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On the Theorem of Denjoy-Sacksteder for Codimension One Foliations without Holonomy

By

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§1. Introduction

A transversally orientable codimension one foliation is defined by a non-singular one form ω satisfying the integrability condition $\omega \wedge d\omega = 0$. If ω is closed then the foliation has very simple properties. In this case the "distance" between two leaves is constant and the holonomy group of each leaf is trivial. Conversely, Sacksteder [11] has obtained the following result which is fundamental for the study of codimension one foliations without holonomy, (we say that a foliation is *without holonomy* if the holonomy group of each leaf is trivial).

Theorem 1.1. *Let M be a compact smooth manifold and $\mathscr{F}$ a transversally orientable codimension one foliation without holonomy of class C^r, $r \geqq 2$, then there exists a topological flow $\varphi: M \times R \rightarrow M$ such that*

(1) *φ preserves $\mathscr{F}$, i.e. $\varphi(\ \ , t)$ sends each leaf of $\mathscr{F}$ into a leaf of $\mathscr{F}$.*
(2) *φ is transversal to $\mathscr{F}$, i.e. $\varphi(x, \mathbf{R})$ is transversal to leaves of $\mathscr{F}$.*

By this theorem we can see that a codimension one foliation $\mathscr{F}$ without holonomy is defined by an 1-cocycle in the Alexander-Spanier cohomology theory with real coefficient (see Roussarie [9]). Though $\mathcal{F}$ is not necessarily defined by a closed one form, we can obtain a foliation defined by closed one form by a small deformation of $\mathcal{F}$, more precisely we have the following Theorem 1.2. which permits us the reduction of almost all problems on codimension one foliations without holonomy to those on foliations defined by closed one forms.

Theorem 1.2. *Let M and $\mathcal{F}$ be as in Theorem* 1.1. *then there exists a nonsingular closed one form ω of class C^r on M and a homeomorphism h of M such that*

(1) *h maps each leaf of $\mathcal{F}$ diffeomorphically onto a leaf of the foliation defined by ω*

(2) *h is isotopic to the identity.*

Moreover h can be chosen arbitrarily near to the identity in the C^0-topology and the cohomology class of ω is unique up to multiplications by non-zero real numbers.

In [11] Theorem 1.1. is deduced from more general results concerning on minimal sets of pseudogroup actions on R and the method of [11] is a generalization of the arguments used to prove the theorems of Poincaré-Bendixon type or of Denjoy-Siegel type. In this paper Theorem 1.1. is considered as a generalization of the theorem of Denjoy-Siegel on flows on the 2-torus ([3], [14]) and we give a geometric proof of Theorem 1.1. Our method permits us to obtain the following topological version of the theorem of Sacksteder.

Theorem 1.3. *Let M and $\mathcal{F}$ be as in Theorem* 1.1. *but we do not assume differentiability conditions on M and $\mathcal{F}$. Suppose that there exists a flow on M transverse to $\mathcal{F}$ then the result of Theorem* 1.1. *holds if and only if $\mathcal{F}$ has no exceptional leaves.*

We remark that the hypothese of Theorem 1.3. is satisfied if $\mathcal{F}$ is of class C^1 and the famous example of Denjoy [3] shows the existence of foliations without holonomy of class C^1 with exceptional leaves.

The program of this paper is the following. In §2 we prepare some results concerning on free group actions on S^1. Main results is Theorem 2.1. which generalize the theorem of Denjoy [3]. In §3 we introduce the notion of holonomy maps for codimension one foliations and show that (Theorem 3.1.) the existence of nontrivial holonomy groups is an obstruction to extending the domain of holonomy maps. By this result we can reduce the problems of pseudogroup actions to those of group actions provided that the foliation is without holonomy. This reduction is done in §4 and the notion of characteristic maps for codimension one foliations without holonomy are defined. Theorem 1.3. is obtained by applying Theorem 2.1. to the characteristic map of $\mathscr{F}$. If the foliation is of class C^2, we can relate the characteristic map to the fundamental group of M by using the notion of Novikov transformation [7] and we can apply Theorem 2.1. to prove Theorem 1.1. Theorem 1.2. is proved in §5, here the essential tool is the theorem of Tischler [15], and some properties of foliations defined by closed non-singular one form are discussed in §5.

§2. Free actions on the circle.

In this section we consider some properties of subgroups of the group of homeomorphisms of the circle S^1 which act freely on S^1.

Let $\mathscr{H}^p(R)$ be the group of periodic homeomorphisms of the real line R, where "periodic" means $f(x+1)=f(x)+1$ for any $x \in R$. We define a map γ from $\mathscr{H}^p(R)$ to R by $\gamma(f)=\lim_{n\to\infty} f^n(x)/n$, where $x \in \mathbf{R}$ and it is well known that the limit exists and is independent of x [5]. We call $\gamma(f)$ the *rotation number* of f. The following properties of rotation numbers are well known or easy to prove

(2.1) *Let f and g be elements of $\mathscr{H}^p(R)$, then $\gamma(fgf^{-1})=\gamma(g)$.*

(2.2) *For any integer n there exists $x_n \in R$ such that $f^n(x_n)=x_n+n\cdot\gamma(f)$.*

(2.3) *γ is continuous and the set of elements of $\mathscr{H}^p(R)$ with rational rotation numbers is dense in $\mathscr{H}^p(R)$, where the topology of $\mathscr{H}^p(R)$ is the uniform topology.*

(2.4) *If f and g commutes then $\gamma(f\circ g)=\gamma(f)+\gamma(g)$.*

Let $\mathscr{H}(S^1)$ be the group of orientation preserving homeomorphisms of the circle $S^1=R/Z$ then there exists a natural projection $\pi: \mathscr{H}^p(R)\to\mathscr{H}(S^1)$ induced from the natural projection $\pi: R\to S^1$. We define a map $\gamma: \mathscr{H}(S^1)\to R/Z$ by $\gamma(f)\equiv\gamma(\tilde{f}) \bmod Z$, where $\tilde{f}$ is an element of $\mathscr{H}^p(R)$ such that $\pi(\tilde{f})=f$. Then the properties (2.1)$\sim$(2.4) and the following (2.5) hold for elements of $\mathscr{H}(S^1)$

(2.5) *$f\in\mathscr{H}(S^1)$ has a periodic point if and only if $\gamma(f)$ is rational.*

Definition 2.1. An element $f\in\mathscr{H}(S^1)$ is a *free element* if f is of finite order or f has no periodic point. An element $\tilde{f}\in\mathscr{H}^p(R)$ is *free* if $\pi\circ\tilde{f}\in\mathscr{H}(S^1)$ is a free element. A subgroup G of $\mathscr{H}(S^1)$ is called a *free subgroup* if any element of G is a free element. This is equivalent to say that no element, except the identity, of G has fixed points.

Proposition 2.1. *Let G be a free subgroup of $\mathscr{H}(S^1)$ then G is commutative and the restriction of γ to G is an injective homomorphism into R/Z.*

Proof. At first we prove the injectivity of $\gamma|G$. Let f and g be elements of G satisfying $\gamma(f)=\gamma(g)=\gamma$.

If γ is rational then by (2.5) and the freeness of G, we have $f^k=g^k=$identity for some integer k and it is easy to construct $h\in\mathscr{H}(S^1)$ such that $h\circ f\circ h^{-1}(x)=x+\gamma$ for $^\forall x\in S^1$. On the other hand, by (2.1) and (2.2), there exists $x_0\in S^1$ such that $h\circ g\circ h^{-1}(x_0)=x_0+\gamma$. By an easy calculation we see that $h^{-1}(x_0)$ is a fixed point of $f^{-1}\circ g$ so by freeness of G we have $f=g$. If γ is irrational, let $\tilde{f}$ and $\tilde{g}$ be elements of $\mathscr{H}^p(R)$ such that $\pi(\tilde{f})=f$, $\pi(\tilde{g})=g$, $0<\tilde{f}(0)<1$ and $0<\tilde{g}(0)<1$. We show that there exists $x\in R$ such that $\tilde{f}(x)=\tilde{g}(x)$. Otherwise, we can suppose $\tilde{f}(x)>\tilde{g}(x)$ for any $x\in R$. Then by (2.3) there exists $h\in\mathscr{H}^p(R)$ with rational rotation number such that $\tilde{f}(x)>\tilde{h}(x)>\tilde{g}(x)$ for any $x\in R$. It is clear that we have $\gamma(\tilde{f})\geqq\gamma(\tilde{h})\geqq\gamma(\tilde{g})$ but $\gamma(\tilde{h})$ is rational and $\gamma(\tilde{f})=$

$\gamma(\tilde{g})$ is irrational. This is a contradiction. Thus there exists $x \in R$ such that $\tilde{f}(x) = \tilde{g}(x)$ then we have $f^{-1} \circ g(\pi(x)) = \pi(x)$ and we have $f = g$. Thus we proved the injectivity of $\gamma | G$.

The commutativity of G follows easily. In fact let f and g be elements of $\mathscr{H}(S^1)$ then by (2.1) we have $\gamma(f \circ g \circ f^{-1}) = \gamma(g)$ so by injectivity of $\gamma | G$ we have $f \circ g \circ f^{-1} = g$. The fact that $\gamma | G$ is a homomorphism follows from (2.4). q.e.d.

Definition 2.2. A subgroup G of $\mathscr{H}(S^1)$ is said to be *topologically conjugate to rotations* if there exists $h \in \mathscr{H}(S^1)$ such that hGh^{-1} is contained in the rotation group $SO(2)$ and we say such a homeomorphism h a *linearization map* of G.

Proposition 2.2. *Let G be a finite free subgroup of $\mathscr{H}(S^1)$ then G is cyclic and is topologically conjugate to rotations.*

Proof. Since $\gamma | G : G \to R/Z$ is injective and G is finite, G is isomorphic to a cyclic group. Let k be the order of G and f a generator of G. Then $f^k =$identity and there exists $h \in \mathscr{H}(S^1)$ such that hfh^{-1} is the rotation of angle $2\pi/k$. It is clear that h is a linearization map of G.

Definition 2.3. Let G be an infinite subgroup of $\mathscr{H}(S^1)$, we define $A(x)$, $x \in S^1$, as the set of accumulation points of the orbit $G \cdot x$.

Lemma 2.1. *Let G be an infinite free subgroup of $\mathscr{H}(S^1)$ then $A(x)$ is independent of x and we can denote it $A(G)$. $A(G)$ is either a nowhere dense perfect set or is the whole circle.*

Proof. We assert that for any $x, y \in S^1$ and $f \in G$, $f \neq$ identity, there exists $g \in G$ such that $g(y)$ is contained in the interval $[x, f(x)]$. In fact, since $\gamma(f) \neq 0$, we have $S^1 = \bigcup_{i=0}^{n} [f^i(x), f^{i+1}(x)]$ for sufficiently large n and, if $y \in [f^i(x), f^{i+1}(x)]$, we have $f^{-i}(y) \in [x, f(x)]$. Let x, y be any elements of S^1 and x_0 an element of $A(x)$. There exists $f_n \in G$, $f_i \neq f_j$ for $i \neq j$, such that $\lim_{n\to\infty} f_n(x) = x_0$. We can suppose that $f_n(x)$

$n=1, 2,\dots$ are arranged as $f_1(x)<f_2(x)<\cdots<f_n(x)<\cdots<x_0$. By the above assertion there exists $g_n\in G$ such that $g_n(y)\in[f_n(x), f_{n+1}(x)]$. Then we have $\lim_{n\to\infty} g_n(y)=x_0$ and $g_n(y)\ n=1, 2,\dots$ are all different. Hence $x_0\in A(y)$ and we have $A(x)=A(y)$. Clearly $A(G)$ is a minimal invariant set of the action of G and is perfect. Since the boundary $\partial A(G)$ of $A(G)$ is also an invariant set, we have $\partial A(G)=\phi$ (in this case $A(G)=S^1$) or $\partial A(G)=A(G)$ (in this case $A(G)$ is nowhere dense).
q.e.d.

The theorem of Denjoy [3] is stated as follows.

Theorem. *Let f be an element of $\mathscr{H}(S^1)$ with irrational rotation number then the group G generated by f is an infinite free subgroup of $\mathscr{H}(S^1)$ and G is topologically conjugate to rotations if and only if $A(G)=S^1$. If f is of class C^2 then $A(G)=S^1$ and G is topologically conjugate to rotations.*

We generalize this theorem as follows.

Theorem 2.1. *Let G be a free subgroup of $\mathscr{H}(S^1)$ then*

(1) *If G is finite, G is topologically conjugate to rotations.*
(2) *If G is infinite, G is topologically conjugate to rotations if and only if $A(G)=S^1$.*
(3) *If all elements of G are of class C^2 and if G is finitely generated then G is topologically conjugate to rotations.*

Proof. (1) is the Proposition 2.2. To prove (3) we can assume that there exists an element $f\in G$ with infinite order. Then the rotation number γ of f is irrational and by the theorem of Denjoy there exists $h\in\mathscr{H}(S^1)$ such that $f'=hfh^{-1}$ is the rotation through the angle $2\pi\gamma$. Let g be an arbitrary element of G and put $g'=hgh^{-1}$ then by Proposition 2.1. f' and g' commute and we have $g'(x+\gamma)=g'(x)+\gamma$. For any $y\in S^1=R/Z$ there exists integers k_i, $i=1, 2,\dots,$ such that $\lim_{i\to\infty} k_i\gamma\equiv y$ (mod. 1) and we have

$$g'(x+y)=\lim g'(x+k_i\gamma)=\lim(g'(x)+k_i\gamma)=g'(x)+y\,.$$

Hence g' is also a rotation of S^1 and we proved (3).

The "only if" part of (2) is trivial. To prove "if" part of (2) we consider two cases. At first we consider the case when there exists $f \in G$ with irrational rotation number. Let G' be the subgroup of $\mathscr{H}(S^1)$ generated by f then we assert that $A(G')=S^1$. Otherwise $A(G')$ is a nowhere dense perfect subset of S^1 and $S^1-A(G')$ consists of countable open intervals $I_n=f^n(I_0)$, $n \in Z$. On the other hand, since $A(G)=S^1$, all orbits of G are dense in S^1. Let us choose $x_0 \in A(G')$ then there exists $g \in G$ such that $g((x_0-\varepsilon, x_0+\varepsilon))$ is a proper subset of I_0 for some $\varepsilon>0$. Since $A(G')$ is nowhere dense and perfect, there exists n such that $I_n=f^n(I_0)$ is contained in $(x_0-\varepsilon, x_0+\varepsilon)$. Thus we see that $gf^n(I_0)$ is a proper subset of I_0 and there exists a fixed point of gf^n in I_0. This contradicts to the freeness of G. Thus $A(G')=S^1$ and by the theorem of Denjoy, f is topologically conjugate to a rotation. The fact that G is topologically conjugate to rotations follows from the proof of (3). In case that all elements of G have rational rotation numbers, we reduce the problem to the above case by showing the existence of $f \in \mathscr{H}(S^1)$ with irrational rotation number which has the property that the group $\bar{G}$ generated by G and f is free. Let $\{f_n\} \subset G$ be a sequence such that $\lim_{n\to\infty} \gamma(f_n)$ exists and is irrational, such a sequence exists because $\gamma(G)$ is an infinite subgroup of R/Z. Choosing a suitable subsequence of $\{f_n\}$, we can assume that, for $x_0 \in S^1$, $\lim_{n\to\infty} f_n(x_0)=y_0$ exists. Then by commutativity of G we have $\lim f_n(g(x_0))=g(y_0)$ for any $g \in G$. Thus $\lim f_n$ is well defined as a map from $G \cdot x_0$ to $G \cdot y_0$ and it is easy to see that $\lim f_n$ preserves the configuration of points of $G \cdot x_0$. Since $A(G)=S^1$, $G \cdot x_0$ and $G \cdot y_0$ are dense in S^1 and $f=\lim f_n$ is a well defined homeomorphism of S^1. By continuity of rotation number map, we see that $\gamma(f)$ is irrational. Let $\bar{G}$ be the group generated by G and f then $\bar{G}$ is commutative because we have, for any $x \in G \cdot x_0$ and $g \in G$, $fg(x)=gf(x)$. To show the freeness of G, suppose that $g \in \bar{G}$ has fixed points. Then $\gamma(g)=0$, and, since by (2.4) $\gamma|\bar{G}$ is a homomorphism and $\gamma(f)$ is irrational, we see that g belongs to G. By freeness of G, g is identity and we proved that $\bar{G}$ is a free subgroup of $\mathscr{H}(S^1)$. Thus we proved (2). q.e.d.

Corollary 2.1. *A compact subgroup G of $\mathscr{H}(S^1)$ is topologically conjugate to rotation.*

Proof. We show that G is a free subgroup of $\mathscr{H}(S^1)$. Otherwise there exists $f \in G$ and $x_1, x_2 \in G$ such that $f(x_1)=x_1$ and $f(x_2) \neq x_2$. Then it is easy to see that the sequence f^n, $n=1, 2,\dots$, has no convergent subsequence and this contradicts to compactness of G. Hence if G is finite, G is topologically conjugate to rotations. If G is infinite, by Proposition 2.1, $\gamma|G$ is an isomorphism and there exists $f \in G$ with irrational $\gamma(f)$. Let G' be the subgroup generated by f, then if $A(G')=S^1$ the Corollary is proved by Theorem 2.1. Otherwise $S^1-A(G')$ is a disjoint union of open intervals $f^n(I_0)$, $n \in Z$. Let us choose $x_1 \in A(G')$, then there exists a sequence $\{n_i\}$ such that $\lim f^{n_i}(x_1)=x_1$. Choose a subsequence $\{m_i\}$ of $\{n_i\}$ such that $\lim f^{m_i}$ converges to $g \in G$, then $g(x_1)=x_1$. But for any $x_2 \in I_0$, since $\lim f^{m_i}(x_2) \in A(G')$, we have $g(x_2) \neq x_2$. This contradicts to freeness of G. q.e.d.

We remark that even if all elements of G are analytic, a linearization map of G cannot be taken, in general, to be diffeomorphic (see Arnold [1]). If $A(G)=S^1$, linearization maps are uniquely determined up to compositions of rotation maps.

Also we remark that in Theorem 2.1. (3) the assumption that G is finitely generated is essential. We will show an example of a free subgroup G of $\mathscr{H}(S^1)$ whose elements are of class C^∞ but not topologically conjugate to rotations.

Let C be the nowhere dense perfect subset of $S^1=[0, 1]/\{0, 1\}$ obtained as follows. Let $I_{01}=\left(0, \frac{1}{4}\right)$, $I_{02}=\left(\frac{1}{2}, \frac{3}{4}\right)$, $\tilde{I}_{01}=\left[\frac{1}{4}, \frac{1}{2}\right]$, $\tilde{I}_{02}=\left[\frac{3}{4}, 1\right]$ and let I_{11}, I_{12} be middle 1/3 intervals of $\tilde{I}_{01}$, $\tilde{I}_{02}$ respectively. We define inductively $\tilde{I}_{kj}$ to be the j-th connected component of $[0, 1]-\bigcup_{0\le l\le k} I_l$, where $I_0=I_{01}\cup I_{02}$ and $I_l=\bigcup_{1\le m\le 2^l} I_{lm}$ for $l\geq 1$, and I_{k+1j} the middle 1/3 interval of $\tilde{I}_{kj}$. Then $C=S^1-\bigcup_{l=0}^{\infty} I_l$ is a nowhere dense perfect subset of S^1. Let f_i, $i=1, 2,\dots$, be diffeomorphisms of S^1 satisfying (1) $f_1(x)=x+\frac{1}{2}$ (2) $f_i^2=f_{i-1}$ for $i\geqq 2$

and (3) f_i maps $\tilde{I}_{ij}$ linearly onto $\tilde{I}_{ij+1}$ for $1 \leq j \leq 2^i$, where we identify $\tilde{I}_{i2^i+1}$ with $\tilde{I}_{i1}$. It is easy to construct such diffeomorphisms by induction. Then the group G generated by f_i, $i=1, 2, \ldots$, is free by properties (1), (2) and by (3) C is invariant under G. Thus $A(G) \subset C$ and by Theorem 2.1. (2) G is not topologically conjugate to rotations.

§3. Holonomy maps.

The arguments of this section are closely related to those of Sacksteder-Schwartz [12]. In this section we use no differentiability conditions. We denote $(M, \mathscr{F}, \varphi)$ a triple consisting of a compact manifold M of dimension n, a codimension one foliation $\mathscr{F}$ on M and a flow $\varphi: M \times R \rightarrow M$ whose orbits are transversal to leaves of $\mathscr{F}$, all are of class C^r, $r \geq 0$. If $\mathscr{F}$ is transversally orientable and of class C^r, $r \geq 1$, there exists a transversal flow φ. If $\mathscr{F}$ is a topological foliation, Siebenmann [13] showed the existence of a complementary codimension $(n-1)$-foliation and the arguments of [13] permit us to use ordinal technics (see, for example, [4]) concerning on transversal curves for codimension one foliations in the topological case.

A *distinguished neighborhood* of $(M, \mathscr{F}, \varphi)$ is an open set U in M with a homeomorphism h from $\bar{U}$ onto $I^{n-1} \times I$ satisfying the following conditions (1) and (2).

(1) $h^{-1}(I^{n-1} \times t)$, $t \in I$, is a connected component of $\bar{U} \cap L$, where L is a leal of $\mathscr{F}$. This set is called a *plaque* and we denote P_x the plaque passing through $x \in U$.

(2) $h^{-1}(p \times I)$, $p \in I^{n-1}$, is a connected component of $\bar{U} \cap \varphi(x, \mathbf{R})$ and is called an *axis*. We denote A_x the axis passing through $x \in U$.

It is clear that for any $x, y \in U$, $P_x \cap A_y$ determines a point in U_x.

The *height* of U is defined by $\sup\{t_1 - t_2 | {}^{\exists}x \in U$ such that $\varphi(x, [t_1, t_2]) \subset U\}$. We remark that for any transversal segment $C = \varphi(x, [t_0, t_1])$ there exists a distinguished neighborhood which contains C.

Let x, y be points of a distinguished neighborhood U, we define $x < y$ if there exists $t > 0$ such that $\varphi(x, t) = A_x \cap P_y$. Clearly if $x < y$

and $y<z$ then $x<z$. Let $\{x_i\}$ be a sequence of points of U which converges to $x \in U$. We denote $x_i \nearrow x$ if $x_1 < x_2 < \cdots < x_n < \cdots < x$. $x_i \searrow x$ is defined similarily.

A curve $l: [t_0, t_1] \to M$ is called a *leaf curve* (*plaque curve*) from $l(t_0)$ to $l(t_1)$ if $l([t_0, t_1])$ is contained in a leaf (plaque) and we assume that, if $t \neq t'$, $l(t) \neq l(t')$.

For a subset S of M we define the *$\mathscr{F}$-extension* Q_s of S by $Q_s = \{x \in M | L_x \cap S \neq \phi\}$ where L_x denotes the leaf passing through x.

Let $l: [0, 1] \to M$ be a leaf curve from $l(0)$ to $l(1)$, we define *holonomy maps* $\bar{\Theta}(l): (-t'_0, t_0) \to \mathbf{R}$ or $\Theta(l): \varphi(l(0), (-t'_0, t_0)) \to \varphi(l(1), \mathbf{R})$ in the following way where t'_0 and t_0 are some positive numbers (possibly infinite) determined by l. We call $(-t'_0, t_0)$ the *domain* of $\bar{\Theta}(l)$. For the definition of $\bar{\Theta}(l)$ and $\Theta(l)$, see Fig. 1.

Let $\Phi: [0, 1] \times R \to M$ be the immersion defined by $\Phi(\tau, t) = \varphi(l(\tau), t)$. Then Φ is transversal to $\mathscr{F}$ and induces a foliation $\bar{\mathscr{F}}$ on $[0, 1] \times R$. The leaves of $\bar{\mathscr{F}}$ are transversal to lines $\tau \times R$, $\tau \in [0, 1]$. For t near to 0 we can define a continuous function $f_t: [0, 1] \to R$ by the property that the leaf $\bar{L}_{(0,t)}$ of $\bar{\mathscr{F}}$ is the graph $\{(\tau, f_t(\tau)) | \tau \in [0, 1]\}$ of f_t. We call the leaf curve l_t defined by $l_t(\tau) = \varphi(l(t), f_t(\tau))$ the *t-lift* of l. For such t we define $\bar{\Theta}(l)(t) = f_t(1)$ and set $t_0 = \sup\{t | \bar{\Theta}(l)(t)$ is well defined$\}$, $-t'_0 = \inf\{t | \bar{\Theta}(l)(t)$ is well defined$\}$. Thus we defined the *holonomy map* $\bar{\Theta}(l): (-t'_0, t_0) \to R$. We define $\Theta(l): \varphi(l(0), (-t'_0, t_0)) \to \varphi(l(1), R)$ by $\Theta(l)(\varphi(l(0), t)) = \varphi(l(1), \bar{\Theta}(l)(t))$. Here we distinguish $\varphi(l(0), t_1)$ and $\varphi(l(0), t_2)$ even if $\varphi(l(0), t_1)$ and $\varphi(l(0), t_2)$ coincide as points of M and, by abuse of language, we call $\Theta(l)$ the *holonomy map* of l. Holonomy maps are of class C^r if $(M, \mathscr{F}, \varphi)$ is of class C^r.

Let l be a closed leaf curve with end points $x \in L$, the germ of $\bar{\Theta}(l)$ at 0 is called the *holonomy* of l. The holonomy of l is determined by the homotopy class of l in $\pi_1(L_x, x)$ and is independent of the choice of φ up to conjugations by origin preserving homeomorphism of R (see Haefliger [4]).

We say that a leaf L *has holonomy* if there exists a closed leaf curve l in L such that for any $\varepsilon > 0$ the restriction of $\bar{\Theta}(l)$ to $(-\varepsilon, \varepsilon)$ is not identity. According to [12], we say that a leaf L_x *has locally* *holonomy pseudogroup* if for any $\varepsilon>0$ there exists a closed leaf curve l with end points x such that the restriction of $\bar{\Theta}(l)$ to $(-\varepsilon, \varepsilon)$ is not identity. We say that a leaf L_x is a *holonomy limit leaf* if for any $\varepsilon>0$ there exists t, $-\varepsilon<t<\varepsilon$, such that the leaf passing through $\varphi(x, t)$ has holonomy.

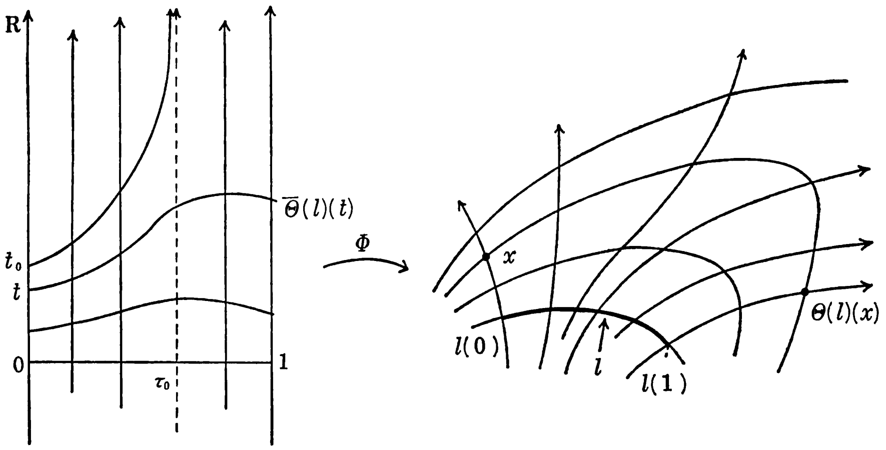
If a leaf L has holonomy then L has locally holonomy pseudogroup and if L has locally holonomy pseudogroup then L is a holonomy limit leaf. At the end of this section we show an example of a leaf without holonomy but has locally holonomy pseudogroup. The author does not know whether the leaves in question in the statements of Theorem 3.1. and Lemma 3.6. have holonomy or not.

Theorem 3.1. *Let $(M, \mathscr{F}, \varphi)$ be as above and l a leaf curve. Let $(-t'_0, t_0)$ be the domain of the holonomy map $\bar{\Theta}(l)$. If t_0 is finite then the leaf L passing through $\varphi(l(0), t_0)$ is a holonomy limit leaf.*

The following lemmas are easy to prove.

Lemma 3.1. *Under the assumption of Theorem* 3.1., $(1)\sim(3)$ *hold.* (*see* Fig. 1.)

(1) *There exists $\tau_0\in[0, 1]$ such that the leaf $\bar{L}_{(0,t_0)}$ of $\bar{\mathscr{F}}$ is asymptotic to the line $\tau_0\times\mathbf{R}$ in $[0, 1]\times\mathbf{R}$ and the holonomy maps*



$([0, 1]\times\mathbf{R}, \bar{\mathscr{F}})$ $(M, \mathscr{F}, \varphi)$

Fig. 1.

$\Theta(l|[\tau_0, 1])$ and $\Theta(l^{-1}|[0, \tau_0])$ are well defined for any $\varphi(l(t_0), t)$, $t \geqq 0$, where $l^{-1}|[0, \tau_0]$ is the leaf curve inverse to $l|[0, \tau_0]$.

(2) *Let l_0: $[0, \tau_0) \to M$ be a leaf curve defined by $l_0(\tau) = \Phi(\tau, f(\tau))$, where f is defined by the property $(\tau, f(\tau)) \in \bar{L}_{(0,t_0)}$, then for any $\varepsilon > 0$ and $\tau_1 < \tau_0$ we have $\lim_{\tau \to \tau_0 - 0} \bar{\Theta}(l_0|[\tau_1, \tau])(-\varepsilon) = -\infty$.*

(3) *$\lim_{\tau \to \tau_0 - 0} l_0(\tau)$ does not exists.*

Lemma 3.2. *Let l be a leaf curve from x to a point of $\varphi(x, R)$, then $\Theta(l)$ is a map from $\varphi(x, (-t_0', t_0))$ to $\varphi(x, \mathbf{R})$. If there exists $t \in (-t_0', t_0)$ such that $\varphi(x, [0, t])$ is a proper subset of $\Theta(l)(\varphi(x, [0, t])$ then there exists $t' \in [0, t]$ such that the leaf passing through $\varphi(x, t')$ has holonomy.*

Lemma 3.3. *Let C be a segment in M transversal to $\mathscr{F}$. Suppose that the end points of C belongs to the same leaf then there exists a closed transversal curve C' such that $Q_c = Q_{c'}$.*

Lemma 3.4. *If C is a closed transversal curve, then Q_c is an open set of M.*

Lemma 3.5. *For any $x \in M$ there exists $t_1 > t_2 > 0$ (or $t_1 < t_2 < 0$) such that $\varphi(x, t_1)$ and $\varphi(x, t_2)$ belong to the same leaf.*

Lemma 3.6. *Let C be a closed transversal curve then either $Q_c = M$ or there exists a leaf L_x in the boundary of Q_c which has locally holonomy pseudogroup.*

Lemma 3.6. is the Theorem 4 of [12], but for completeness we will give a brief proof. Let φ be a transversal flow which has C as an orbit. If $Q_c \neq M$, there exists $x \in \partial Q_c$ such that $\varphi(x, (0, \delta])$ is contained in Q_c for some small $\delta > 0$ (if δ is negative we reverse the flow φ). Since Q_c is an open neighborhood of C, we can suppose that the distance between C and L_x is greater than 2δ. Therefore, for any sequence of positive numbers $\{t_i\}$ which converges to zero, there exists leaf curves l_i from x to y_i such that $\bar{\Theta}(l_i)(t_i)=2\delta$. Choosing a subsequence of $\{y_i\}$, we can suppose that $\{y_i\}$ converges to y. Choose a distinguished neighborhood of y containing $\varphi(y, [0, 2\delta])$, then all y_i belongs to the same plaque by the condition $\varphi(x, (0, t_i])\subset Q_c$. Let l'_i be a leaf curve which is the composition of l_i and a plaque curve from y_i to y. Put $y'=\varphi(y, \delta)$ then there exists $0<t'_i<t_i$ such that $\Theta(l'_i)(\varphi(x, t'_i))=y'$ for sufficiently large i. For any $\varepsilon>0$ choose i and j so that $0<t'_j<t'_i<\varepsilon$ then $\bar{\Theta}(l'_i\circ l'^{-1}_j)(t'_i)=t'_j$, thus the leaf L_x has locally holonomy pseudogroup.

Proof of Theorem 3.1. Let l_0 and τ_0 be as in Lemma 3.1. Since M is compact there exists a sequence $0<\tau_1<\tau_2<\cdots<\tau_n<\cdots<\tau_0$ such that $\lim \tau_i=\tau_0$ and $x_i=l_0(\tau_i)$ converges to some $x\in M$. Let us choose a distinguished neighborhood U of x and we assume x_i belongs to U for any i.

Case 1. There exists a subsequence $\{x_{i_n}\}$ of $\{x_i\}$ such that $x_{i_n}\nearrow x$. In this case we can assume that $x_i\nearrow x$. We fix some i and for $j>i$ let l_j be the restriction of l_0 to $[\tau_i, \tau_j]$. For any $\varepsilon>0$ there exists j such that $-\bar{\Theta}(l_j)(-\varepsilon)$ is greater than the height of U by Lemma 3.1. Then, since $\varphi(x_i, -\varepsilon)$ belongs to U if ε is small, the leaf curve $l'_j\circ l_j$ satisfies the condition of Lemma 3.2 where l'_j is a plaque curve from $P_{x_i}\cap A_{x_j}$ to x_i. Hence there exists $\varepsilon'<\varepsilon$ such that the leaf passing through $\varphi(x_i, -\varepsilon')$ has holonomy. Since ε can be chosen arbitrarily small, the leaf $L_{x_i}=L_{\varphi(l(0),\tau_0)}$ is a holonomy limit leaf.

Case 2. (Fig. 2.) $x_i\searrow x$ and there exists $t<0$ such that $y=\varphi(x, t)$ belongs to L_{x_i}. In this case we can assume that $\varphi(x, [t, 0])$ is an segment in M (otherwise the problem is reduced to the case 1). As in the case 1 we fix i and define leaf curves l_j for $j>i$. We choose a leaf curve l' from y to x_i then for sufficiently small $\varepsilon>0$, the holonomy map $\bar{\Theta}(l')$ is well defined at $-\varepsilon$ and $\bar{\Theta}(l)(-\varepsilon)$ is sufficiently small. Let us choose a distinguished neighborhood V containing $\varphi(x, [t-\varepsilon, 0])$. Then for sufficiently large j, x_j belongs to V and $-\bar{\Theta}(l_j)(\bar{\Theta}(l')(-\varepsilon))$ is well defined and is greater than the height of V. Choose

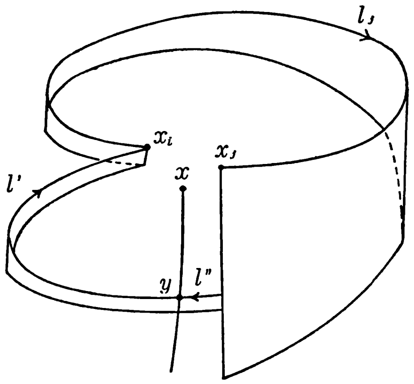


Fig. 2.

a plaque curve l'' in V from $P_y \cap A_{x_j}$ to y, then the leaf curve $l'' \circ l' \circ l_j$ satisfies the condition of Lemma 3.2. and, since we can choose $\bar{\Theta}(l')(-\varepsilon)$ arbitrarily near to 0. L_{x_i} is a holonomy limit leaf.

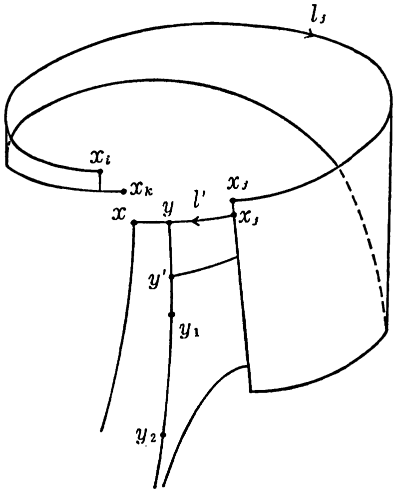


Fig. 3.

Case 3. (Fig. 3) $x_i \searrow x$ and $\varphi(x, (-\infty, 0))$ does not intersect L_{x_i}. We remark that, by Lemma 3.1. (2), it is sufficient to show that the leaf passing through $\varphi(x_i, -\varepsilon_i)$ has holonomy where $\{\varepsilon_i\}$ is a sequence of non-negative real numbers which is bounded. We fix i then there exists k such that $x_i > x_k > x$ and we define $\varepsilon > 0$ by $P_{x_k} \cap A_{x_i} = \varphi(x_i, -\varepsilon)$.

If j is sufficiently large, $-\bar{\Theta}(l_j)(-\varepsilon)$ is greater than the height of U where l_j is the restriction of l_0 to $[\tau_i, \tau_j]$. Let l': $[0, 1]\to M$ be a plaque curve from $x'_j=P_x\cap A_{x_j}$ to x. Since $\varphi(x, (-\infty, 0))$ does not intersect L_{x_i}, the holonomy map $\bar{\Theta}(l')$ is not defined at $\bar{\Theta}(l_j)(-\varepsilon)$. Therefore, by Lemma 3.1. (1), there exists $0<\tau_0<1$ such that $\Theta(l'|[\tau_0, 1])$ and $\Theta(l'^{-1}|[0, \tau_0])$ are well defined on $\varphi(y, (-\infty, 0))$ where $y=l'(\tau_0)$. We remark that $\Theta(l'^{-1}|[0, \tau_0])(\varphi(y, (-\infty, 0))$ is contained in $\Theta(l_j)(\varphi(x_i, [-\varepsilon, 0])$. By Lemma 3.5. there exists $y_1=\varphi(y, t_1)$ and $y_2=\varphi(y, t_2)$, $t_2<t_1<0$ such that y_1 and y_2 belongs to the same leaf. Let C be the transversal segment $\varphi(y, [t_2, t_1])$ then by Lemma 3.4. there exists a closed transversal curve C' satisfying $Q_c=Q_{c'}$. We assert that Q_c does not contain y. Otherwise Q_c contains x and, since Q_c is open, Q_c contains x_n for large n. Then there exists $x'=\varphi(y, t_3)$, $t_2\leqq t_3\leqq t_1$ such that x' belongs to L_{x_i} and $x''=\Theta(l|[\tau_0, 1])(x')\in\varphi(x, (-\infty, 0))$ belongs to L_{x_i}, this is a contradiction. Hence, by Lemma 3.6., there exists $y'=\varphi(y, t_4)$, $t_1\leqq t_4\leqq 0$ such that the leaf Ly' has holonomy, and the leaf passing through $\Theta(l_j^{-1})(\Theta(l'^{-1}|[0, \tau_0])(y'))=\varphi(x_i, \varepsilon_i)$ for some ε_i, $-\varepsilon<\varepsilon_i\leqq 0$ has holonomy. Since $|\varepsilon_i|$ is smaller than the heitgh of U, this completes the proof of Theorem 3.1. q.e.d.

Corollary 3.1. *Let $C=\varphi(x, [0, a])$ be a transversal segment in M. Suppose that no leaf in Q_c has holonomy then leaves in Q_c are homeomorphic.*

Proof. Let L_t be the leaf passing through $\varphi(x, t)$, $t\in[0, a]$. We define a map f from L_0 to L_t as follows. For a point y of L_0, choose a leaf curve l_y from x to y and define $f(y)$ by $f(y)=\Theta(l_y)(\varphi(x, t))$. f is well defined by Theorem 3.1. and is independent of the choice of l_y by Lemma 3.2. It is clear that f is a homeomorphism.

Corollary 3.2. *Suppose that all leaves of $\mathscr{F}$ do not have holonomy, then for any closed transversal curve C we have $Q_c=M$.*

Proof. This is a direct consequence of Lemma 3.6.

Now let us show an example of a foliation mentioned before.

Let V be the closed orientable two dimensional manifold of genus 2, $M=V\times S^1$ and let φ be a flow on M defined by $\varphi((x, \tau), t)=(x, \tau+t)$, $x\in V, \tau\in S^1$. Let f and g be diffeomorphisms of S^1 and G be the subgroup of $\mathscr{H}(S^1)$ generated by f and g. Then by the method of Sacksteder ([10]) or [8]) we can construct a foliation $\mathscr{F}(f, g)$ on M whose leaves are transversal to $x\times S^1$, $x\in V$, and any holonomy map $\Theta(l)$, where l is a leaf curve joining two points of $x\times S^1$, is a diffeomorphism of $x\times S^1$ which coincides with an action of an element of G. In our example, f and g are defined as follows. Let φ be a smooth function on [0, 1] which is identically zero near boundary and is monotone increasing on $\left[0, \frac{1}{2}\right]$, monotone decreasing on $\left[\frac{1}{2}, 1\right]$. We define a diffeomorphism f of $S^1=[0, 1]/0\sim 1$ by $f(x)\equiv x+\varepsilon\varphi(x) \bmod 1$, where ε is sufficiently small. We choose a sequence of intervals $I_0\supset I_1\supset I_2\cdots$ of S_1 such that $\bigcap_{n\geqq 0} I_n=\left\{\frac{1}{2}\right\}$ and $f^i(I_0)\cap f^j(I_0)=\phi$ for $i\neq j$. There exists a diffeomorphism g of S^1 satisfying $g(x)=x$ for $x\in S^1-\bigcup_{n\geqq 0} f^n(I_0)$ and for $x\in f^n(I_{n+1})$ and there exists $y\in f^n(I_n)-f^n(I_{n+1})$ such that $g(y)\neq y$. Then the leaf of the foliation $\mathscr{F}(f, g)$ passing through $x\times\frac{1}{2}$ does not have holonomy but has locally holonomy pseudogroup.

§4. Characteristic map and Novikov transformation.

Let $(M, \mathscr{F}, \varphi)$ be as in §3 and in this section we suppose that $\mathscr{F}$ is without holonomy. Then, by Theorem 3.1. for any leaf curve l, $\bar{\Theta}(l)$ is a homeomorphism of R and, by Lemma 3.2. $\bar{\Theta}(l)$ is determined by its end points.

Definition 4.1. For a point x of M, a set of real numbers $\bar{G}_x$ is defined by $\bar{G}_x=\{\tau\in R|\varphi(x, \tau)\in L_x\}$. For an element τ of $\bar{G}_x$ we define a homeomorphism $\bar{\chi}_x(\tau)$ of R by $\bar{\chi}_x(\tau)(t)=\tau+\bar{\Theta}(l_\tau)(t)$, where l_τ is a leaf curve from x to $\varphi(x, \tau)$.

Thus we have a set $\bar{G}_x$ and a map $\bar{\chi}_x$ from $\bar{G}_x$ to $\mathscr{H}(R)$ where $\mathscr{H}(R)$ is the group of homeomorphisms of R. The following properties are easy to prove (for convenience we denote $\bar{G}$ or $\bar{\chi}$ omitting the subscript x).

(1) If $\bar{\chi}(\tau)(t)=t$ for some t then $\tau=0$ and $\bar{\chi}(0)=$identity.

(2) If $\bar{\chi}(\tau)(t)=\bar{\chi}(\tau')(t)$ for some t then $\tau=\tau'$.

(3) $\bar{\chi}(\tau')\circ\bar{\chi}(\tau)=\bar{\chi}(\tau'')$ where $\tau, \tau'\in G$ and $\tau''=\bar{\chi}(\tau')(\bar{\chi}(\tau)(0))$.

(4) $\bar{\chi}(\tau)^{-1}=\bar{\chi}(\tau')$ where $\tau'=\bar{\Theta}(l_\tau^{-1})(-\tau)$.

(5) If $\varphi(x, t_1)$ and $\varphi(x, t_2)$ belong to the same leaf of $\mathscr{F}$ then there exists $\tau\in\bar{G}$ such that $\bar{\chi}(\tau)(t_1)=t_2$.

We define a multiplication in $\bar{G}$ by $\tau'\cdot\tau=\tau''$ where τ'' is defined by (3). Then from properties (1), (3), (4), $\bar{G}$ is a group and $\bar{\chi}$ is a monomorphism from $\bar{G}$ to $\mathscr{H}(R)$.

Proposition 4.1. *$\bar{G}_x$ is abelian and $\bar{\chi}_x(\bar{G}_x)$ acts on R without fixed points.*

Proof. From the definition of multiplication in $\bar{G}$, it is easy to see that $\bar{G}$ is an ordered group where the order in $\bar{G}$ is induced from the order of R. Moreover $\bar{G}$ is an Archimedean ordered group, i.e. for any τ and τ' different from the unit, there exists n such that $\tau^n>\tau'$. In fact, if there does not exist such n, there exists $\lim\tau^n=\tau_0$ and the holonomy map $\Theta(l_\tau)$ has a fixed point $\varphi(x, \tau_0)$, this contradicts to the assumption that $\mathscr{F}$ is without holonomy. Thus by the theorem of Hölder (see [2]) $\bar{G}$ is isomorphic to a subgroup of R and, in particular, $\bar{G}$ is commutative.

Since M is compact, we can suppose that the flow φ has a closed trajectory C of period 1 and we fix a point $x_0\in C$.

Lemma 4.1. *For any $\tau\in\overline{G_{x_0}}$ we have $\bar{\chi}(\tau)(t+1)=\bar{\chi}(\tau)(t)+1=\bar{\chi}(\tau+1)(t)$.*

That is to say $\bar{\chi}(\overline{G_{x_0}})$ is contained in $\mathscr{H}^p(R)$.

Proof. Since $\varphi(x_0, 1)=x_0$, 1 is an element of $\overline{G_{x_0}}$ and a leaf curve l joining x_0 and $\varphi(x_0, 1)$ is closed. For any closed leaf curve l, the holonomy map $\bar{\Theta}(l)$ is the identity map. Thus $\bar{\chi}(1)(t)=t+1$ and, since $\bar{\chi}(\tau)$ commutes with $\bar{\chi}(1)$, we have $\bar{\chi}(\tau)(t+1)=\bar{\chi}(\tau)(t)+1=\bar{\chi}(\tau+1)(t)$.

Definition 4.2. Let G_{x_0} be the intersection of C and L_{x_0}, for an element x of G_{x_0} we define a homeomorphism $\chi(x)$ of C by $\chi(x)(\varphi(x_0, t))=\Theta(l_x)(\varphi(x_0, t))$, where l_x is a leaf curve from x_0 to x.

$\chi(x)$ is well defined, because, if $x=\varphi(x_0, \tau)$ we have $\chi(x)(\varphi(x_0, t))=\varphi(x_0, \bar{\chi}(\tau)(t))$ and by Lemma 4.1. this does not depend on choices of τ and t.

We identify C with S^1 and we consider χ as a map from G_{x_0} to $\mathscr{H}(S^1)$. Then properties analogous to (1)~(5) hold for χ and we can define a group structure on G_{x_0} and χ is an injective homomorphism whose image $\chi(G_{x_0})$ is a free subgroup of $\mathscr{H}(S^1)$. Define a homomorphism π' from $\overline{G_{x_0}}$ to G_{x_0} by $\pi'(\tau)=\varphi(x_0, \tau)$ then the following diagram is commutative.

$$\begin{array}{ccc} \overline{G_{x_0}} & \xrightarrow{\bar{\chi}} & \mathscr{H}^p(R) \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ G_{x_0} & \xrightarrow{\chi} & \mathscr{H}(S^1) \end{array}$$

We call the homomorphisms χ or $\bar{\chi}$, the *characteristic map* of $\mathscr{F}$.

Proof of Theorem 1.3. Let C be a periodic trajectory of φ of period 1 and x_0 a point of C. Since $\mathscr{F}$ has no exceptional leaf, G_{x_0} is finite or dense in C. G_{x_0} can be identified with an orbit of the action of $\chi(G_{x_0})$ on S_1 and by Theorem 2.1. there exists a linearization map $h\in\mathscr{H}(S^1)$ of $\chi(G_{x_0})$. Let $\tilde{h}$ be a lift of h to $\mathscr{H}^p(R)$ then $\tilde{h}\chi(\tau)\tilde{h}^{-1}$, $\tau\in\overline{G_{x_0}}$, is a translation of $\mathbf{R}$. We define a transversal flow φ' which preserves $\mathscr{F}$ in the following way. At first, for any point $x_1=\varphi(x_0, t_1)$ of C we define $\varphi'(x_1, t)$ by $\varphi(x_0, \tilde{h}^{-1}(t+\tilde{h}(t_1))$. It is clear that $\varphi'(x_1, t)$ does not depend on the choice of t_1. Let l be a leaf curve with end points belonging to C, we assert that $\varphi'(\Theta(l)x_1, t)$ coincides with $\Theta(l)\varphi'(x_1, t)$. In fact there exists $\tau\in\overline{G_{x_0}}$ such that $\Theta(l)\varphi(x_0, t)=\varphi(x_0, \bar{\chi}(\tau)t)$ and, using the relation $\tilde{h}\bar{\chi}(\tau)\tilde{h}^{-1}(t)=t+a$ for some $a\in R$, we have $\varphi'(\Theta(l)x_1, t)=\varphi'(\varphi(x_0, \bar{\chi}(\tau)t_1), t)=\varphi(x_0, \tilde{h}^{-1}(t+\tilde{h}\bar{\chi}(\tau)t_1))=\varphi(x_0, \tilde{h}^{-1}(t+\tilde{h}(t_1)+a))=\varphi(x_0, \bar{\chi}(\tau)(\tilde{h}^{-1}(t+\tilde{h}(t_1)))=\Theta(l)\varphi(x_0, \tilde{h}^{-1}(t+\tilde{h}(t_1)))=\Theta(l)$ $\varphi'(x_1, t)$. For any point $x\in M$ we choose a leaf curve l from x

to a point x' of C, this is possibly by Corollary 3.2. We define $\varphi'(x, t)$ by $\Theta(l^{-1})\varphi'(x', t)$. Let l' be another leaf curve from x to a point x'' of C then $\Theta(l'^{-1})\varphi'(x'', t)=\Theta(l'^{-1})\varphi'(\Theta(l^{-1}\circ l')x', t)=\Theta(l^{-1})\varphi(x', t)$ and $\varphi'(x, t)$ does not depend on the choice of l. It is easy to see that φ' is a flow on M which preserves leaves of $\mathscr{F}$. q.e.d.

Let $(M, \mathscr{F}, \varphi)$ be as above and from now on we assume that M, $\mathscr{F}$ and φ are of class C^r, $r\geqq 2$. Let $(\tilde{M}, \tilde{\mathscr{F}}, \tilde{\varphi})$ be the universal covering of $(M, \mathscr{F}, \varphi)$, that is to say p: $\tilde{M}\to M$ is the universal covering of M and $\tilde{\mathscr{F}}$ and $\tilde{\varphi}$ are the foliation and the flow on $\tilde{M}$ induced from $\mathscr{F}$ and φ by p. Novikov has proved the following theorem ([7]).

Theorem 4.1. *$\tilde{M}$ is diffeomorphic to $\tilde{L}\times\mathbf{R}$, where $\tilde{L}$ is the universal covering of a leaf L of $\mathscr{F}$, $\{x\}\times\mathbf{R}$ is an orbit of $\tilde{\varphi}$ and $\tilde{L}\times\{t\}$ is a leaf of $\tilde{\mathscr{F}}$.*

Proof. At first we remark that, for any leaf curve $\tilde{l}$ of $\tilde{\mathscr{F}}$, the holonomy map $\bar{\Theta}(\tilde{l})$ is defined on R. In fact if l is a leaf curve of $\mathscr{F}$ which is the projection of $\tilde{l}$, clearly we have $\bar{\Theta}(\tilde{l})=\bar{\Theta}(l)$: $R\to R$. Let $\tilde{C}$ be an orbit of $\tilde{\varphi}$, we assert that $Q_{\tilde{C}}=\tilde{M}$. Otherwise, considering an orbit $\tilde{C}'$ of $\tilde{\varphi}$ passing through a point of $\partial Q_{\tilde{C}}$, there exists $\tilde{x}, \tilde{y}\in\tilde{C}'$ such that $\tilde{x}$ belongs to $Q_{\tilde{C}}$ but $\tilde{y}$ does not belong to $Q_{\tilde{C}}$. Consider a leaf curve $\tilde{l}$ from $\tilde{x}$ to a point of $\tilde{C}$, then the holonomy map $\Theta(\tilde{l})$ is not defined at $\tilde{y}$. This contradicts to above remark and we proved $Q_{\tilde{C}}=\tilde{M}$. If the orbit $\tilde{C}$ passes through a leaf $\tilde{L}$ at two points $\tilde{x}$ and $\tilde{y}$, then by Lemma 3.3. there exists a closed transversal curve $\tilde{l}$. Then $l=p\circ\tilde{l}$ is a closed transversal curve in M which is homotopic to zero. By standard arguments (see [4]) this implies the existence of a leaf with holonomy. Thus any orbit $\tilde{C}$ of $\tilde{\varphi}$ passes through any leaf one and only one time and $\tilde{M}$ is diffeomorphic to the product of a leaf $\tilde{L}$ of $\tilde{\mathscr{F}}$ and the real line R. Since $\pi_1(\tilde{L})$ is trivial, $\tilde{L}$ is the universal covering of L.

Corollary 4.1. *For any curve l in M from x to y, there exists a real number t_l such that l is homotopic relative $\{x, y\}$ to a curve which is a join of $\varphi(x, [0, t_l])$ with a leaf curve from $\varphi(x, t_l)$ to y.*

Moreover t_l is uniquely determined by the homotopy class relative $\{x, y\}$ of l.

Proof. Let $\tilde{l}$ be a lift of l to $\tilde{M}$ with end points $\tilde{x}$ and $\tilde{y}$. Then by Theorem 4.1. there exists t_l such that $\varphi(\tilde{x}, t_l)$ belongs to $\tilde{L}_{\tilde{y}}$. Then, if $\tilde{l}'$ is a leaf curve from $\varphi(\tilde{x}, t_l)$ to $\tilde{y}$, $\tilde{l}$ is homotopic to the join of $\varphi(x, [0, t_l])$ with $p\circ\tilde{l}'$. The uniqueness of t_l is clear.

For an element α of $\pi_1(M, x_0)$, Novikov has defined a transformation $q(\alpha)$ of R as follows. We fix a point $\tilde{x}_0$ of $p^{-1}(x_0)$ and we denote $\tilde{L}_t$ the leaf passing through $\tilde{\varphi}(\tilde{x}_0, t)$. Then by Theorem 4.1. all leaves of $\tilde{\mathscr{F}}$ are indexed by R. Let α be an element of $\pi_1(M, x_0)$ then α induces a covering transformation $\tilde{\alpha}: \tilde{M}\to\tilde{M}$ and $\tilde{\alpha}$ preserves the leaves of $\tilde{\mathscr{F}}$. We define a diffeomorphism $q(\alpha)$ of R by the relation $\tilde{L}_{q(\alpha)(t)} = \tilde{\alpha}(\tilde{L}_t)$. Then $q: \pi_1(M, x_0)\to \mathrm{Diff}(R)$, where $\mathrm{Diff}(R)$ is the group of diffeomorphisms of R, is a homomorphism. We call q the *Novikov transformation* of $\mathscr{F}$.

The Novikov transformation is related to the characteristic map $\bar{\chi}_{x_0}$ by the following lemma.

Lemma 4.2. *Let α be an element of $\pi_1(M, x_0)$ and l a representative of α. Then t_l belongs to $\overline{G_{x_0}}$ and we have $q(\alpha)=\bar{\chi}_{x_0}(t_l)$ where t_l is the real number defined by Corollary* 4.1.

Proof. Let $\tilde{l}$ be the lift of l with initial point $\tilde{x}_0$ and end point $\tilde{x}_1$. We choose a leaf curve $\tilde{l}'$ from $\tilde{x}_1$ to $\tilde{y}=\tilde{\varphi}(\tilde{x}_0, t_l)$ then $l'=p\circ\tilde{l}'$ is a leaf curve from x_0 to $p\circ\tilde{y}=\varphi(x_0, t_l)$ and t_l belongs to $\overline{G_{x_0}}$. From the definition of covering transformations, we have $\tilde{\alpha}(\tilde{\varphi}(\tilde{x}_0, t)) = \tilde{\varphi}(\tilde{x}_1, t)$ and by considering the t-lift of $\tilde{l}'$ we see that $\tilde{\varphi}(\tilde{x}_1, t)$ and $\tilde{\varphi}(\tilde{y}, \bar{\Theta}(\tilde{l}')(t))$ belong to the same leaf. Since $\tilde{\varphi}(\tilde{y}, \bar{\Theta}(\tilde{l}')(t))=\tilde{\varphi}(x_0, t_l+\bar{\Theta}(\tilde{l}')(t))$, we have $q(\alpha)(t)=t_l+\bar{\Theta}(\tilde{l}')(t)=t_l+\bar{\Theta}(l')(t)=\bar{\chi}_{x_0}(t_l)(t)$.

Lemma 4.3. *Let us define a map δ from $\pi_1(M, x_0)$ to $\overline{G_{x_0}}$ by $\delta(\alpha) = q(\alpha)(0)$, $\alpha\in\pi_1(M, x_0)$ then we have $\bar{\chi}_{x_0}\circ\delta=q$ and δ is a surjective homomorphism. Moreover* $\ker q=\ker\delta=i_*\pi_1(L_{x_0}, x_0)$ *where i is the inclusion of the leaf L_{x_0} into M and i_* is injective.*

Proof. By Lemma 4.1. $q(\alpha)(0)=\bar{\chi}_{x_0}(t_l)(0)=t_l \in \overline{G_{x_0}}$, thus δ is well defined. To prove that δ is a homomorphism, let α_i, $i=1, 2$, be an element of $\pi_1(M, x_0)$, l_i a representative of α_i and $\tilde{l}_i$ the lift of l_i with initial point $\tilde{x}_0$ and end point $\tilde{x}_i$. By considering the covering transformation $\tilde{\alpha}_1$, it is easy to see that the end point of $\tilde{l}_1 \circ \tilde{\alpha}_1(\tilde{l}_2)$ and $\tilde{\varphi}(\tilde{x}_1, t_{l_2})$ belongs to the same leaf and from the definition of $\bar{\chi}_{x_0}$, $\tilde{\varphi}(\tilde{x}_1, t_{l_2})$ and $\tilde{\varphi}(\tilde{x}_0, \bar{\chi}_{x_0}(t_{l_1})(t_{l_2}))$ belong to the same leaf. Thus we proved that $q(\alpha_1\alpha_2)(0)=\bar{\chi}_{x_0}(t_{l_1})(t_{l_2})$ but from the definition of the multiplication in G_{x_0}, this shows that δ is a homomorphism. The fact $\bar{\chi}_{x_0} \circ \delta = q$ follows by the same consideration. To prove that δ is surjective, let t_1 be an element of G_{x_0}, we define a curve l_1 in M by $l_1(t)=\varphi(x_0, t \cdot t_1)$, $0 \leqq t \leqq 1$, and we choose a leaf curve l_2 from $\varphi(x_0, t_1)$ to x_0. Let α be the homotopy class of the join of l_1 with l_2 then it is clear that $\delta(\alpha)=t_1$. Clearly $i_*\pi_1(L_{x_0}, x_0)$ is contained in the kernel of δ. Let us suppose that $\delta(\alpha)=0$ then by Lemma 4.1. α is represented by a leaf curve in L_{x_0}. Thus $\ker \delta = i_*\pi_1(L_{x_0}, x_0)$. The injectivity of i_* follows from the fact that $\tilde{L}_{\tilde{x}_0}$ is the universal covering of L_{x_0}.

Lemma 4.4. *Let us suppose that the trajectory $\varphi(x_0, R)$ is periodic of period* 1 *and define a homomorphism π'' from $\pi_1(M, x_0)$ to G_{x_0} by $\pi''=\pi' \circ \delta$ then π'' is surjective and its kernel is generated by $i_*\pi_1(L_{x_0}, x_0)$ and the periodic trajectory $\varphi(x_0, \mathbf{R})$. The following diagram is well defined and is commutative.*

$$\begin{array}{ccc} \pi_1(M, x_0) & \xrightarrow{q} & \mathrm{Diff}^p(R) \\ \downarrow{\scriptstyle \pi''} & & \downarrow{\scriptstyle \pi} \\ G_{x_0} & \xrightarrow{\chi_{x_0}} & \mathrm{Diff}(S^1) \end{array}$$

The proof is straightforward from the preceeding lemmas.

Proof of Theorem 1.1. We can suppose that a trajectory $\varphi(x_0, R)$ is periodic of period 1. Since $\pi_1(M, x_0)$ is finitely generated, G_{x_0} is finitely generated. By Theorem 2.1. there exists a linearlization map $h \in \mathscr{H}(S^1)$ for $\chi_{x_0}(G_{x_0})$ and the theorem follows from the proof of Theorem 1.3.

We remark that the foliation $\mathscr{F}$ is defined by a non-singular closed 1-form if and only if the linearization map h is differentiable.

§5. Foliations defined by closed 1-forms.

As is remarked above, a foliation $\mathscr{F}$ without holonomy is not necessarily defined by a closed 1-form, but we can consider $\mathscr{F}$ as a foliation defined by a closed 1-form if we change the differential structure of M. More precisely we have the following proposition.

Proposition 5.1. *Let $(M, \mathscr{F}, \varphi)$ be as in §4, we assume $(M, \mathscr{F}, \varphi)$ is of class C^r, $r \geqq 2$. Then there exists a differentiable manifold $\bar{M}$ and a foliation $\bar{\mathscr{F}}$ defined by a closed non-singular 1-form $\bar{\omega}$ on $\bar{M}$ of class C^r. $\bar{M}$ and $\bar{\mathscr{F}}$ satisfy the following conditions.*

(1) *$\bar{M}$ is identical to M as a topological manifold. We denote h the identity map from $\bar{M}$ to M.*
(2) *h sends each leaf of $\bar{\mathscr{F}}$ diffeomorphically onto a leaf of $\mathscr{F}$.*

Proof. We choose a coordinate system $\{(U_\lambda\colon x_\lambda^1, x_\lambda^2, \dots, x_\lambda^n)\}_{\lambda \in \Gamma}$ on M such that

(1) U_λ is a distinguished neighborhood of $(\mathscr{F}, \varphi)$.
(2) A plaque is defined by $x_\lambda^n = c$ and an axis is defined by $x_\lambda^i = c_i$, $i = 1, 2, \dots, n-1$.

Then in $U_\lambda \cap U_\mu$ we have $x_\mu^i = \varphi_{\mu\lambda}^i(x_\lambda^1, \dots, x_\lambda^{n-1})$, $i = 1, 2, \dots, n-1$, and $x_\mu^n = \varphi_{\mu\lambda}^n(x_\lambda^n)$ where $\varphi_{\mu\lambda}^i$ are differentiable functions. We choose a point y_λ of U_λ for each $\lambda \in \Gamma$ and define a continuous function $\bar{x}_\lambda^n$ on U_λ by $\bar{x}_\lambda^n(y) = t$ if $\varphi'(y_\lambda, t)$ belongs to P_y where φ' is a flow on M which preserves $\mathscr{F}$. Then in $U_\lambda \cap U_\mu$ we have $\bar{x}_\mu^n = \bar{x}_\lambda^n + c_{\mu\lambda}$ for some constant $c_{\mu\lambda}$. Hence $\{(U_\lambda; x_\lambda^1, \dots, x_\lambda^{n-1}, \bar{x}_\lambda^n)\}$ defines a differentiable structure on M and we denote $\bar{M}$ the manifold M with this differentiable structure. Define $\bar{\omega} = d\bar{x}_\lambda^n$ on U_λ then $\bar{\omega}$ is defined on $\bar{M}$ and the foliation $\bar{\mathscr{F}}$ defined by $\bar{\omega}$ satisfies the desired properties. q.e.d.

The following theorem of Joubert-Moussu [6], which is an improvement of the theorem of Tischler [15], is useful.

Theorem 5.1. *Let ω be a non-singular closed 1-form on a compact manifold M and X be a vector field on M such that $\omega(X)\equiv 1$. Then there exists a submersion f of M onto S^1 such that the fibres of f are transversal to X. Moreover the integral manifolds of ω are covering spaces of the fibres of f.*

Proof of Theorem 1.2. Let $\overline{M}$, $\bar{\mathscr{F}}$ and $\bar{\omega}$ be as in Proposition 5.1. and $\bar{X}$ a vector field on $\overline{M}$ defined by $\bar{X}=\frac{\partial}{\partial X_\lambda^n}$ on U_λ. By Theorem 5.1. there exists a submersion f of $\overline{M}$ onto S^1 such that the fibres of f are transversal to $\bar{X}$. Define a function $a(x)$ on $\overline{M}$ satisfying $f_*(a(x)\bar{X})=\frac{\partial}{\partial t}$ where t is the natural coordinate of S^1, then the 1-parameter transformation $\bar{\varphi}$ of $a(x)\bar{X}$ preserves fibres of f and the homeomorphism h of $\overline{M}$ onto M sends each trajectory of $\bar{\varphi}$ onto a trajectory of φ. We fix a fibre F of f and a tubular neighborhood U of F. For any point x of F there exists a neighborhood U_x of x in F such that we can define a diffeomorphism π_x of U_x into a plaque $\bar{P}_x$ of $\bar{\mathscr{F}}$ by $\pi_x(y)=\bar{P}_x\cap\bar{A}_y$ where $\bar{P}_x$ and $\bar{A}_y$ are a plaque and an axis respectively of an distinguished neighborhood of $(\bar{\mathscr{F}}, \bar{X})$ which is contained in U, we call π_x a local projection. We choose a tirangulation of F and we suppose that for each $(n-1)$-simplex σ a local projection $\pi_\sigma(=\pi_x$ for some $x\in\sigma)$ is well defined on a neighborhood U_σ of σ in F. Then by Proposition 5.1. (2), $h\circ\pi_\sigma$ is a diffeomorphism of U_σ into a leaf of $\mathscr{F}$ and for any $x\in U_\sigma$, $h\circ U_\sigma(x)$ and $h(x)$ belongs to the same axis for $\mathscr{F}$.

Assertion. *There exists a differentiable imbedding h' of F into $h(U)$ which is transversal to φ and satisfies* (*) *for any $x\in U_\sigma$, $h'(x)$ and $h\circ\pi_\sigma(x)$ belongs to the same axis.*

We prove the assertion by skeletonwise induction. Suppose that there exists a differentiable imbedding h' of a neighborhood U_k in F of the k-skeleton of F into $h(U)$ which satisfies the condition (*) for any $x\in U_k\cap U_\sigma$ where σ is an $(n-1)$-simplex of F. Let τ be a $(k+1)$-

simplex, we choose an $(n-1)$-simplex σ containing τ as its face, then in a neighborhood of $\partial\tau$ a differentiable function $f_{\sigma\tau}$ is defined by $\varphi(h\circ\pi_\sigma(x), f_{\sigma\tau}(x))=h'(x)$. Let $\bar{f}_{\sigma\tau}$ be a differentiable function defined on a neighborhood of τ which agrees with $f_{\sigma\tau}$ on a neighborhood of $\partial\tau$ and define an imbedding of a neighborhood of τ to $h(U)$ by $h'(x)=\varphi(h\circ\pi_\sigma(x), \bar{f}_{\sigma\tau}(x))$. Thus we obtained a differentiable imbedding h' of a neighborhood of the $(k+1)$-skeleton of F into $h(F)$ satisfying $(*)$ and we proved the assertion.

Let us choose real numbers $0=t_0<t_1<\cdots<t_n<1$ and we consider t_i as a point of S^1. Let F_i be the fibre $f^{-1}(t_i)$ and U_i be a tubular neighborhood of F_i and we assume $U_i\cap U_j=\phi$ for $i\neq j$. Then by the assertion there exists differentiable imbeddings h'_i of F_i into $h(U_i)$ which satisfy $(*)$ and it is easy to see that we can extend $\bigcup_{i=0}^{n} h'_i\colon \cup F_i\to M$ to a diffeomorphism h' of $\overline{M}$ onto M and h' can be choosen arbitrarily near to h if we choose $\delta=\max\{t_{i+1}-t_i\}$ sufficiently small. Then the foliation $(h'^{-1})^*\bar{\mathscr{F}}$ is defined by the closed 1-form $(h'^{-1})^*\bar{\omega}$ and the homeomorphism $h'\circ h^{-1}$ satisfies the condition of Theorem 1.2.

q.e.d.

The rest of this section is devoted to the study of foliations defined by closed non-singular 1-forms. Let $(M, \mathscr{F}, \varphi)$ be as before and we assume that $\mathscr{F}$ is defined by a closed 1-form ω. Let C be a closed orbit of φ and x_0 a point of C then the linearization map h of the characteristic map χ_{x_0} is differentiable and if $\int_c \omega=1$ the restriction of ω to C is h^*dt where t is the natural coordinate of $S^1=R/Z$. Moreover if φ is defined by a vector field X satisfying $\omega(X)\equiv 1$ then φ preserves $\mathscr{F}$ and we can choose h to be the identity map of S^1.

Proposition 5.2. *For $\alpha\in\pi_1(M, x_0)$, the Novikov transformation $q(\alpha)$ belongs to $\mathscr{H}^p(R)$ and its rotation number $\gamma\circ q(\alpha)$ is related to ω by the following formula.*

$$\gamma\circ q(\alpha)=\frac{1}{k}\int_\alpha \omega \qquad \text{where } k=\int_c \omega\,.$$

Proof. We can assume $k=1$. Let $\tilde{h}$ be a diffeomorphism of R

which is a lift of the linearization map h of χ_{x_0}. By Corollary 4.1. we can choose a representative of α which is the join of a part of the trajectory $l_1=\varphi(x, [0, t_1])$ and a leaf curve l_2 from $\varphi(x_0, t_1)$ to x_0 where $t_1=q(\alpha)(0)$ by the definition of $q(\alpha)$. Then we have $\int_\alpha \omega = \int_{l_1} h^*dt = \tilde{h}(t_1)-\tilde{h}(0)$. On the other hand, by Lemma 4.4. and the definition of $\tilde{h}$, we have $\tilde{h}q(\alpha)\tilde{h}^{-1}(t)=t+\gamma\circ q(\alpha)$ for any $t\in R$. Thus $\tilde{h}(t_1)=\tilde{h}q(\alpha)(0)=\tilde{h}(0)+\gamma\circ q(\alpha)$ and we have $\int_\alpha \omega=\gamma\circ q(\alpha)$.

Proposition 5.3. (1) *Define a homomorphism j of $\pi_1(M, x_0)$ to R by $j(\alpha)=\int_\alpha \omega$, then the following sequence is exact and the image of j is free abelian of rank k, $k\geqq 1$.*

$$1 \longrightarrow \pi_1(L_{x_0}, x_0) \xrightarrow{i_*} \pi_1(M, x_0) \xrightarrow{j} R.$$

(2) *If $k=1$, then the leaves of $\mathscr{F}$ are fibres of a fibration of M onto S^1 and if $k\geqq 2$, all leaves of $\mathscr{F}$ are everywhere dense in M.*
(3) $\pi_i(L_{x_0})\cong\pi_i(M)$ *for* $i\geqq 2$.

Proof. (1) is clear from Proposition 5.2. and Lemma 4.3. If $k=1$ then by Proposition 5.2. $\gamma\circ q(\alpha)$ is rational for any $\alpha\in\pi_1(M, x_0)$ and it follows that the group G_{x_0} is finite. Then it is easy to see that there exists a closed transversal curve C' which passes through each leaf at only one time and M is a fibration over C' whose fibres are leaves of $\mathscr{F}$. If $k\geqq 2$ then there exists $\alpha\in\pi_1(M, x_0)$ such that $\gamma\circ q(\alpha)$ is irrational. So all orbits of $\chi(G_{x_0})$ are dense in C and all leaves of $\mathscr{F}$ are dense in M. (3) follows easily from Theorem 5.1.

Proposition 5.4. *Let ω_0 and ω_1 be non-singular closed 1-forms on a compact manifold M and $\mathscr{F}_{\omega_i}$ a foliation defined by ω_i, $i=0, 1$. If ω_0 and ω_1 define the same cohomology class in $H^1(M, R)$ then $\mathscr{F}_{\omega_0}$ and $\mathscr{F}_{\omega_1}$ are concordant.*

Proof. $\omega_1=\omega_0+df$ and we can assume that f is positive on M. Let ω be an 1-form on $M\times I$ defined by $\omega=p^*\omega_0+d(t\cdot p^*f)$ where P

is the projection of $M \times I$ on M and t is the coordinate of $I=[0, 1]$. Then it is clear that ω is a non-singular closed 1-form on $M \times I$ and the foliation $\mathscr{F}_\omega$ defined by ω is a concordance between $\mathscr{F}_{\omega_0}$ and $\mathscr{F}_{\omega_1}$.

Corollary 5.1. *Let ω_0 and ω_1 be as above then the leaf L_1 of $\mathscr{F}_{\omega_1}$ and L_0 of $\mathscr{F}_{\omega_0}$ have the same homotopy type.*

Proof. Let ω be as above and L be a leaf of $\mathscr{F}_\omega$. By Proposition 5.3. the injection of L_i to L induces isomorphisms of homotopy groups and L_0 and L_1 are homotopy equivalent to L.

Corollary 5.2. *Moreover if* $\dim M \geqq 6$ $\pi_1(M)$ *is abelian and the Whitehead group $Wh(\pi_1(M))$ vanishes then L_0 and L_1 are diffeomorphic.*

Proof. Let ω be as above and f be a submersion of $M \times I$ onto S^1 which satisfies the condition of Theorem 5.1. and f_i $(i=0, 1)$ be the restriction of f to $M \times \{i\}$. Then by Corollary 5.1. the fibres of f_0 and f_1 are homotopy equivalent and by the conditions of Corollary 5.2. the fibres of f_0 and f_1 are diffeomorphic. Since L_i is a covering space of the fibre F_i of f_i which correspond to the same subgroup of $\pi_1(F_0) \cong \pi_1(F_1)$, L_0 and L_1 are diffeomorphic.

Theorem 5.2. *Let ω_0 and ω_1 be non-singular closed 1-forms on a compact manifold M which define the same cohomology class. Suppose that there exists a vector field X on M such that $\omega_0(X)$ and $\omega_1(X)$ never vanish, then the foliations $\mathscr{F}_{\omega_0}$ and $\mathscr{F}_{\omega_1}$ are differentiably isotopic.*

Proof. We can assume $\omega_0(X) \equiv 1$ and the one parameter group φ defined by X has a periodic orbit C of period 1. Let $(\tilde{M}, \tilde{\mathscr{F}}_{\omega_i}, \tilde{\varphi})$ be the universal covering of $(M, \mathscr{F}_{\omega_i}, \varphi)$ $(i=0, 1)$ and choose a point $\tilde{x}_0$ of $\tilde{M}$ which projects to a point x_0 of C. Let $q(\alpha)$ be the Novikov transformation of $\mathscr{F}_{\omega_0}$ for $\alpha \in \pi_1(M, x_0)$ then it is easy to see that $q(\alpha)(t)=t+\int_\alpha \omega_0$. Let h be a linearization map of the characteristic map χ_{x_0} of $\mathscr{F}_{\omega_1}$ and $\tilde{h}$ a diffeomorphism of R which is a lift of h. Then the Novikov transformation $q'(\alpha)$ is calculated, by using Proposition 5.2., as $\tilde{h}q'(\alpha)\tilde{h}^{-1}(t)=t+\int_\alpha \omega_1$. Let $\tilde{x}$ be a point of $\tilde{L}_t$ where $\tilde{L}_t$ is a leaf of $\mathscr{F}_{\omega_0}$ indexed by the theorem of Novikov. There exists unique $\tau(\tilde{x})\in R$ such that $\tilde{\varphi}(\tilde{x}, \tau(\tilde{x}))$ belongs to the leaf of $\tilde{\mathscr{F}}_{\omega_1}$ passing through $\tilde{\varphi}(x_0, \tilde{h}^{-1}(t))$. Define a diffeomorphism $\tilde{g}$ of $\tilde{M}$ by $\tilde{g}(\tilde{x})=\tilde{\varphi}(\tilde{x}, \tau(\tilde{x}))$. Then, using the relation $\int_\alpha \omega_0=\int_\alpha \omega_1$, it is easy to see that $\tilde{g}$ commutes with covering transformations and $\tilde{g}$ induces a diffeomorphism g of M. It is clear that g sends each leaf of $\mathscr{F}_{\omega_0}$ to a leaf of $\mathscr{F}_{\omega_1}$ and g is isotopic to identity.

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