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Some remarks on the existence of independent solutions for homogeneous first order differential system

By

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§1. Introduction

In an open set U containing the origin in R^n we consider homogeneous first order partial differential operators

$$L_k=\sum_{j=1}^{n} a_k^j(x)\frac{\partial}{\partial x_j} \qquad (k=1, \dots, m)$$

with coefficients in $C^1(U)$. Let $A(x)=(a_k^j(x))_{\substack{j=1,\cdots,n\\ k=1,\cdots,m}}$ be a (m, n)-matrix. Let an integer $r_0=n-\max_{x\in U} \operatorname{rank} A(x)$. In this note we investigate the condition for the existence of independent solutions of C^2-class for the system of differential equations $L_k(f)=0$ $(k=1, \dots, m)$ in a neighborhood of the origin contained in U; here r (at least C^1-class) functions $f_1(x), \dots, f_r(x)$ are called, by definition, independent if $df_1\wedge\cdots\wedge df_r\neq 0$ holds.

When the coefficients are real-valued in $C^1(U)$, we see that it is equivalent to say that there exists a regular change of coordinates around the origin in R^n by which all the operators L_k can be transformed into the operators in $n-r$ new coordinates variables with r real parameters, where r denotes the number of independent solutions for $L_k(f)=0$

$(k=1, \ldots, m)$.

To make our problem clear, let us consider the case of single equation $L(f)=\sum_{j=1}^{n} a_j(x)\frac{\partial f}{\partial x_j}=0$. Suppose all the $a_j(x)$ are real-valued in $C^1(U)$ otherwise they are analytic in U. We know that if one of them is not zero at the origin, there exist $n-1$ independent solutions of C^2-class for that in a neighborhood of the origin. Thus our problem is to investigate the condition for the existence of $r(r\leqq n-1)$ independent solutions of C^2-class in a neighborhood of the origin when $a_j(0)=0$ $(j=1, \ldots, n)$. We want to present a point of view to this problem.

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§2. A theorem

Firstly we state a lemma which is basic in later discussion:

Lemma. *Let us assume that there exist r $(r\leqq n-1)$ independent solutions of C^2-class $f_j(x)$ $(j=1, \ldots, r)$ for $L(f)=\sum_{j=1}^{n} a_j(x)\frac{\partial f}{\partial x_j}=0$ in a neighborhood of the origin contained in U, where $a_j(x)$ $(j=1, \ldots, n)$ are $C^1(U)$. Then there exist $n-r$ homogeneous first order partial differential operators P_j $(j=1, \ldots, n-r)$, which can be determined only by those r independent solutions $f_j(x)$ $(j=1, \ldots, r)$ in a neighborhood of the origin V, satisfying the following conditions:*

(1) *L is expressed as a linear combination of P_j; namely there exist functions $c_j(x)$ $(j=1, \ldots, n-r)$ in $C^1(V)$ such that*

$$L=\sum_{j=1}^{n-r} c_j(x)P_j \text{ in } V;$$

(2) $P_j(f_\lambda)=0$ for $j=1, \ldots, n-r$, $\lambda=1, \ldots, r$ in V;

and furthermore

(3) $\{P_j\}_{j=1,\cdots,n-r}$ *is Jacobi's system.*

Proof. Relabelling the variables if necessary, we may suppose $D\equiv$

$\frac{\partial(f_1, \ldots, f_r)}{\partial(x_1, \ldots, x_r)} \neq 0$ in a neighborhood of the origin V contained in U. Then from the system of equations:

$$\begin{cases} a_1 \frac{\partial f_1}{\partial x_1} + \ldots + a_r \frac{\partial f_1}{\partial x_r} = -a_{r+1} \frac{\partial f_1}{\partial x_{r+1}} - \ldots - a_n \frac{\partial f_1}{\partial x_n}, \\ \cdot \quad \cdots \quad \cdot \quad \cdot \quad \cdots \quad \cdot \\ \cdot \quad \cdots \quad \cdot \quad \cdot \quad \cdots \quad \cdot \\ \cdot \quad \cdots \quad \cdot \quad \cdot \quad \cdots \quad \cdot \\ a_1 \frac{\partial f_r}{\partial x_1} + \ldots + a_r \frac{\partial f_r}{\partial x_r} = -a_{r+1} \frac{\partial f_r}{\partial_{r+1}} - \ldots - a_n \frac{\partial f_r}{\partial x_n}, \end{cases}$$

we can express $a_1(x), \ldots, a_r(x)$ as the linear combinations of $a_{r+1}(x), \ldots, a_n(x)$ with coefficients in $C_1(V)$, say, $a_j(x) = \sum_{i=r+1}^{n} a_j(x)\ c_j^i(x)$ $(j=1, \ldots, r)$; more precisely:

$$c_j^\lambda(x) = -\frac{1}{D} \frac{\partial(f_1, \ldots, f_{\lambda-1}, f_\lambda, f_{\lambda+1}, \ldots, f_r)}{\partial(x_1, \ldots, x_{\lambda-1}, x_j, x_{\lambda+1}, \ldots, x_r)}$$
$$j=r+1, \ldots, n;\ \lambda=1, \ldots, r.$$

Consequently, we have

$$L = a_{r+1}(x)\left(\frac{\partial}{\partial x_{r+1}} + \sum_{i=1}^{r} c_{r+1}^i \frac{\partial}{\partial x_i}\right) + \ldots + a_n(x)\left(\frac{\partial}{\partial x_n} + \sum_{i=1}^{r} c_n^i \frac{\partial}{\partial x_i}\right).$$ Set $H_j \equiv \frac{\partial}{\partial x_j} + \sum_{i=1}^{r} c_j^i(x) \frac{\partial}{\partial x_i}$ for $j=r+1, \ldots, n$.

And denote $P_k \equiv H_{k+r}$ $(k=1, \ldots, n-r)$. Then there remains to prove (2) and (3) for these P_k $(k=1, \ldots, n-r)$. Firstly we can easily verify that for $j=r+1, \ldots, n$

$$P_{j-r}(f) = H_j(f) = \frac{1}{D} \begin{vmatrix} \frac{\partial f}{\partial x_j}, & \frac{\partial f}{\partial x_1}, & \ldots, & \frac{\partial f}{\partial x_r} \\ \frac{\partial f_1}{\partial x_j}, & \frac{\partial f_1}{\partial x_1}, & \ldots, & \frac{\partial f_1}{\partial x_r} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \frac{\partial f_r}{\partial x_j}, & \frac{\partial f_r}{\partial x_1}, & \ldots, & \frac{\partial f_r}{\partial x_r} \end{vmatrix},$$

which show that (2) holds. On the other hand, $P_1, \dots, P_{n-r}$ are clearly linearly independent in V. Since they have r independent solutions $f_j(x)$ $(j=1, \dots, r)$, $\{P_j\}_{j=1,\dots,n-r}$ is Jacobi's system. q.e.d.

Now, let r be a positive integer such that $r \leqq r_0$. Then, from this lemma we have the following

Theorem *Let L_k $(k=1, \dots, m)$ be homogeneous first order partial differential operators*

$$L_k = \sum_{j=1}^{n} a_k^j(x) \frac{\partial}{\partial x_j}$$

where the coefficients are real-valued in $C^1(U)$, otherwise they are analytic in U $(k=1, \dots, m;\ j=1, \dots, n)$. Then there exist r independent solutions of C^2-class for the system of differential equations $L_k(f)=0$ $(k=1, \dots, m)$ in a neighborhood of the origin when and only when there exist $n-r$ homogeneous first order partial differential operators P_j $(j=1, \dots, n-r)$ with real-valued coefficients of C^1-class, otherwise with complex-valued ones analytic in a neighborhood of the origin respectively satisfying the following:

(1) *L_k $(k=1, \dots, m)$ are expressed as the linear combinations of P_j $(j=1, \dots, n-r)$; namely in a neighborhood of the origin it holds that*

$$L_k = \sum_{j=1}^{n-r} c_k^j(x) P_j,$$

where the coefficients are real-valued functions of C^1-class, otherwise complex-valued ones analytic;

and moreover

(2) *$\{P_j\}_{j=1,\dots,n-r}$ is Jacobi's system.*

Proof. This is easy. The necessity is an immediate consequence of the above lemma. In fact, let $f_1(x), \dots, f_r(x)$ be a system of independent solutions for $L_k(f)=0$ $(k=1, \dots, m)$. Then it suffices to apply the lemma to each $L_k(f)=0$, taking account of the fact that P_j are determined only by $f_i(x)$ $(i=1, \dots, r)$.

The sufficiency is shown as follows: Since $P_j(f)=0$ $(j=1, \dots, n-r)$ is Jacobi's system, this system has r independent solutions $f_1(x), \dots, f_r(x)$ of C^2-class in a neighborhood of the origin. These $f_j(x)$ are the solutions of $L_k(f)=0$ $(k=1, \dots, m)$. q.e.d.

§3. Remarks

1. We can restate the theorem as follows: Under the same assumptions and notations as the theoerm, there exist r independent solutions for $L_k(f)=0$ $(k=1, \dots, m)$ in a neighborhood of the origin if and only if there exist $\{j_1, \dots, j_r\} \subset \{1, \dots, n\}$ and real-valued functions $b_{j}^{j_s}(x)$ of C^1-class in a neighborhood of the origin or complex-valued ones analytic respectively according as the coefficients are real-valued ones of C^1-class or complex-valued ones analytic $(s=1, \dots, r;\ j \in \{1, \dots, n\} - \{j_1, \dots, j_r\} \equiv \mathrm{I})$ such that

$$a_k^\lambda(x) = \sum_{j \in I} a_k^j(x) b_k^\lambda(x) \quad \text{for } \lambda \notin I \text{ and } k=1, \dots, m; \tag{1.1}$$

$$\frac{\partial b_j^\mu}{\partial x_i} + \sum_{\lambda \notin I} b_i^\lambda \frac{\partial b_j^\mu}{\partial x_\lambda} = \frac{\partial b_i^\mu}{\partial x_j} + \sum_{\lambda \notin I} b_i^\lambda \frac{\partial b_i^\mu}{\partial x_\lambda} \quad i, j \in I, \ \mu \notin I. \tag{1.2}$$

2. When $a_k^j(x)$ are complex-valued in $C^1(U)$ and not always analytic, the condition stated in the theorem remains a necessary one in order that there exist r independent solutions of C^2-class for $L_k(f)=0$ $(k=1, \dots, m)$ in a neighborhood of the origin. In the actual case we do not know any satisfactory sufficient condition. But, under the conditions (1) and (2) of the theorem, the analyticity in the r suitable variables assures the existence of r independent solutions of C^2-class for $L_k(f)=0$ $(k=1, \dots, m)$; for this, we refer to A. Andreotti and C. D. Hill [1].

3. Let us consider the case $n=2$. Namely we consider the operators $L_k = a_k^1(x)\frac{\partial}{\partial x_1} + a_k^2(x)\frac{\partial}{\partial x_2}$ $(k=1, \dots, m)$ with real-valued coefficients in $C^1(U)$, otherwise with complex-valued ones analytic in U. The theorem states that there exists a solution of C^2-class for $L_k(f)=0$

($k=1, \dots, m$) such that grad $f(0)\neq 0$ in a neighborhood of the origin when and only when there exist a C^1-class or analytic function $b(x)$ ($c(x)$) in a neighborhood of the origin according as the coefficients are real-valued functions in $C^1(U)$, or complex-valued ones analytic in U, satisfying the following:

$$a_k^1(x)=b(x)a_k^2(x) \quad (\text{or } a_k^2(x)=c(x)a_k^1(x)) \quad k=1, \dots, m.$$

In other words the functions:

$$a_k^1(x)/a_k^2(x) \quad \text{or} \quad (a_k^2(x)/a_k^1(x)) \quad k=1, \dots, m$$

defined where $a_k^2(x)\neq 0$ (or $a_k^1(x)\neq 0$) for $k=1, \dots, m$ are the restrictions of the function which is in C^1 or analytic in a neighborhood of the origin to the places where $a_k^2(x)\neq 0$ (or $a_k^1(x)\neq 0$).

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Reference

[1] A. Andreotti and C. D. Hill, Complex characteristic coordinates and tangential Cauchy-Riemann equations, Annali della Scuola Norm. Sup. di Pisa, 1972 (**26**) 299–324.