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On the representation of $SL_2(F_q)$ in the space of Hilbert modular forms

By

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Introduction. In his paper [9], Hecke studied the representation of the special linear group $SL_2(\mathbf{F}_p)$ over a finite field $\mathbf{F}_p$ in the space of elliptic modular forms of weight 2, and especially showed that the difference of the multiplicities of some two irreducible reprsentations of $SL_2(\mathbf{F}_p)$ in that representation is equal to the class number of some imaginary quadratic field. One can consider a similar problem for the case of Hilbert modular cusp forms. In fact, if the weight $k\geqslant 4$, we can use Selberg's trace formula and can generalize a part of Hecke's theorem. If $k=2$, we cannot use the trace formula, but some other method (e. g. Hirzebruch's resolution of cusp singularities of Hilbert modular surfaces) is available for the case of a real quadratic field, and we have another generalization of Hecke's result.

While preparing the manuscript, the author was communicated by H. Yoshida that he had obtained the same result as ours for $k\geqslant 4$, as a part of his doctorial thesis at Princeton University [16] (see below Theorem 1). Furthermore his method of the proof is practically identical to ours, so we omit the whole details of the proof for $k\geqslant 4$ and in this note we shall give the proof only for $k=2$.

For the description of our result, we start from a few definitions. Let F be a totally real algebraic number field, and $\mathfrak{o}_F$ be its maximal order. Let $\sigma_1, \cdots, \sigma_n$ be all the distinct isomorphisms of F into $\mathbf{R}$, and $\mathfrak{H}$ be the complex upper half plane. Then $SL_2(\mathfrak{o}_F)$ acts on $\mathfrak{H}^n$ by

$$\gamma(z_1, \cdots, z_n) = \left(\frac{a^{\sigma_1} z_1 + b^{\sigma_1}}{c^{\sigma_1} z_1 + d^{\sigma_1}}, \cdots\cdots, \frac{a^{\sigma_n} z_n + b^{\sigma_n}}{c^{\sigma_n} z_n + d^{\sigma_n}} \right)$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathfrak{o}_F)$ and $(z_1, \cdots, z_n) \in \mathfrak{H}^n$, and $SL_2(\mathfrak{o}_F)$ gives a discontinuous group of transformations on $\mathfrak{H}^n$; and the volume of $\mathfrak{H}^n/SL_2(\mathfrak{o}_F)$ is finite. We fix a prime ideal $\mathfrak{p}$ of F which is prime to $6 = 2 \times 3$ and the different $D(F/\mathbf{Q})$ of F over $\mathbf{Q}$. We define a subgroup $\Gamma(\mathfrak{p})$ of $SL_2(\mathfrak{o}_F)$ of finite index by

$$\Gamma(\mathfrak{p}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathfrak{o}_F) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathrm{mod.}\ \mathfrak{p} \right\}.$$

Let k be an even positive integer and for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathfrak{o}_F)$ and $z = (z_1, \cdots, z_n) \in \mathfrak{H}^n$, put $j_k(\gamma, z) = \Pi (c^{\sigma_i} z_i + d^{\sigma_i})^{-k}$. For a function on $\mathfrak{H}^n$ we put $f|[\gamma]_k = f(z) j_k(\gamma, z)$. Let $S_k(\Gamma(\mathfrak{p}))$ denote the space of cusp forms of weight k with respect to $\Gamma(\mathfrak{p})$, i. e., the set of all holomorphic functions on $\mathfrak{H}^n$ which satisfy,

i) $f|[\gamma]_k = f$ for all $\gamma \in \Gamma(\mathfrak{p})$.

ii) $f(z)$ vanishes at every cusps of $\Gamma(\mathfrak{p})$.

For $\gamma \in SL_2(\mathfrak{o}_F)$ and $f \in S_k(\Gamma(\mathfrak{p}))$, $f|[\gamma]_k$ is also contained in $S_k(\Gamma(\mathfrak{p}))$, and the map $\gamma \longrightarrow [\gamma]_k$ defines a representation π of $SL_2(\mathfrak{o}_F)/\Gamma(\mathfrak{p})$ in the space $S_k(\Gamma(\mathfrak{p}))$. Put $q = N\mathfrak{p}$ and let $\mathbf{F}_q$ denote the finite field with q elements, then $SL_2(\mathfrak{o}_F)/\Gamma(\mathfrak{p})$ is isomorphic to $SL_2(\mathbf{F}_q)$.

Now the irreducible representations of $SL_2(\mathbf{F}_q)$ have been classified by Schur [13], and following the notation of Hecke [9], they are called $\mathfrak{G}_1$, $\mathfrak{G}_q$, $\mathfrak{G}^i_{(q+1)/2}$ $(i=1, 2)$, $\mathfrak{G}^i_{(q-1)/2}$ $(i=1, 2)$, $\mathfrak{G}^j_{q+1}$ $(1 \leqslant j \leqslant (q-3)/2)$, $\mathfrak{G}^l_{q-1}$ $(1 \leqslant l \leqslant (q-1)/2)$, where the suffix r of $\mathfrak{G}^i_r$ indicates that $\mathfrak{G}^i_r$ has degree r. Here to distinguish $\mathfrak{G}^1_{(q+1)/2}$ (resp. $\mathfrak{G}^1_{(q-1)/2}$) from $\mathfrak{G}^2_{(q+1)/2}$ (resp. $\mathfrak{G}^2_{(q-1)/2}$) we denote by $\mathfrak{G}^1_{(q+1)/2}$

(resp. $\mathfrak{S}^1_{(q-1)/2}$) the representation satisfying $\operatorname{tr}(\mathfrak{S}^1_{(q+1)/2}(\begin{pmatrix}1&1\\0&1\end{pmatrix}))=(1+\sqrt{(-1)^{(q-1)/2}q})/2$ (resp. $\operatorname{tr}(\mathfrak{S}^1_{(q-1)/2}(\begin{pmatrix}1&1\\0&1\end{pmatrix}))=(-1+\sqrt{(-1)^{(q-1)/2}q})/2$). As we assume k is even, $\pi(\begin{pmatrix}-1&\\&-1\end{pmatrix})$ is the identity automorphism of $S_k(\Gamma(\mathfrak{p}))$. Hence by the character table [13], we see that $\mathfrak{S}^i_{(q-1)/2}$ does not appear in π if $q\equiv 1 \bmod. 4$ and that $\mathfrak{S}^i_{(q+1)/2}$ does not appear in π if $q\equiv 3 \bmod 4$. In view of these facts we denote by y_i the multiplicity of $\mathfrak{S}^i_{(q+1)/2}$ if $q\equiv 1 \bmod. 4$ and that of $\mathfrak{S}^i_{(q-1)/2}$ if $q\equiv 3 \bmod. 4$.

In the case of $F=\mathbf{Q}$, $k=2$ and $\mathfrak{p}=(p)$, Hecke [9] determined the multiplicity of each irreducible representation in π and especially showed that $y_1-y_2=-\frac{1}{p}\sum_{a=1}^{p-1}a\left(\frac{a}{p}\right)$, where $\left(\frac{}{p}\right)$ is the quadratic residue symbol mod. p. By Dirichlet's formula for the class number of an imaginary quadratic field, the above value is equal to 0 if $p\equiv 1 \bmod. 4$ and is equal to the class number of $\mathbf{Q}(\sqrt{-p})$ if $p\equiv 3 \bmod. 4$. Furthermore in the case of $F=\mathbf{Q}$, $k\geqslant 2$ and even, Eichler [2] calculated y_1-y_2 by using his "trace formula" and showed that

$$y_1-y_2=\frac{1}{\sqrt{(-1)^{(p-1)/2}}p}\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\frac{\exp(2\pi\sqrt{-1}\,a/p)}{1-\exp(2\pi\sqrt{-1}\,a/p)}$$

and that this is equal to $-\frac{1}{p}\sum_{a=1}^{p-1}a\left(\frac{a}{p}\right)$.

Our purpose in this note is to generalize these results to the case $n\geqslant 2$. In fact, as mentioned above, H. Yoshida and the present author have proved independently the following

Theorem 1. *If* $n\geqslant 2$ *and* k *is even and* $k\geqslant 4$, *we have*

$$|y_1-y_2|=2^{n-1}\sum h_{K_j}/h$$

where in the summation $\sum$ *of the right hand side,* K_j *runs ouns all the totally imaginary quadratic extensions of* F *with the relative discriminant* $\mathfrak{p}$, *and* h_{K_j} *and* h *denote the class number of* K_j *and* F *respectively.*

A brief sketch of our proof will be seen in §1.

Now, we shall explain the situation and the method for the peculiar case $k=2$ and $n=2$. In the course of the proof of Theorem 1, firstly y_1-y_2 is represented as the value at 1 of some sort of L-function (hence it is an "infinite sum") and is shown to be equal to a sum of relative class numbers. In the case of $F=\mathbf{Q}$, $k\geqslant 4$, this L-function is equal to $\frac{1}{\sqrt{(-1)^{(p-1)/2}p}}\times\frac{\sqrt{-1}\,p}{2\pi}L\left(s, \left(\frac{\ }{p}\right)\right)\left(1-\left(\frac{-1}{p}\right)\right)$. On the other hand in the proof of Hecke and Eichler, y_1-y_2 is represented as a "finite sum" and then this finite sum is shown to be equal to a class number by means of Dirichlet's formula for the class number of an imaginary quadratic field. In view of these facts, it seems interesting for us to calculate y_1-y_2 by some other method than Selberg's trace formula. Actually in the case of $n=2$ and $k=2$, we can compute y_1-y_2 by using Hirzebruch's resolution of singularities of Hilbert modular surfaces [11] [12] and Atiyah's Lefschetz fixed point formula for vector bundles over compact complex manifold [1]. In this way we can represent y_1-y_2 as a finite sum. For simplicity in this note we assume some additional condition on F and $\mathfrak{p}$ (see the text).

Unfortunately, as the trace formula is not available for weight 2, we cannot know whether the "finite sum" equals to a sum of relative class numbers of totally imaginary quadratic extensions of F. But Eichler's result tells that the value y_1-y_2 does not depend on the weight k, it may be reasonable to expect that "finite sum expression" of y_1-y_2 obtained by our method gives a "finite sum expression" of the value at 1 of some sort of L-function.

In §1 we give some preliminary results and a brief outline of the proof of Theorem 3. In §2 we review Hirzebruch's resolution of Hilbert modular surfaces and, as an application, we prove Theorem 2, which asserts that some automorphism of the compactification X of $\mathfrak{H}\times\mathfrak{H}/\Gamma(\mathfrak{p})$ can be extended to the resolution $\tilde{X}$ of X. In §3 we calculate y_1-y_2 explicitly by using Theorem 2 and Atiyah's fixed point theorem.

The author wishes to express his hearty thanks to his friend A. Fujiki who took an interest in the present problem and gave him many helpful suggestions.

§ **1.** Let F, $\mathfrak{p}$, $S_k(\Gamma(\mathfrak{p}))$, $SL_2(\mathbf{F}_q)$, and π be as in Introduction. We denote by $\left(\frac{\ }{\mathfrak{p}}\right)$ the quadratic residue symbol mod. $\mathfrak{p}$, and let η be an element of $\mathfrak{o}_F$ such that $\left(\frac{\eta}{\mathfrak{p}}\right)=-1$. Looking at the character table [13], it can readily be seen that

$$(1) \qquad y_1-y_2=\frac{1}{\sqrt{(-1)^{(q-1)/2}q}}(\mathrm{tr}\ (\pi(\varepsilon))-\mathrm{tr}\ (\pi(\varepsilon'))),$$

where $\varepsilon=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $\varepsilon'=\begin{pmatrix}1 & \eta\\ 0 & 1\end{pmatrix}$. Hence to know the value y_1-y_2, it is sufficient to calculate $\mathrm{tr}(\pi(\varepsilon))$ and $\mathrm{tr}(\pi(\varepsilon'))$.

For $z=(z_i)$, $z'=(z'_i)\in\mathfrak{H}^n$, put

$$K(z,\ z')=\prod_{i=1}^{n}\left(\frac{z_i-\bar{z}'_i}{2\sqrt{-1}}\right)^{-k},$$

where — denotes the complex conjugation. Then by Selberg's trace formula (c. f. Godement [6], Shimizu [14]), for $k\geqslant 4$ we have

$$(2) \qquad \mathrm{tr}(\pi(\varepsilon))=\frac{(k-1)^n}{(4\pi)^n}\int_{\mathscr{F}}\frac{\sum_{\gamma\in\Gamma(\mathfrak{p})}K(\varepsilon z,\ \gamma z)\ \overline{j(\gamma,\ z)}}{K(z,\ z)}dz$$

$$(2') \qquad \mathrm{tr}(\pi(\varepsilon'))=\frac{(k-1)^n}{(4\pi)^n}\int_{\mathscr{F}}\frac{\sum_{\gamma\in\Gamma(\mathfrak{p})}K(\varepsilon' z,\ \gamma z)\ \overline{j(\gamma,\ z)}}{K(z,\ z)}dz,$$

where $\mathscr{F}$ is a fundamental domain of $\Gamma(\mathfrak{p})$ in $\mathfrak{H}^n$, and $dz=\prod_{i=1}^{n}\frac{dx_idy_i}{y_i^2}$, $z_i=x_i+\sqrt{-1}\,y_i$. By means of Shimizu's method, we can calculate the integrals on the right hand side of (2) and (2′). As to the explicit calculation, we refer to [16]. From this calculation we obtain Theorem 1 stated in the Introduction.

In the following we compute $\mathrm{tr}(\pi(\varepsilon))$ and $\mathrm{tr}(\pi(\varepsilon'))$ by another method in the case of $n=2$ and $k=2$. First we describe the outline of it. Let X be the complete normal algebraic surface which is obtained from $\mathfrak{H}\times\mathfrak{H}/\Gamma(\mathfrak{p})$ by a usual compactification at finitely many cusps of $\Gamma(\mathfrak{p})$. Following Hirzebruch's method we can resolve the singularities of X and we obtain a complete non-singular algebraic surface $\tilde{X}$. Now ε(resp. ε') induces the automorphism f_ε(resp. $f_{\varepsilon'}$) of X. First of all in §2 we show that f_ε (resp. $f_{\varepsilon'}$) can be extended to the biregular automorphism $\tilde{f}_\varepsilon$ (resp. $\tilde{f}_{\varepsilon'}$) of $\tilde{X}$ respectively. By Freitag ([3], [4], [5]), $S_2(\Gamma(\mathfrak{p}))$ is isomorphic to the space $H^0(\tilde{X}, \Omega^2)$ of 2-forms on X and $\mathrm{tr}(\pi(\varepsilon))$ (resp. $\mathrm{tr}(\pi(\varepsilon'))$) is equal to $\mathrm{tr}(\tilde{f}_\varepsilon | H^0(\tilde{X}, \Omega^2))$ (resp. $\mathrm{tr}(\tilde{f}_{\varepsilon'} | H^0(\tilde{X}, \Omega^2))$), where $\mathrm{tr}(\tilde{f}_\varepsilon | H^0(\tilde{X}, \Omega^2))$ (resp. $\mathrm{tr}(\tilde{f}_{\varepsilon'} | H^0(\tilde{X}, \Omega^2))$) denotes the trace of the linear map which is induced by $\tilde{f}_\varepsilon$ (resp. $\tilde{f}_{\varepsilon'}$) in the space $H^0(X, \Omega^2)$. Now $\mathrm{tr}(\tilde{f}_\varepsilon | H^0(\tilde{X}, \Omega^2))$ (resp. $\mathrm{tr}(\tilde{f}_{\varepsilon'} | H^0(\tilde{X}, \Omega^2))$) can be calculated by Atiyah's Lefschetz fixed point theorem for vector bundles. In §3 we shall give an explicit computation of $\mathrm{tr}(\tilde{f}_\varepsilon | H^0(\tilde{X}, \Omega^2))$ and $\mathrm{tr}(\tilde{f}_{\varepsilon'} | H^0(\tilde{X}, \Omega^2))$ under some condition on F and $\mathfrak{p}$. The above description is the main idea of the present investigation. Here we make some preliminary remarks. First of all we assume that F is a real quadratic field and that an integral ideal $\mathfrak{a}$ of F satisfies $(\mathfrak{a}, D(F/\mathbf{Q})(6))=1$.

Remark 1. Let X be the complete algebraic variety obtained from $\mathfrak{H}\times\mathfrak{H}/\Gamma(\mathfrak{a})$ by the compactification. Then X does not have singularities other than cusp singularities.

For this it is sufficient to show that $\Gamma(\mathfrak{a})$ does not contain any elliptic element. Suppose $\Gamma(\mathfrak{a})$ contains an elliptic element $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let N be the order of σ, then there exists an N-th root of unity ζ_N such that $\mathrm{tr}\,\sigma=\zeta_N+\zeta_N^{-1}$. Let $\mathfrak{p}$ denote a prime factor of $\mathfrak{a}$. From $\sigma\equiv\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$mod. $\mathfrak{p}$, p is ramified in $\mathbf{Q}(\zeta_N)$and it follows that $p|N$ and $F\supset\mathbf{Q}(\zeta_N+\zeta_N^{-1})$, where p is the residual characteristic of $\mathfrak{p}$. Since we have assumed that $\mathfrak{p}$ is prime to $D(F/\mathbf{Q})$ and p is fully ramified in $\mathbf{Q}(\zeta_p+\zeta_p^{-1})$, we must have $p=2$, or 3. This contradicts the assumption $(p, 6)=1$.

Remark 2. Let γ be an element of $SL_2(\mathfrak{o}_F)$. Then, since $\Gamma(\mathfrak{a})$ is a normal subgroup of $SL_2(\mathfrak{o}_F)$, γ induces an automorphism of $\mathfrak{H}\times\mathfrak{H}/\Gamma(\mathfrak{a})$ and this automorphism can be extended to that of X. Let f be the automorphism of X defined by an element $\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}$, $a\in\mathfrak{o}_F$, then we see by the same argument as above that f has not any elliptic fixed point (the fixed point which is a inner point of $\mathfrak{H}\times\mathfrak{H}$).

Before making the last remark, we shall give some elementary consideration on the cusps of $\Gamma(\mathfrak{a})$. It is known [15] that the $SL_2(\mathfrak{o}_F)$-equivalence classes of cusps is in one to one correspondence with the ideal classes of F. Let $\mathfrak{a}_1, \cdots, \mathfrak{a}_h$ be a complete system of representatives of the ideal classes of F. Here we assume that $\mathfrak{a}_1=\mathfrak{o}_F$ and that $\mathfrak{a}_i$ is integral and prime to $\mathfrak{a}$. Then a cusp which corresponds to $\mathfrak{a}_i$ is given by $\frac{\alpha}{\beta}$ for some α and β, where α and β are two elements of $\mathfrak{o}_F$ such that $(\alpha, \beta)=\mathfrak{a}_i$. If $\beta=0$ we consider $\frac{\alpha}{\beta}$ the infinite point of $\mathfrak{H}^n$. As to $\Gamma(\mathfrak{a})$-equivalence classes, we have

Lemma 1. *Let $\frac{\alpha}{\beta}$ and $\frac{\alpha'}{\beta'}$ be two cusps such that $(\alpha, \beta)=(\alpha', \beta')=\mathfrak{a}_i$. Then $\frac{\alpha}{\beta}$ and $\frac{\alpha'}{\beta'}$ are $\Gamma(\mathfrak{a})$-equivalent to each other if and only if there exists an element e of the unit group E of F such that $\alpha\equiv e\alpha'$, $\beta\equiv e\beta'$* mod. $\mathfrak{a}$.

Proof. If $\frac{\alpha}{\beta}$ and $\frac{\alpha'}{\beta'}$ are $\Gamma(\mathfrak{a})$-equivalent, then it is easy to see that there exists $e\in E$ which satisfies the above condition. Conversely if $\alpha\equiv e\alpha'$, $\beta\equiv e\beta'$ mod. $\mathfrak{a}$, Hilfsatz 1 in [8] asserts that there exists $\gamma\in\Gamma(\mathfrak{a})$ such that $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\gamma\begin{pmatrix}e\alpha'\\ e\beta'\end{pmatrix}$. As $\frac{e\alpha'}{e\beta'}=\frac{\alpha'}{\beta'}$, we see $\frac{\alpha}{\beta}$ and $\frac{\alpha'}{\beta'}$ are $\Gamma(\mathfrak{a})$-equivalent to each other.

By this Lemma we can determine all the $\Gamma(\mathfrak{a})$-equivalence classes of cusps.

For each $\lambda=\frac{\alpha}{\beta}$, $(\alpha, \beta)=\mathfrak{a}_i$, there exist such elements ξ, ν in $\mathfrak{a}_i^{-1}$ with $\det\begin{pmatrix}\alpha & \xi\\ \beta & \nu\end{pmatrix}=1$ (c. f. [15]). In the following we fix the representatives $\lambda=\frac{\alpha}{\beta}$ of the $\Gamma(\mathfrak{a})$-equivalence classes of the cusps, and fix such elements ξ, ν as above for each λ and put $\gamma_\lambda=\begin{pmatrix}\alpha & \xi\\ \beta & \nu\end{pmatrix}$.

Lemma 2. *Notation being as above, we assume* $\mathfrak{a}$ *to be square free. Then for a cusp* $\lambda=\frac{\alpha}{\beta}$, $(\alpha, \beta)=\mathfrak{a}_i$, *the stabilizer group of* λ *in* $\Gamma(\mathfrak{a})$ *is equal to*

$$\left\{\begin{pmatrix}\alpha & \xi\\ \beta & \nu\end{pmatrix}\begin{pmatrix}e & m\\ 0 & e^{-1}\end{pmatrix}\begin{pmatrix}\alpha & \xi\\ \beta & \nu\end{pmatrix}^{-1}\middle|\, e\in E(\mathfrak{a}),\ m\in\mathfrak{a}\mathfrak{a}_i^{-2}\right\},$$

where $E(\mathfrak{a})=\{e\in E\,|\,e\equiv 1 \text{ mod. } \mathfrak{a}\}$ *and,* ξ, ν *are elements of* $\mathfrak{a}_i^{-1}$ *mentioned above.*

Proof. The stabilizer group of λ in $SL_2(\mathfrak{o}_F)$ is equal to $\left\{\begin{pmatrix}\alpha & \xi\\ \beta & \nu\end{pmatrix}\begin{pmatrix}e & m\\ 0 & e^{-1}\end{pmatrix}\begin{pmatrix}\alpha & \xi\\ \beta & \nu\end{pmatrix}^{-1}\middle|\, e\in E,\ m\in\mathfrak{a}_i^{-2}\right\}$. If $\begin{pmatrix}\alpha & \xi\\ \beta & \nu\end{pmatrix}\begin{pmatrix}e & m\\ 0 & e^{-1}\end{pmatrix}\begin{pmatrix}\alpha & \xi\\ \beta & \nu\end{pmatrix}^{-1}$ is contained in $\Gamma(\mathfrak{a})$, then from

$$\begin{pmatrix}\alpha & \xi\\ \beta & \nu\end{pmatrix}\begin{pmatrix}e & m\\ 0 & e^{-1}\end{pmatrix}\begin{pmatrix}\alpha & \xi\\ \beta & \nu\end{pmatrix}^{-1}=\begin{pmatrix}\alpha\nu e-\alpha\beta m-\beta\xi e^{-1} & -\alpha\xi e+\alpha^2 m+\alpha\xi e^{-1}\\ \beta\nu e-\beta^2 m-\beta\nu e^{-1} & -\beta\xi e+\alpha\beta m+\alpha\nu e^{-1}\end{pmatrix},$$

we see $\alpha\nu e-\alpha\beta m-\beta\xi e^{-1}\equiv 1$ mod. $\mathfrak{a}$ and $-\beta\xi e+\alpha\beta m+\alpha\nu e^{-1}\equiv 1$ mod $\mathfrak{a}$. From this we have $(\alpha\nu-\beta\xi)e+(\alpha\nu-\beta\xi)e^{-1}\equiv 2$ mod. $\mathfrak{a}$, $(e+e^{-1})\equiv$

2 mod. $\mathfrak{a}$, and $(e-1)^2 \equiv 0$ mod. $\mathfrak{a}$. By the assumption on $\mathfrak{a}$ we obtain $e \equiv 1$ mod. $\mathfrak{a}$. Then the above element is contained in $\Gamma(\mathfrak{a})$ if and only if $\alpha\beta m \equiv \alpha^2 m \equiv \beta^2 m \equiv 0$ mod. $\mathfrak{a}$ and this condition is equivalent to $m \in \mathfrak{a}_i^{-2}$.

Remark 3. In this remark we assume that the class number of F is one and $\mathfrak{a}$ is a prime ideal $\mathfrak{p}=(\mu)$ of F. The fixed points of f_ι (resp. $f_{\iota'}$) are equal to the cusps which is $\Gamma(\mathfrak{p})$-equivalent to $\frac{\alpha}{\mu}$, $\alpha \in \mathfrak{o}_F$, $(\alpha, \mu)=\mathfrak{o}_F$. In fact in Remark 2 we noted that inner points of $\mathfrak{H}\times\mathfrak{H}$ are not fixed by f_ι and $f_{\iota'}$. Let $\frac{\alpha}{\beta}$, $(\alpha, \beta)=\mathfrak{o}_F$, be a fixed cusp of f_ι (resp. $f_{\iota'}$), then there exists an element $\gamma \in \Gamma(\mathfrak{p})\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$ (resp. $\Gamma(\mathfrak{p})\begin{pmatrix}1 & \eta\\0 & 1\end{pmatrix}$) which fixes the cusp $\frac{\alpha}{\beta}$. Since $\Gamma(\mathfrak{p})$ $\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$ (resp. $\Gamma(\mathfrak{p})\begin{pmatrix}1 & \eta\\0 & 1\end{pmatrix}$) is contained in $SL_2(\mathfrak{o}_F)$ and in view of result on the stabilizer group of $\frac{\alpha}{\beta}$ in $SL_2(\mathfrak{o}_F)$ (c. f. [15]), there exist $e \in E$ and $m \in \mathfrak{o}_F$ such that $\gamma=\begin{pmatrix}\alpha & \xi\\\beta & \nu\end{pmatrix}\begin{pmatrix}e & m\\0 & e^{-1}\end{pmatrix}\begin{pmatrix}\alpha & \xi\\\beta & \nu\end{pmatrix}^{-1}$, where ξ and ν are the fixed elements of $\mathfrak{o}_F$ for $\frac{\alpha}{\beta}$ with $\det\begin{pmatrix}\alpha & \xi\\\beta & \nu\end{pmatrix}=1$. Since

$$\gamma=\begin{pmatrix}\alpha & \xi\\\beta & \nu\end{pmatrix}\begin{pmatrix}e & m\\0 & e^{-1}\end{pmatrix}\begin{pmatrix}\alpha & \xi\\\beta & \nu\end{pmatrix}^{-1}=\begin{pmatrix}\alpha\nu e-\alpha\beta m-\beta\xi e^{-1} & -\alpha\xi e+\alpha^2 m+\alpha\xi e^{-1}\\\beta\nu e-\beta^2 m-\beta\nu e^{-1} & -\beta\xi e+\alpha\beta m+\alpha\nu e^{-1}\end{pmatrix},$$

we see that

(a) $\alpha\eta e-\alpha\beta m-\beta\xi e^{-1}=\alpha(\eta e-\beta m-\eta e^{-1})+e^{-1}\equiv 1$

(b) $\beta(\nu e-\beta m-\nu e^{-1})\equiv 0$ mod. $\mathfrak{p}$.

From (b) it follows that $\beta\equiv 0$ mod. $\mathfrak{p}$ or $\eta e-\beta m-\eta e^{-1}\equiv 0$ mod. $\mathfrak{p}$. If we assume $\beta \not\equiv 0$ mod. $\mathfrak{p}$ so $\eta e-\beta m-\eta e^{-1}\equiv 0$ mod. $\mathfrak{p}$, then by (a) we see $e^{-1}\equiv 1$ mod. $\mathfrak{p}$ and $\beta m\equiv 0$ mod. $\mathfrak{p}$. As we assume $\beta \not\equiv 0$ mod. $\mathfrak{p}$, it follows that $m\equiv 0$ mod. $\mathfrak{p}$ and $-\alpha\xi e+\alpha^2 m+\alpha\xi e^{-1}\equiv 0$ mod. $\mathfrak{p}$. But this contradicts to the assumption that $\gamma\in\Gamma(\mathfrak{p})\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$ (resp. $\Gamma(\mathfrak{p})$

$\begin{pmatrix}1 & \eta\\0 & 1\end{pmatrix}$). Hence $\beta\equiv 0 \bmod. \mathfrak{p}$ and we have proved our assertion. Moreover we see easily that if $\beta\equiv 0 \bmod. \mathfrak{p}$ the set of all the elements in $\Gamma(\mathfrak{p})\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$ (resp. $\Gamma(\mathfrak{p})\begin{pmatrix}1 & \eta\\0 & 1\end{pmatrix}$) which fix the cusp $\frac{\alpha}{\beta}$ is equal to $\left\{\begin{pmatrix}\alpha & \xi\\\beta & \eta\end{pmatrix}\begin{pmatrix}e & m\\0 & e^{-1}\end{pmatrix}\begin{pmatrix}\alpha & \xi\\\beta & \eta\end{pmatrix}^{-1}\middle| e\in E(\mathfrak{p}),\ m\in\mathfrak{o}_F,\ \alpha^2m\equiv 1 \bmod. \mathfrak{p}\right\}$ (resp. $\left\{\begin{pmatrix}\alpha & \xi\\\beta & \eta\end{pmatrix}\begin{pmatrix}e & m\\0 & e^{-1}\end{pmatrix}\begin{pmatrix}\alpha & \xi\\\beta & \eta\end{pmatrix}^{-1}\middle| e\in E(\mathfrak{p}),\ m\in\mathfrak{o}_F,\ \alpha^2m\equiv\eta \bmod. \mathfrak{p}\right\}$), where $E(\mathfrak{p}) = \{e\in E\,|\,e\equiv 1 \bmod. \mathfrak{p}\}$.

Lastly we give the restatement of Lemma 1 under the condition on F and $\mathfrak{a}$ in this remark.

Lemma 1′. *$\Gamma(\mathfrak{p})$-equivalence classes of the cusps $\lambda=\frac{\alpha}{\beta}$ such as $\beta\in\mathfrak{p}$, $(\alpha, \beta)=\mathfrak{o}_F$ are in one to one correspondence with $(\mathfrak{o}_F/\mathfrak{p})^\times/\tilde{E}$, where $\tilde{E}$ denotes the subgroup of $(\mathfrak{o}_F/\mathfrak{p})^\times$ generated by the classes represented by some element of E. Let $\{\alpha_i\}$ be a complete system of the representatives of $(\mathfrak{o}_F/\mathfrak{p})^\times/\tilde{E}$, then we can choose $\frac{\alpha_i}{\mu}$ as the complete system of the representatives of $\Gamma(\mathfrak{p})$-equivalence classes of such cusps $\lambda=\frac{\alpha}{\beta}$ with $\beta\in\mathfrak{p}$ and $(\alpha, \beta)=\mathfrak{o}_F$.*

In the following we fix such $\frac{\alpha_i}{\mu}$ as in Lemma 1′ as the representatives of $\Gamma(\mathfrak{p})$-equivalence classes of the cusps as above.

§ **2.** Now we briefly review Hirzebruch's result on the resoution of cusp singularities. We restrict ourselves to the case $n=2$ and assume F is a real quadratic field. Two data M and V determine a cusp, where M is a submodule of F of rank 2 over $\mathbf{Z}$, and V is a subgroup of the group of all totally positive units E_+ of F of finite index and satisfies $VM\subset M$. Let $\Gamma_{(M,V)}$ be the group $\left\{\begin{pmatrix}e & m\\0 & 1\end{pmatrix}\middle| e\in V,\ m\in M\right\}$, then $\Gamma_{(M,V)}$ acts on $\mathfrak{H}\times\mathfrak{H}$ as follows;

$$\gamma(z_1, z_2) = (ez_1+m,\ e'z_2+m'),$$

where $(z_1, z_2)\in\mathfrak{H}\times\mathfrak{H}$, $\gamma=\begin{pmatrix} e & m \\ 0 & 1 \end{pmatrix}$ and $'$ denotes the conjugation of F over $\mathbf{Q}$. Here we note that $\Gamma_{(M,V)}$ acts on $\mathfrak{H}\times\mathfrak{H}$ freely. Let K be a positive real number, then the subset $\mathfrak{H}_K=\{z\in\mathfrak{H}\times\mathfrak{H}|\ y_1y_2\geqslant K\}$ of $\mathfrak{H}\times\mathfrak{H}$ is stable under the action of $\Gamma_{(M,V)}$ and the qutient space N of $\mathfrak{H}_K$ by $\Gamma_{(M,V)}$ can be compactified by adding one point (∞) ((∞) is the cusp given by the data M and V). We can give on $N\cup(\infty)$ a structure of a normal complex space with one isolated singularity at the cusp (∞). We call this singularity "a cusp singularity". Let M^0 be the dual module of M, i. e.

$M^0=\{a\in|\, am+a'm'\in\mathbf{Z},\ {}^{\forall}m\in M\}$, then the local ring $\mathcal{O}_{(M,V)}$ for the cusp (M, V) consists of all "convergent" Fourier series

$$f(z_1, z_2)=\sum_{m_0\in M_0} a(m_0)\exp\left(2\pi\sqrt{-1}\,(m_0z_1+m_0'z_2)\right),$$

with coefficients $a(m_0)$ $(\in\mathbf{C})$ satisfying $a(m_0)\neq 0$ only if both $m_0>0$ and $m_0'>0$, or $m_0=0$ and $a(em_0)=a(m_0)$ for $e\in V$. Here "convergent" means that f converges for $\mathrm{Im}(z_1)\,\mathrm{Im}(z_2)>C$ for a positive constant C depending on f ([11], [12]).

For a sequence of positive integers $((b_1, \cdots, b_r))$ such as $b_i\geqslant 2$ and $b_i\geqslant 3$ for at least one b_i, Hirzebruch constructed a "canonical" isolated singularity and showed that this singularity is isomorphic to a cusp singularity for some (M, V). Further he showed that every cusp singularity is isomorphic to a "canonical" isolated singularity for some sequence of positive integers as above. Before explaining the relation between $((b_1, \cdots, b_r))$ and (M, V), and the construction of canonical singularities, we quote some algebraic lemmas from [12].

Lemma 3. (*Hirzebruch*) (1) *For every module M in F, there exists a totally positive element $a\in F$ such that $aM=\mathbf{Z}+\mathbf{Z}w$, where w is an element of F which satisfies $0<w'<1<w$. Further this condition on w, $0<w'<1<w$, implies that w can be expanded to a purely periodic continued fraction.*

(2) *Let* $((b_1,\cdots,b_r))$ *be such a sequence of positive integers as above and extend the definition of* b_k *for* $k\in\mathbf{Z}$ *by* $b_k=b_j$ *if* $k\equiv j \bmod. r$. *Then the continued fraction* $b_1-\cfrac{1}{b_2-\cfrac{1}{\ddots}}$ *converges to a real quadratic irrational number* w *which satisfies* $0<w'<1<w$. *Put* $\dfrac{p_k}{q_k}=b_1-\cfrac{1}{b_2-\cfrac{1}{\ddots\, b_{k-1}-\cfrac{1}{b_k}}}$ *and let* s *be the length of primitive cycle of* $((b_1,\cdots,b_r))$ ([11], [12]), *then the group of totally positive units* E_+ *of the real quadratic field* $\mathbf{Q}(w)$ *has a generator* $e_0=p_s-q_sw$, *and* $e_0^t=p_{ts}-q_{ts}w$.

First we note if a is a totally positive element of F, then the cusp singularities which are given by (M, V) and by (aM, V) are isomorphic to each other. Hence by Lemma 3. (1) we may suppose M is of the form $M=\mathbf{Z}+\mathbf{Z}w$ with $0<w'<1<w$. By Lemma 3. (2) $((b_1,\cdots,b_r))$ determines a submodule $M=\mathbf{Z}+\mathbf{Z}w$ of a real quadratic field, and let s be the length of the primitive period of $((b_1,\cdots,b_r))$ and V denotes the subgroup of the totally positive units E_+ of $\mathbf{Q}(w)$ of index $t=\frac{r}{s}$, then the "canonical" singularity given by $((b_1,\cdots,b_r))$ is isomorphic to the cusp singularity determined by (M, V). Conversely for the cusp given by (M, V), where $M=\mathbf{Z}+\mathbf{Z}w$, $0<w'<1<w$ and $[E_+ : V]=t$, $((b_1,\cdots,b_s))$ denotes the primitive cycle of the continued fraction of w. Then the above cusp singularity is isomorphic to the canonical singularity determined by the sequence $((b_1,\cdots,b_s,b_1,\cdots,b_s,\cdots\cdots,b_1,\cdots,b_s))$ (t-times).

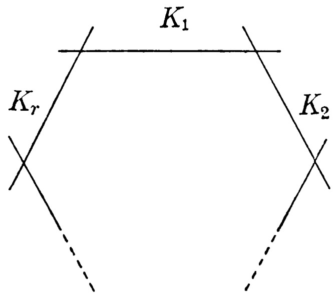
Now we give the construction of canonical singularities. For $((b_1,\cdots,b_r))$, extend the definition b_k for $k\in\mathbf{Z}$ as above. For $k\in\mathbf{Z}$, R_k denotes a copy of $\mathbf{C}^2$ with complex coordinates (u_k, v_k) and R_k' (resp. R_k'') denotes $R_k-\{u_k=0\}$ (resp. $R_k-\{v_k=0\}$). We define biholomorphic maps $\varphi_{k-1}: R'_{k-1} \longrightarrow R''_k$ by the equations $u_k = u_{k-1}^{b_k} v_{k-1}$ and $v_k = 1/u_{k-1}$. If we make the identifications given by φ_{k-1}'s on the disjoint union $\cup R_k$, we get a complex manifold Y in which we have a string of compact rational curves S_k non-singularly embedded. S_k is given by $u_k = 0$ in the k-th coordinate system and by $v_{k-1} = 0$ in the $(k-1)-th$ coordinate system. S_k and S_{k+1} intersect in just one point transversally. S_i and S_k $(i<k)$ do not intersect, if $k-i \neq 1$. The self intersection number $S_k \cdot S_k$ equals $-b_k$. The complex manifold Y admits a biholomorphic map $T: Y \longrightarrow Y$ which sends a point with coordinates (u_k, v_k) in the k-th coordinate system to the point with the same coordinates in the $(k+r)$-th coordinate system. Thus $T(S_k) = S_{k+r}$. On the other hand there exists a tubular neighborhood Y^0 of $\cup S_k$ on which the infinite cyclic group $Z = \{T^n | n \in \mathbf{Z}\}$ operates freely such that Y^0/Z is a complex manifold in which r rational curves $K_1, \cdots, K_r$ $(K_1 \cup \cdots \cup K_r = \cup S_k/Z)$ are embedded. The tubular neighborhood Y^0 is given as follows; let F be a function on Y which is given in $k(>0)$-th coordinate by

$$F = \frac{(w_k \log |u_k| + \log |v_k|)(w'_k \log |u_k| + \log |v_k|)}{|w_1 w'_1| \cdots\cdots |w_k w'_k|}.$$

Then Y^0 is the subset of points y of Y which satisfy $w_k \log |u_k| + \log |v_k| < 0$, $w'_k \log |u_k| + \log |v_k| < 0$ for some k-th coordinate system and $F(y) > C$ for some fixed constant C, where w_k for $k \in \mathbf{Z}$ is defined by the continued fractions

$$w_k = b_{k+1} - \cfrac{1}{b_{k+2} - \cfrac{}{\ddots}}.$$

By the condition of b_i, the intersection matrix of the curves $K_1, \cdots, K_r$ is negative-definite. According to Grauert [7] the curves $K_1, \cdots, K_r$ can be blown down to a singular point x in a complex space. By the construction, x has a cyclic resolution as follows;



Lastly we give a description of the local ring $\mathcal{O}_{((b_1,\dots,b_r))}$ of x and the relation between the local ring of the corresponding cusp and that of x. $\mathcal{O}_{((b_1,\dots,b_r))}$ consists of all functions which are holomorphic in some neighborhood of $\cup K_i$. We write down these function in 0-th coordinate system. Let $f=\sum a_j(u, v)^j$ be a power series in two variables, where $j\in \mathbf{Z}\times\mathbf{Z}$, $(u, v)^j=u^{j_1}v^{j_2}$ for $j=(j_1, j_2)$ and we define $Tj=\begin{pmatrix} -q_{r-1} & p_{r-1} \\ -q_r & p_r \end{pmatrix}\begin{pmatrix} j_1 \\ j_2 \end{pmatrix}$. Then f is contained in $\mathcal{O}_{((b_1,\dots,b_r))}$ if and only if f satisfies following conditions;

i) $a_{Tj}=a_j$ and $a_j=0$ only for $j=(0, 0)$ or j such as $j_1>0$, $j_2>0$, $w'<\frac{j_1}{j_2}<w$.

ii) f converges for (u, v) such as
$(w\log|u|+\log|v|)(w'\log|u|+\log|v|)>C_f$,
$w\log|u|+\log|v|<0$, and $w'\log|u|+\log|v|<0$,
where C_f is a constant depending on f.

Put (3) $\begin{cases} u=\exp(2\pi\sqrt{-1}\,\zeta_1) \\ v=\exp(2\pi\sqrt{-1}\,\zeta_2) \end{cases}$ and (4) $\begin{cases} w\zeta_1+\zeta_2=z_1 \\ w'\zeta_1+\zeta_2=z_2, \end{cases}$

then $\mathcal{O}_{((b_1,\dots,b_r))}$ can be described by means of z_1, z_2. For $j=(j_1, j_2)$, define x_1, x_2 by $\begin{cases} x_1w+x_2w'=j_1 \\ x_1+x_2=j_2 \end{cases}$. Then $x_1=x_2'$ and $x_1\in M^0$. Let V be as above, then $\mathcal{O}_{((b_1,\dots,b_r))}$ consists of all Fourier series

$f=\sum_{m_0\in M_0} c(m_0)\exp(2\pi\sqrt{-1}(m_0z_1+m_0'z_2))$ in z_1, z_2 which satisfy following conditions;

i) $c(m_0)=c(em_0)$ for all $e\in V$ and $c(m_0)=0$ only if both $m_0>0$, $m_0'>0$ or $m_0=0$.

ii) f converges for $\mathrm{Im}(z_1)\mathrm{Im}(z_2)>C_f$, where C_f is a constant depending on f.

As remarked before, this is isomorphic to the local ring of the cusp given by (M, V).

Let $\mathcal{O}_{(M,V)}$ be a cusp and $\bar{e}$ (resp $\bar{m}$) be an element of E (resp. F) such as $\bar{e}M=M$ (resp. $(e-1)\bar{m}\in M$ for all $e\in V$). Then $\bar{e}$ (resp. $\bar{m}$) defines a map $g_{\bar{e}}$ (resp. $g_{\bar{m}}$) from $\mathcal{O}_{(M,V)}$ to itself given by;

$$g_{\bar{e}}: (\exp(2\pi\sqrt{-1}\,z_1),\ \exp(2\pi\sqrt{-1}\,z_2))$$
$$\longrightarrow(\exp(2\pi\sqrt{-1}\,\bar{e}^2z_1),\ \exp(2\pi\sqrt{-1}\,\bar{e}'^2z_2))$$
$$(\text{resp. } g_{\bar{m}}: (\exp(2\pi\sqrt{-1}\,z_1),\ \exp(2\pi\sqrt{-1}\,z_2)$$
$$\longrightarrow(\exp(2\pi\sqrt{-1}\,(z_1+\bar{m})),\ \exp(2\pi\sqrt{-1}\,(z_2+\bar{m}')))).$$

We note that the maps $(z_1, z_2)\longrightarrow(\bar{e}^2z_1, \bar{e}'^2z_2)$ and $(z_1, z_2)\longrightarrow(z_1+m, z_2+m')$ make stable the neighborhood $\mathfrak{H}_K=\{\mathrm{Im}(z_1)\mathrm{Im}(z_2)>K\}$, and it is easy to see that the maps $g_{\bar{e}}$ and $g_{\bar{m}}$ induce maps from $\mathcal{O}_{(M,V)}$ to itself. As to these maps $g_{\bar{e}}$ and $g_{\bar{m}}$ we prove the following theorem.

Theorem 2. *$g_{\bar{e}}$ and $g_{\bar{m}}$ can be extended to the resolution of the cusp $\mathcal{O}_{(M,V)}$.*

Proof. If for a totally positive number a of F, (aM, V) satisfies the condition of Lemma 3 (1), $\mathcal{O}_{(M,V)}$ is isomorphic to $\mathcal{O}_{(aM,V)}$ by the mapping

$$(\exp(2\pi\sqrt{-1}\,z_1),\ \exp(2\pi\sqrt{-1}\,z_2))$$
$$\longrightarrow(\exp(2\pi\sqrt{-1}\,az_1),\ \exp(2\pi\sqrt{-1}\,a'z_2)).$$

By this isomorphism the maps $g_{\bar{e}}$ and $g_{\bar{m}}$ are transformed to the maps $g_{\bar{e}}'$ and $g_{\bar{m}}'$ given by

$$g'_{\bar{e}}:(\exp(2\pi\sqrt{-1}\,z_1),\ \exp(2\pi\sqrt{-1}\,z_2))$$
$$\longrightarrow(\exp(2\pi\sqrt{-1}\,\bar{e}^2 z_1),\ \exp(2\pi\sqrt{-1}\,\bar{e}'^2 z_2))$$
$$g'_{\bar{m}}:(\exp(2\pi\sqrt{-1}\,z_1),\ \exp(2\pi\sqrt{-1}\,z_2))$$
$$\longrightarrow(\exp(2\pi\sqrt{-1}\,(z_1+a\bar{m})),\ \exp(2\pi\sqrt{-1}\,(z_2+(a\bar{m})')))$$

and $\bar{e}$ and $\bar{m}$ satisfy $\bar{e}(aM)=aM$ and $(e-1)a\bar{m}\in aM$ for all $e\in V$ respectively. Hence we may assume that (M, V) satisfies the condition in Lemma 3(1) and we use the same notation as before. Let $e_0=p_s-q_s w$ be the generator of E_+ and l be the integer such as $\bar{e}^2=e_0^l$, where s denotes the length of the primitive cycle of w as before. $\tilde{g}_{\bar{e}}$ denotes the map from Y to itself which sends a point with coordinates (u_k, v_k) in k-th coordinate system to the point with coordinate (u_k, v_k) in $(k+ls)$-th coordinate system. Then this map $g_{\bar{e}}$ makes stable the tubular neighborhood Y^0 and induces a map from Y^0/Z to itself. It is easy to see that $\tilde{g}_{\bar{e}}$ maps $\bigcup K_i$ to itself and induces in $\mathcal{O}_{(M,V)}$ the map $g_{\bar{e}}$ and our assertion is proved for $\tilde{g}_{\bar{e}}$. Now we prove our assertion for $g_{\bar{m}}$. For a positive integer k, define positive integers p_k and q_k by

$$\frac{p_k}{q_k}=b_1-\cfrac{1}{b_2-\cfrac{}{\ddots\, b_{k-1}-\cfrac{1}{b_k}}}\,.$$

Then p_k and q_k satisfy

$$\begin{pmatrix}-q_{k-1} & p_{k-1}\\ -q_k & p_k\end{pmatrix}=\begin{pmatrix}0 & 1\\ -1 & b_k\end{pmatrix}\cdots\cdots\begin{pmatrix}0 & 1\\ -1 & b_1\end{pmatrix},$$

where we define $p_0=1$ and $q_0=0$. We extend the definition of p_k and q_k to negative integers. For $k=-1$ put $p_{-1}=0$, $q_{-1}=-1$, and for $k<-1$ define p_k, q_k inductively by

$$\begin{pmatrix}-q_{k-1} & p_{k-1}\\ -q_k & p_k\end{pmatrix}=\begin{pmatrix}b_{k+1} & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}b_{k+2} & -1\\ 1 & 0\end{pmatrix}\cdots\cdots\begin{pmatrix}b_0 & -1\\ 1 & 0\end{pmatrix}.$$

Define a map which is given in k-th coordinate system by

$$(u_k, v_k) \longrightarrow \left(\exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\left(\bar{m}\frac{p_k - q_k w'}{w-w'}\right)\right) u_k,\right.$$

$$\left.\exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\left(\bar{m}\frac{-p_{k-1} + q_{k-1} w'}{w-w'}\right)\right) v_k\right),$$

then it is easy to see that this map is compatible with the identification of $\cup R_k$ by φ_k's and induces an automorphism of the complex manifold Y. We denote this map by $\tilde{g}_{\bar{m}}$. By the definition of a Z-equivalent tubular neighborhood, $\tilde{g}_{\bar{m}}$ makes stable all such tubular neighborhoods Y^0. Further $\tilde{g}_{\bar{m}}$ maps $\cup K_i$ to itself. Now we show that if $k \equiv k' \bmod. r$, then

$$(5)\quad \begin{cases} \exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\left(\bar{m}\dfrac{p_k - q_k w'}{w-w'}\right)\right) = \exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\left(\bar{m}\dfrac{p_{k'} - q_{k'} w'}{w-w'}\right)\right) \\ \exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\left(\bar{m}\dfrac{-p_{k-1} + q_{k-1} w'}{w-w'}\right)\right) \\ \qquad = \exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\left(\bar{m}\dfrac{-p_{k'-1} + q_{k'-1} w'}{w-w'}\right)\right). \end{cases}$$

If this is shown, the map from Y^0 to itself induced by $\tilde{g}_{\bar{m}}$ defines an automorphism of Y^0/Z and by (3), (4) we see easily that this map induces $g_{\bar{m}}$ in $\mathcal{O}_{(M,V)}$. Hence $g_{\bar{m}}$ can be extended to the resolution of the cusp and our assertion is proved for $g_{\bar{m}}$. Now if $k \equiv k' \bmod. r$, then there exists an integer l such as

$$\begin{pmatrix} -q_{k-1} & p_{k-1} \\ -q_k & p_k \end{pmatrix} = \begin{pmatrix} -q_{k'-1} & p_{k'-1} \\ -q_{k'} & p_{k'} \end{pmatrix} \begin{pmatrix} -q_{r-1} & p_{r-1} \\ -q_r & p_r \end{pmatrix}^l.$$

It is enough to prove (5) for $l=1$. For $l=1$ we have

$$\begin{cases} p_{k-1} = -q_{k'-1}\, p_{r-1} + p_{k'-1}\, p_r \\ q_{k-1} = -q_{k'-1}\, q_{r-1} + p_{k'-1}\, q_r \\ p_k \ \ = -q_{k'}\, p_{r-1} + p_{k'}\, p_r \\ q_k \ \ = -q_{k'}\, q_{r-1} + q_{k'}\, q_r. \end{cases}$$

By $w=b_1-\cfrac{1}{b_2-\cfrac{1}{\ddots\, b_r-\cfrac{1}{w}}}$, we see $\dfrac{p_r w-p_{r-1}}{q_r w-q_{r-1}}=w$,

hence $p_{r-1}-q_{r-1}w'=w'(p_r-q_r w')$. From these two relations we obtain $p_k-q_k w'=(p_r-q_r w')(p_{k'}-q_{k'}w')$ and $p_{k-1}-q_{k-1}w'=(p_r-q_r w')(p_{k'-1}-q_{k'-1}w')$. But $p_r-q_r w'\in V$ and by the assumption on $\bar{m}$ we have $(p_r-q_r w'-1)\bar{m}\in M$. Moreover $\dfrac{1}{w-w'}$ and $\dfrac{w'}{w-w'}$ are the basis of the dual module M^0 of M and we have

$$\mathrm{tr}\left(\bar{m}\frac{p_k-q_k w'}{w-w'}\right)=\mathrm{tr}\left(\bar{m}\frac{(p_r-q_r w')(p_{k'}-q_{k'}w')}{w-w'}\right)\equiv\mathrm{tr}\left(\bar{m}\frac{p_{k'}-q_{k'}w'}{w-w'}\right)$$

$$\mathrm{mod.}\ \mathbf{Z}.$$

Similarly $\mathrm{tr}\left(\bar{m}\dfrac{p_{k-1}-q_{k-1}w'}{w-w'}\right)\equiv\mathrm{tr}\left(\bar{m}\dfrac{p_{k'-1}-q_{k'-1}w'}{w-w'}\right)$ mod. $\mathbf{Z}$.

Thus we finish the proof of Theorem 2.

Let $\mathfrak{a}$ be a square-free integral ideal of F such as $(\mathfrak{a}, D(F/\mathbf{Q})(6))=1$, and X be the complete variety obtained form $\mathfrak{H}\times\mathfrak{H}/\Gamma(\mathfrak{a})$ by the compactification. Then by Remark 2 X has no singularity other than cusp singularities. For the representative $\lambda=\dfrac{\alpha}{\beta}$ of each $\Gamma(\mathfrak{a})$-equivalence class of the cusps, transform this cusp to (∞) by the element γ_λ fixed before, then by Lemma 2 this map induces an isomorphism from the local ring of the point on X represented by $\lambda=\dfrac{\alpha}{\beta}$ to the local ring of the cusp at (∞) given by the data $M=\mathfrak{a}\mathfrak{a}_i^{-2}$ and $V=E(\mathfrak{a})^2$, where $\mathfrak{a}_i=(\alpha, \beta)\,(1\leqslant i\leqslant h)$ and $E(\mathfrak{a})=\{e\in E|e\equiv 1 \text{ mod. } \mathfrak{a}\}$ as before. Following Hirzebruch we can resolve this singularity, hence we can resolve the singularities of the variety X and obtain a non-singular surface. We donote this surface by $\tilde{X}$. Now $\Gamma(\mathfrak{a})$ is a normal subgroup of $SL_2(\mathfrak{o}_F)$ hence an element

γ of $SL_2(\mathfrak{o}_F)$ induces an isomorphism from $\mathfrak{H}\times\mathfrak{H}/\Gamma(\mathfrak{a})$ to itself given by $(z_1, z_2)\longrightarrow(\gamma z_1, \gamma' z_2)$. This isomorphism can be extended to X, and we denote this map by f. Then as a corollary of Theorem 2 we can prove the following.

Corollary. *f can be extended to $\tilde{X}$ as an biholomorphic automorphism of $\tilde{X}$.*

Proof. Let ψ be the canonical map from $\tilde{X}$ to X, then ψ induces an isomorphism from $\tilde{X}-\bigcup\psi^{-1}(\mathfrak{s})$ to $X-\bigcup\mathfrak{s}$, where $\bigcup\mathfrak{s}$ denotes all the points of X represented by the cusps. Hence f can be extended to $\tilde{X}-\bigcup\psi^{-1}(\mathfrak{s})$ as an automorphism of $\tilde{X}-\bigcup\psi^{-1}(\mathfrak{s})$. If a point $\mathfrak{s}$ of $\bigcup\mathfrak{s}$ is mapped to $\mathfrak{s}'$ of $\bigcup\mathfrak{s}$, then f induces an isomorphism from $\mathcal{O}_{\mathfrak{s}'}$ to $\mathcal{O}_{\mathfrak{s}}$. Let $\lambda=\frac{\alpha}{\beta}$ and $\lambda'=\frac{\alpha'}{\beta'}$ denote the representatives of $\mathfrak{s}$ and $\mathfrak{s}'$ as before. Then f induces an isomorphism from $\mathcal{O}_{\lambda'}$ to $\mathcal{O}_{\lambda}$ given by an element $\bar{\gamma}$ of $SL_2(\mathfrak{o}_F)$ such as $\bar{\gamma}\lambda=\lambda'$. By γ_λ and $\gamma_{\lambda'}$, $\mathcal{O}_\lambda$ and $\mathcal{O}_{\lambda'}$ are isomorphic to $\mathcal{O}_{(M,V)}$, where $M=\mathfrak{a}\mathfrak{a}_i^{-2}$, $V=E(\mathfrak{a})$ and $\mathfrak{a}_i=(\alpha, \beta)=(\alpha', \beta')$. Hence composing these maps we obtain an automorphism of $\mathcal{O}_{(M,V)}$. This automorphism is induced by $\gamma_{\lambda'}^{-1}\bar{\gamma}\gamma_\lambda$ and by the definition $\gamma_{\lambda'}^{-1}\bar{\gamma}\gamma_\lambda$ is of the form $\begin{pmatrix} e & m \\ 0 & e^{-1}\end{pmatrix}$ for some element e of E and for some element m of $\mathfrak{a}_i^{-2}$. By the way e and $e^{-1}m$ satisfy the condition on $\bar{e}$ and $\bar{m}$ in Theorem 2 and we have two automorphisms g_e and $g_{e^{-1}m}$ of $\mathcal{O}_{(M,V)}$. By $\begin{pmatrix} e & m \\ 0 & e^{-1}\end{pmatrix}=\begin{pmatrix} e & 0 \\ 0 & e^{-1}\end{pmatrix}\begin{pmatrix} 1 & e^{-1}m \\ 0 & 1\end{pmatrix}$, the automorphism of $\mathcal{O}_{(M,V)}$ induced by $\gamma_{\lambda'}^{-1}\bar{\gamma}\gamma_\lambda$ is equal to $g_{e^{-1}m}g_e$ and by Theorem 2, g_e and $g_{e^{-1}m}$ can be extended to the resolution of $\mathcal{O}_{(M,V)}$, hence $g_{e^{-1}m}g_e$ can be extended also to that of $\mathcal{O}_{(M,V)}$. By the definition of $\tilde{X}$ this shows that f can be extended to $\psi^{-1}(\mathfrak{s})$ and our corollary is proved.

§ **3.** First we explain the relation between the cusp forms and 2-forms on $\tilde{X}$. We use the same notation as before, and let $\mathfrak{p}$ be a prime ideal of F such as $(\mathfrak{p}, D(F/\mathbf{Q})(6))=1$ and X be the compactification of $\mathfrak{H}\times\mathfrak{H}/\Gamma(\mathfrak{p})$ and $\tilde{X}$ be the non-singular variety obtained by the resolution of singularities. For $\tilde{X}$ and $U=X-\cup\mathfrak{S}$, $\Omega^2_{\tilde{X}}$ and Ω^2_U denote the sheaves of holomorphic 2-forms on $\tilde{X}$ and U respectively. Let $M_2(\Gamma(\mathfrak{p}))$ be the space of modular forms of weight 2 with respect to $\Gamma(\mathfrak{p})$, i. e. the space consisting of holomorphic functions on $\mathfrak{H}\times\mathfrak{H}$ which satisfy only the condition 1^0 in the defintion of cusp forms. Then for $\omega_U\in H^0(U, \Omega^2_U)$, we can associate an element g of $M_2(\Gamma(\mathfrak{p}))$ by $\omega_U=g(z_1, z_2)dz_1\wedge dz_2$ and this map is an isomorphism ([3], [4], [5]). Connecting this map with the restriction map from $H^0(\tilde{X}, \Omega^2_{\tilde{X}})$ to $H^0(U, \Omega^2_U)$, we obtain a map from $H^0(\tilde{X}, \Omega^2_{\tilde{X}})$ to $M_2(\Gamma(\mathfrak{p}))$. According to Freitag ([4] Satz 3, [5] Satz 3. 2), this induces an isomorphism from $H^0(\tilde{X}, \Omega^2_{\tilde{X}})$ to $S_2\Gamma(\mathfrak{p})$ and we see that this isomorphism is compatible with the extention of f. Hence if we denote f_ε and $f_{\varepsilon'}$ the automorphism of X induced by ε and ε' respectively, by Theorem 2 f_ε and $f_{\varepsilon'}$ can be extended to $\tilde{X}$. We denote these maps $\tilde{f}_\varepsilon$ and $\tilde{f}_{\varepsilon'}$ respectively. Then we have

$$\mathrm{tr}(f_\varepsilon|S_2\Gamma(\mathfrak{p}))=\mathrm{tr}(\tilde{f}_\varepsilon|H^0(\tilde{X}, \Omega^2_{\tilde{X}}))$$

$$\mathrm{tr}(f_{\varepsilon'}|S_2\Gamma(\mathfrak{p}))=\mathrm{tr}(\tilde{f}_{\varepsilon'}|H^0(\tilde{X}, \Omega^2_{\tilde{X}}))$$

where $\mathrm{tr}(f|*)$ denotes the trace of f in the space $*$. Now we calculate $\mathrm{tr}(\tilde{f}_\varepsilon|H^0(\tilde{X}, \Omega^2_{\tilde{X}}))$ and $\mathrm{tr}(f_{\varepsilon'}|H^0(\tilde{X}, \Omega^2_{\tilde{X}}))$ using Atiyah's Lefschetz fixed point theorem ([1] Theorem 4. 6). For simplicity we assume that the class number of F is equal to one and that $\mathfrak{p}$ is generated by a totally positive element μ of F. We apply this theorem to our case taking the canonical bundle K of $\tilde{X}$ as the vector bundle V in that theorem. As the order of f_ε (resp. $f_{\varepsilon'}$) is finite, we may apply this theorem. This theorem asserts that the alternating sum

$$\sum_{i=0}^{2}(-1)^i\,\mathrm{tr}(\tilde{f}_\varepsilon|H^i(\tilde{X}, \Omega^2_{\tilde{X}}))\quad(\text{resp. }\sum_{i=0}^{2}(-1)^i\,\mathrm{tr}(\tilde{f}_{\varepsilon'}|H^i(X, \Omega^2_{\tilde{X}})))$$

can be expressed in terms of the number determined by the fixed point set of f_ε (resp. $f_{\varepsilon'}$), i. e.

$$\sum_{i=0}^{2}(-1)^i \operatorname{tr}(f_\varepsilon|H^i(\tilde{X},\ \Omega^2_{\tilde{X}}))=\sum\nu(\operatorname{Fix}(\tilde{f}_\varepsilon)_j)$$

$$(\text{resp. } \sum_{i=0}^{2}(-1)^i \operatorname{tr}(\tilde{f}_{\varepsilon'}|H^i(\tilde{X},\ \Omega^2_{\tilde{X}}))=\sum\nu(\operatorname{Fix}(\tilde{f}_{\varepsilon'})_j)),$$

where the sum extends over all the connected components $\operatorname{Fix}(\tilde{f}_\varepsilon)_j$ (resp. $\operatorname{Fix}(\tilde{f}_{\varepsilon'})_j$) of the fixed point set of $\tilde{f}_\varepsilon$ (resp. $\tilde{f}_{\varepsilon'}$) and the number $\nu(\operatorname{Fix}(\tilde{f}_\varepsilon)_j)$ (resp. $\nu(\operatorname{Fix}(\tilde{f}_{\varepsilon'})_j)$) is determined by $\operatorname{Fix}(\tilde{f}_\varepsilon)_j$ (resp. $\operatorname{Fix}(\tilde{f}_{\varepsilon'})_j$). As to the definition of $\nu(\operatorname{Fix}(\tilde{f}_\varepsilon)_j)$ and $\nu(\operatorname{Fix}(\tilde{f}_{\varepsilon'})_j)$, we refer to [1], [10]. Let $\mathcal{O}_{\tilde{X}}$ be the structure sheaf of $\tilde{X}$, then by Serre duality we see that

$$\operatorname{tr}(\tilde{f}_\varepsilon|H^2(\tilde{X},\ \Omega^2_{\tilde{X}}))=\operatorname{tr}(\tilde{f}_\varepsilon|H^0(\tilde{X},\ \mathcal{O}_{\tilde{X}})) \quad \text{and}$$
$$\operatorname{tr}(\tilde{f}_\varepsilon|H^1(\tilde{X},\ \Omega^2_{\tilde{X}}))=\operatorname{tr}(\tilde{f}_\varepsilon|H^1(\tilde{X},\ \mathcal{O}_{\tilde{X}})).$$

We see $\operatorname{tr}(\tilde{f}_\varepsilon|H^2(\tilde{X},\ \Omega^2_{\tilde{X}}))=1$ and by [5] Satz 7. 1, $H^1(X,\ \mathcal{O}_{\tilde{X}})=0$ hence $\operatorname{tr}(\tilde{f}_\varepsilon|H^1(\tilde{X},\ \Omega^2_{\tilde{X}}))=0$. Similar results hold for $\tilde{f}_{\varepsilon'}$ also and we obtain

$$\begin{aligned}(6)\qquad &\operatorname{tr}(\tilde{f}_\varepsilon|H^0(\tilde{X},\ \Omega^2_{\tilde{X}}))-\operatorname{tr}(\tilde{f}_{\varepsilon'}|H^0(\tilde{X},\ \Omega^2_{\tilde{X}}))\\ &=\sum\nu(\operatorname{Fix}(\tilde{f}_\varepsilon)_j)-\sum\nu(\operatorname{Fix}(f_{\varepsilon'})_j).\end{aligned}$$

Now we study the fixed point set of $\tilde{f}_\varepsilon$ (resp. $\tilde{f}_{\varepsilon'}$). By Remark 3, f_ε (resp. $f_{\varepsilon'}$) fixes only the cusps $\Gamma(\mathfrak{p})$-equivalent to $\frac{\alpha}{\mu}$, where $(\mu,\ \alpha)=\mathfrak{o}_F$, hence the fixed point set of $\tilde{f}_\varepsilon$ (resp. $\tilde{f}_{\varepsilon'}$) lies in $\bigcup_{\mathfrak{s}\sim\frac{\alpha}{\mu}}\psi^{-1}(\mathfrak{s})$, where the sum extends over all the points of X represented by $\frac{\alpha}{\mu}$'s. We write down the action of f_ε (resp. $\tilde{f}_{\varepsilon'}$) on $\bigcup_{\mathfrak{s}\sim\frac{\alpha}{\mu}}\psi^{-1}(\mathfrak{s})$ explicitly.

By $\gamma_\lambda\left(\lambda=\frac{\alpha}{\mu}\right)$, $\mathcal{O}_\lambda$ is isomorphic to $\mathcal{O}_{(M,V)}$, where $M=\mathfrak{p}$ and $V=E(\mathfrak{p})^2$. Under this isomorphism the automorphism of $\mathcal{O}_\lambda$ induced by $\tilde{f}_\varepsilon$ (resp. $\tilde{f}_{\varepsilon'}$) is transformed to that of $\mathcal{O}_{(M,V)}$ given by

$$\begin{aligned}&(\exp(2\pi\sqrt{-1}\,z_1),\ \exp(2\pi\sqrt{-1}\,z_2))\\ &\longrightarrow\left(\exp\left(2\pi\sqrt{-1}\left(z_1+\left(\frac{1}{\alpha^2}\right)\right)\right),\ \exp\left(2\pi\sqrt{-1}\left(z_2+\left(\frac{1}{\alpha^2}\right)'\right)\right)\right)\end{aligned}$$

(resp. $(\exp(2\pi\sqrt{-1}\,z_1),\ \exp(2\pi\sqrt{-1}\,z_2))$

$$\longrightarrow\left(\exp\left(2\pi\sqrt{-1}\left(z_1+\left(\frac{\eta}{\alpha^2}\right)\right)\right),\ \exp\left(2\pi\sqrt{-1}\left(z_2+\left(\frac{\eta}{\alpha^2}\right)'\right)\right)\right)$$

where $\left(\frac{1}{\alpha^2}\right)$ (resp. $\left(\frac{\eta}{\alpha^2}\right)$) denotes an element of $\mathfrak{o}_F$ such as $\alpha^2\left(\frac{1}{\alpha^2}\right)\equiv 1 \bmod. \mathfrak{p}$ (resp. $\alpha^2\left(\frac{\eta}{\alpha^2}\right)\equiv\eta \bmod. \mathfrak{p}$). Of course this map does not depend on the choice of $\left(\frac{1}{\alpha^2}\right)$ (resp. $\left(\frac{\eta}{\alpha^2}\right)$). By the isomorphism $\mathcal{O}_{(\mu\mathfrak{o}_F, E(\mathfrak{p})^2)} \xrightarrow{\sim} \mathcal{O}_{(\mathfrak{o}_F, E(\mathfrak{p})^2)}$ given by

$$(\exp(2\pi\sqrt{-1}\,z_1),\ \exp(2\pi\sqrt{-1}\,z_2))$$
$$\longrightarrow\left(\exp\left(2\pi\sqrt{-1}\,\frac{z_1}{\mu}\right),\ \exp\left(2\pi\sqrt{-1}\,\frac{z_2}{\mu'}\right)\right),$$

the above automorphism of $\mathcal{O}_{(\mu\mathfrak{o}_F, E(\mathfrak{p})^2)}$ is transformed to that of $\mathcal{O}_{(\mathfrak{o}_F, E(\mathfrak{p})^2)}$ given by

$$(\exp(2\pi\sqrt{-1}\,z_1),\ \exp(2\pi\sqrt{-1}\,z_2))$$
$$\longrightarrow\left(\exp\left(2\pi\sqrt{-1}\left(z_1+\frac{1}{\mu}\left(\frac{1}{\alpha^2}\right)\right)\right),\ \exp\left(2\pi\sqrt{-1}\left(z_2+\frac{1}{\mu'}\left(\frac{1}{\alpha^2}\right)'\right)\right)\right)$$

(resp. $(\exp(2\pi\sqrt{-1}\,z_1),\ \exp(2\pi\sqrt{-1}\,z_2))$

$$\longrightarrow\left(\exp\left(2\pi\sqrt{-1}\left(z_1+\frac{1}{\mu}\left(\frac{\eta}{\alpha^2}\right)\right)\right),\ \exp\left(2\pi\sqrt{-1}\left(z_2+\frac{1}{\mu'}\left(\frac{\eta}{\alpha^2}\right)'\right)\right)\right).$$

For $F=\mathbf{Q}(\sqrt{N})$, where N is a square-free rational integer, let R be the largest one among the rational integers smaller than $\sqrt{N}$ if $N\equiv 2$, $3 \bmod. 4$ and the largest odd one among the integers smaller than $\sqrt{N}$ if $N\equiv 1 \bmod. 4$. Put $w=R+1+\sqrt{N}$ if $N\equiv 2$, $3 \bmod. 4$ and $w=(R+2+\sqrt{N})/2$ if $N\equiv 1 \bmod. 4$. Then w satisfies $0<w'<1<w$, $\mathfrak{o}_F=\mathbf{Z}+\mathbf{Z}w$ and can be expanded to a purely periodic continued fraction. Let $((b_1, \cdots, b_s))$ be its primitive cycle and t be the index $[E_+ : E(\mathfrak{p})^2]$, and define b_k for $k\in\mathbf{Z}$ by $b_i=b_k$ if $i\equiv k$ mod. s. First we consider the automorphism of $\mathcal{O}_{(M,V)}$ induced by f_*.

By the proof of Theorem 2 we see that the extened automorphism of the resolution of the cusp $\mathcal{O}_{(M,V)}$ is given in k-th coordinate system by

$$(u_k, v_k) \longrightarrow \left(\exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\frac{1}{\mu}\left(\frac{1}{\alpha^2}\right)\frac{p_k - q_k w'}{w-w'}\right)u_k,\right.$$
$$\left.\exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\frac{1}{\mu}\left(\frac{1}{\alpha^2}\right)\frac{-p_{k-1}+q_{k-1}w'}{w-w'}\right)v_k\right).$$

Hence if $\exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\ \frac{1}{\mu}\left(\frac{1}{\alpha^2}\right)\frac{p_k-q_kw'}{w-w'}\right) \neq 1$ and $\exp\Big(2\pi\sqrt{-1}\ \mathrm{tr}$ $\frac{1}{\mu}\left(\frac{1}{\alpha^2}\right)\frac{-p_{k-1}+q_{k-1}w'}{w-w'}\Big) \neq 1$, then (0, 0) is the only one fixed point of $\tilde{f}_t$ in R_k. On the other hand if $\exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\ \frac{1}{\mu}\left(\frac{1}{\alpha^2}\right)\frac{p_k-q_kw'}{w-w'}\right)=1$ (resp. $\exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\frac{1}{\mu}\left(\frac{1}{\alpha^2}\right)\frac{-p_{k-1}+q_{k-1}w'}{w-w'}\right)=1$), then S_{k+1} (resp. S_k) is the fixed point set of $\tilde{f}_t$ in R_k. Thus the fixed point set of $\tilde{f}_t$ in $\psi^{-1}(\mathfrak{s})$ is equal to $(\bigcup_i (K_i \cdot K_{i+1})) \cup (\bigcup_j K_j)$, where i runs over such indeces as $\exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\ \frac{1}{\mu}\left(\frac{1}{\alpha^2}\right)\frac{p_i-q_iw'}{w-w'}\right)\neq 1$ and $\exp\Big(2\pi\sqrt{-1}$ $\mathrm{tr}\frac{1}{\mu}\left(\frac{1}{\alpha^2}\right)\frac{p_{i-1}-q_{i-1}w'}{w-w'}\Big)\neq 1$, and j runs over such indeces as $\exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\frac{1}{\mu}\left(\frac{1}{\alpha^2}\right)\frac{-p_{j-1}+q_{j-1}w'}{w-w'}\right)=1$. Here we put $K_i = K_{i'}$ for $i, i' \in \mathbf{Z}$ if $i \equiv i'$ mod. st. The contribution of $K_i \cdot K_{i+1}$ to the sum $\sum \nu(\mathrm{Fix})\tilde{f}_t)_j)$ is equal to

$$\frac{\exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\frac{1}{\mu}\left(\frac{1}{\alpha^2}\right)\frac{p_k-q_kw'}{w-w'}\right)}{\left(1-\exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\frac{1}{\mu}\left(\frac{1}{\alpha^2}\right)\frac{p_k-q_kw'}{w-w'}\right)\right)}$$
$$\times\frac{\exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\frac{1}{\mu}\left(\frac{1}{\alpha^2}\right)\frac{-p_{k-1}+q_{k-1}w'}{w-w'}\right)}{\left(1-\exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\frac{1}{\mu}\left(\frac{1}{\alpha^2}\right)\frac{-p_{k-1}+q_{k-1}w'}{w-w'}\right)\right)}.$$

And the contribution of K_j to the sum $\sum\nu(\mathrm{Fix}(\tilde{f}_\iota)_j)$ is equal to

$$\left\{\exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\frac{1}{\mu}\left(\frac{1}{\alpha^2}\right)\frac{p_k-q_kw'}{w-w'}\right)(1-c-d)(1+c/2)\right.$$

$$\left.\times\frac{1}{1-\exp\left(-2\pi\sqrt{-1}\ \mathrm{tr}\frac{1}{\mu}\left(\frac{1}{\alpha^2}\right)\frac{p_k-q_kw'}{w-w'}\right)(1-d)}\right\}[K_j],$$

where c is the first Chern class of K_j and d is that of the normal bundle of K_j. As to the definition of $\{\ \ \}[K_j]$ we refer to [1]. Notice that $c[K_j]=2(1-\pi)$, where π is the genus of K_j, and that $d[K_j]$ is the self-intersection number of K_j. Now in our case π is equal to zero and $d[K_j]$ is equal to $-b_j$. Hence we see that the above value is equal to

$$\frac{\exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\frac{1}{\mu}\left(\frac{1}{\alpha^2}\right)\frac{p_j-q_jw'}{w-w'}\right)}{1-\exp\left(-2\pi\sqrt{-1}\ \mathrm{tr}\frac{1}{\mu}\left(\frac{1}{\alpha^2}\right)\frac{p_j-q_jw'}{w-w'}\right)}$$

$$\times\left(-1+\frac{b_j}{1-\exp\left(-2\pi\sqrt{-1}\ \mathrm{tr}\frac{1}{\mu}\left(\frac{1}{\alpha_2}\right)\frac{p_j-q_jw'}{w-w'}\right)}\right).$$

Similar results hold for $\tilde{f}_{\iota'}$. For $\alpha\ni\mathfrak{o}_F$ we put

$$\nu(\alpha)=\sum_i\frac{\exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\frac{\alpha}{\mu}\cdot\frac{p_i-q_iw'}{w-w'}\right)}{\left(1-\exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\frac{\alpha}{\mu}\cdot\frac{p_i-q_iw'}{w-w'}\right)\right)}$$

$$\times\frac{\exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\frac{\alpha}{\mu}\cdot\frac{-p_{i-1}+q_{i-1}w'}{w-w'}\right)}{\left(1-\exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\frac{\alpha}{\mu}\cdot\frac{-p_{i-1}+q_{i-1}w'}{w-w'}\right)\right)}$$

$$+\sum_j \frac{\exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\,\frac{\alpha}{\mu}\cdot\frac{p_j-q_jw'}{w-w'}\right)}{1-\exp\left(-2\pi\sqrt{-1}\ \mathrm{tr}\,\frac{\alpha}{\mu}\cdot\frac{p_j-q_jw'}{w-w'}\right)}$$

$$\times\left(-1+\frac{b_j}{1-\exp\left(-2\pi\sqrt{-1}\ \mathrm{tr}\,\frac{\alpha}{\mu}\cdot\frac{p_j-q_jw'}{w-w'}\right)}\right),$$

where i runs over such indeces as $1\leqslant i\leqslant st$ and

$$\exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\,\frac{\alpha}{\mu}\cdot\frac{p_i-q_iw'}{w-w'}\right)\neq 1,$$

$$\exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\,\frac{\alpha}{\mu}\cdot\frac{-p_{i-1}+q_{i-1}w'}{w-w'}\right)\neq 1,$$

and j runs over such indeces as $1\leqslant j\leqslant st$ and

$$\exp\left(2\pi\sqrt{-1}\ \mathrm{tr}\,\frac{\alpha}{\mu}\cdot\frac{-p_{j-1}+q_{j-1}w'}{w-w'}\right)=1.$$

Then we see easily that $\nu(\varepsilon\alpha)=\nu(\alpha)$ for all totally positive unit ε. For the point $\mathfrak{s}$ of X represented by the cusp $\frac{\alpha}{\mu}$ the contribution of the fixed point set in $\phi^{-1}(\mathfrak{s})$ to the sum $\sum\nu(\mathrm{Fix}(\tilde{f}_\iota)_j)$ (resp. $\sum\nu(\mathrm{Fix}(\tilde{f}_{\iota'})_j)$) is equal to $\nu\left(\left(\frac{1}{\alpha^2}\right)\right)$ (resp. $\nu\left(\left(\frac{\eta}{\alpha^2}\right)\right)$). Hence we obtain

$$\sum\nu(\mathrm{Fix}(\tilde{f}_\iota)_j)=\frac{1}{[E:E(\mathfrak{p})]}\sum_{\substack{\alpha \bmod \mathfrak{p}\\ \alpha\not\equiv 0 \bmod \mathfrak{p}}}\left(1+\left(\frac{\alpha}{\mathfrak{p}}\right)\right)\nu(\alpha)\tag{6}$$

and

$$\sum\nu(\mathrm{Fix}(\tilde{f}_{\iota'})_j)=\frac{1}{[E:E(\mathfrak{p})]}\sum_{\substack{\alpha \bmod \mathfrak{p}\\ \alpha\not\equiv 0 \bmod \mathfrak{p}}}\left(1-\left(\frac{\alpha}{\mathfrak{p}}\right)\right)\nu(\alpha)\tag{7}$$

where $\left(\frac{}{\mathfrak{p}}\right)$ denotes the quadratic residue symbol mod. $\mathfrak{p}$ Thus by (1), (5), (6) and (7) we obtain the following theorem.

Theorem 3. *If the class number of the real quadratic field F is equal to one and the prime ideal $\mathfrak{p}$ is generated by a totally positive element then for $k=2$ we have,*

$$y_1-y_2=\frac{1}{\sqrt{(-1)^{(q-1)/2}q}}\ \frac{2}{[E:E(\mathfrak{p})]}\sum_{\alpha \bmod \mathfrak{p}}\left(\frac{\alpha}{\mathfrak{p}}\right)\nu(\alpha).$$

Example. We shall show an example of Theorem 3 by taking $F=\mathbf{Q}(\sqrt{5})$ and $\mu=\frac{7-\sqrt{5}}{2}$. In this case $w=\frac{3+\sqrt{5}}{2}=((3))$, $e_0=\frac{3-\sqrt{5}}{2}$, $p_i-q_iw=e_0^i$ for $i\geqslant 0$, $[E:E(\mathfrak{p})]=10$, and $[E_+:E(\mathfrak{p})^2]=5$. For any $\alpha\in\mathbf{Z}$, $(\alpha,\ 11)=1$,

$$\nu(\alpha)=\sum_{i=1}^{5}\frac{\exp\left(2\pi\sqrt{-1}\frac{\alpha}{11}\operatorname{tr}\frac{(e_0')^i\mu'}{\sqrt{5}}\right)}{1-\exp\left(2\pi\sqrt{-1}\frac{\alpha}{11}\operatorname{tr}\frac{(e_0')^i\mu'}{\sqrt{5}}\right)}$$

$$\times\frac{\exp\left(2\pi\sqrt{-1}\frac{-\alpha}{11}\operatorname{tr}\frac{(e_0')^{i-1}\mu'}{\sqrt{5}}\right)}{1-\exp\left(2\pi\sqrt{-1}\frac{-\alpha}{11}\operatorname{tr}\frac{(e_0')^{i-1}\mu'}{\sqrt{5}}\right)}$$

$$=\frac{1}{11^2}\sum_{i=1}^{5}\left(\sum_{l=1}^{10} l\exp\left(-2\pi\sqrt{-1}\frac{\alpha l}{11}\operatorname{tr}\frac{(e_0')^i\mu'}{\sqrt{5}}\right)\right)$$

$$\times\left(\sum_{m=1}^{10} m\exp\left(2\pi\sqrt{-1}\frac{\alpha m}{11}\operatorname{tr}\frac{(e_0')^{i-1}\mu'}{\sqrt{5}}\right)\right).$$

Hence

$$\sum_{\alpha=1}^{10}\left(\frac{\alpha}{11}\right)\nu(\alpha)=\frac{1}{11^2}\sum_{i=1}^{5}\sum_{l,m=1}^{10} lm\sum_{\alpha=1}^{10}\left(\frac{\alpha}{11}\right)$$

$$\exp\left(-2\pi\sqrt{-1}\frac{\alpha}{11}\left(l\operatorname{tr}\frac{(e_0')^i\mu'}{\sqrt{5}}-m\operatorname{tr}\frac{(e_0')^{i-1}\mu'}{\sqrt{5}}\right)\right)$$

$$=-\frac{1}{11^2}\sum_{i=1}^{5}\sum_{\alpha=1}^{10}\left(\frac{\alpha}{11}\right)\exp\left(2\pi\sqrt{-1}\frac{\alpha}{11}\right)$$

$$\times \sum_{l,m=1}^{10} lm \left(\frac{l \operatorname{tr} \frac{(e_0')^i \mu'}{\sqrt{5}} - m \operatorname{tr} \frac{(e_0')^{i-1} \mu'}{\sqrt{5}}}{11} \right)$$

$$= -\frac{1}{11^2} \sum_{i=1}^{5} \sum_{\alpha=1}^{10} \left(\frac{\alpha}{11} \right) \exp\left(2\pi\sqrt{-1} \frac{\alpha}{11} \right) \times 0 .$$

Thus,

$$y_1 - y_2 = 0 .$$

Actually, this agrees with the fact that there does not exist a totally imaginary quadratic extension $\mathbf{Q}(\sqrt{5})$ with the relative discriminant $\left(\frac{7-\sqrt{5}}{2} \right)$.

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