $c_2 \in (0 :_R L_2)$ such that $b_1 - b_2 = c_1 + c_2$. Let $r = b_1 - c_1 = b_2 + c_2$. For all $z \in L_1$,

$$\beta(z) = b_1z = (b_1 - c_1)z = rz,$$

and for all $z \in L_2$,

$$\beta(z) = b_2z = (b_2 + c_2)z = rz.$$

It follows that $\beta(z) = rz$ for all $z \in L_1 + L_2 = L$. This completes the proof. □

Theorem 4.4. *Let R be any ring, and let M be a Noetherian R-module such that*

(a) $Rm = (0 :_M (0 :_R Rm))$ *for all* $m \in M$, *and*

(b) $(0 :_R N \cap K) = (0 :_R N) + (0 :_R K)$ *for all submodules N and K of M.*

Then M is quasi-injective.

Proof. By Lemma 4.3. □

Lemma 4.5. *Let R be any ring, let M be an R-module, let n be a positive integer, and let the R-module $V = M^{(n)}$ be the direct sum of n copies of M. Suppose that*

(a) $Rm = (0 :_M (0 :_R Rm))$ *for all* $m \in M$, *and*

(b) *for each $v \in V$, every homomorphism $\varphi : Rv \to M$ can be lifted to a homomorphism $\theta : V \to M$.*

Then $L = (0 :_M (0 :_R L))$ for every n-generated submodule L of M.

Proof. Let L be any n-generated submodule of M. There exist elements $x_i \in L$ $(1 \le i \le n)$ such that $L = Rx_1 + \cdots + Rx_n$. Clearly $L \subseteq (0 :_M (0 :_R L))$. Let $m \in (0 :_M (0 :_R L))$. Now let v denote the element $(x_1, \cdots, x_n)$ of V and $\varphi : Rv \to M$ the mapping defined by $\varphi(rv) = rm$ $(r \in R)$. If $r \in R$ and $rv = 0$ then $rx_i = 0$ $(1 \le i \le n)$ and hence $rL = 0$ so that $rm = 0$. Thus φ is well-defined and is clearly a homomorphism. By (b) there exists a homomorphism $\theta : V \to M$ such that θ lifts φ to V. For each $1 \le i \le n$ let $\mu_i : Rx_i \to M$ be the homomorphism defined by $\mu_i(sx_i) = \theta(0, \dots, 0, sx_i, 0, \dots, 0)$ $(s \in R)$, where $(0, \dots, 0, sx_i, 0, \dots, 0)$ is the element of V with ith component sx_i and all other components 0. By (a) and Lemma 4.2, for each $1 \le i \le n$ there exists an $r_i \in R$ such that $\mu_i(x_i) = r_i x_i$. Now

$$\begin{aligned} m &= \varphi(v) = \varphi(x_1, \dots, x_n) = \theta(x_1, \dots, x_n) \\ &= \theta(x_1, 0, \dots, 0) + \cdots + \theta(0, \dots, 0, x_n) \\ &= \mu_1(x_1) + \cdots + \mu_n(x_n) = r_1 x_1 + \cdots + r_n x_n \in L. \end{aligned}$$

Thus $(0 :_M (0 :_R L)) \subseteq L$, and hence $L = (0 :_M (0 :_R L))$. □

Theorem 4.6. *Let R be any ring, and let M be a quasi-injective R-module. Then the following statements are equivalent.*

(i) $Rm = (0 :_M (0 :_R Rm))$ *for all* $m \in M$.

(ii) $L = (0 :_M (0 :_R L))$ *for every finitely generated submodule* L *of* M.

Proof. Since M is quasi-injective it follows that, for every positive integer n, M is $M^{(n)}$-injective (see, for example, [**4**, Proposition 16.13 (2)]). Apply Lemma 4.5. □

Corollary 4.7. *Let R be any ring, and let M be a Noetherian quasi-injective R-module. Then M is a comultiplication module if and only if $Rm = (0 :_M (0 :_R Rm))$ for all $m \in M$.*

Proof. By Theorems 1.5 and 4.6 □

Corollary 4.8. *Let R be any ring, and let M be a Noetherian R-module such that $(0 :_R N \cap K) = (0 :_R N) + (0 :_R K)$ for all submodules*

N and K of M. Then M is a comultiplication module if and only if $Rm = (0 :_M (0 :_R Rm))$ *for all* $m \in M$.

Proof. By Theorem 4.4 and Corollary 4.7. $\square$

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