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Radiation conditions and spectral theory for 2-body Schrödinger operators with "oscillating" long-range potentials III

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Introduction

In the recent works ([1], [2], [3]) we have developed a spectral theory for the Schrödinger operators $-\Delta+V(x)$, in an exterior domain Ω of $\boldsymbol{R}^n$, with some real "oscillating" long-range potentials $V(x)$. Roughly speaking, $V(x)$ is called "*oscillating*" *long-range* if it behaves as $r=|x|\to\infty$ like

$$V(x)=0(1),\ \partial_r V(x)=0(r^{-1})\ (\partial_r=\partial/\partial r)\ \text{ and } \tag{0.1}$$

$$\partial_r^2 V(x)+aV(x)=0(r^{-1-\delta})\ \text{ for some }\ a\geqq 0\ \text{ and }\ \delta>0.$$

For example, the potential

$$V(x)=\frac{c\sin br}{r}\ (b,\ c \text{ are non-zero real})$$

satisfies the above conditions with $a=b^2$ and $\delta=1$.

In this paper we shall modify our previous results to the case of potentials which consist of the sum of several "oscillating" long-range potentials. Note that the last condition of (0.1) is not satisfied by the potential

$$V(x)=\frac{c_1\sin b_1 r}{r}+\frac{c_2\sin b_2 r}{r}$$

(b_1, b_2, c_1, c_2 are non-zero real) unless $b_1=b_2$. So the results of [2] and [3] are not directly applied to this type of potentials, and it is necessary to make some modification.

For this purpose we return to the semi-abstract theory developed in the first half of [2], where we gave a sufficient condition under which the principle of limiting absorption are justified for the exterior boundary-value problem

$$\begin{cases} \{-\Delta+V(x)-\zeta\}u=f(x) \quad \text{in} \quad \Omega \\ Bu=\begin{Bmatrix} u \quad \text{or} \\ \nu\cdot\nabla u+d(x)u \end{Bmatrix}=0 \quad \text{on} \quad \partial\Omega\,. \end{cases} \tag{0.2}$$

Here ζ is a complex number, $\nu=(\nu_1, \cdots, \nu_n)$ is the outer unit normal to the boundary $\partial\Omega$, ∇ is the gradient in $\boldsymbol{R}^n$ and $d(x)$ is a real-valued smooth function on $\partial\Omega$.

The condition is summarized as follows:

Assumption 1 ([2]). There exist real constants $\delta>0$, Λ_δ and a real function $\gamma(\lambda)$ of $\lambda>\Lambda_\delta$ such that

(0.3) $$0<\gamma(\lambda)<\min\{4\delta, 2\}$$

and the following growth property holds: Let $u\in H^2_{\mathrm{loc}}(\bar{\Omega})$ satisfy

(0.4) $$\{-\Delta+V(x)-\lambda\}u=0 \quad \text{in} \quad \Omega$$

with $\lambda>\Lambda_\delta$. If we have the inequality

(0.5) $$\int_{B(R_0)}(1+r)^{-1+\beta}|u|^2dx<\infty$$

for some $\beta>\gamma(\lambda)/2$ and $R_0>0$, where $B(R_0)=\{x\in\Omega\,;\ |x|>R_0\}$, then u must identically vanish in Ω.

Assumption 2 ([2]). Let

(0.6) $$\Pi^\pm_\delta=\{\zeta=\lambda\pm i\tau\in\boldsymbol{C}\,;\ \lambda>\Lambda_\delta \quad \text{and} \quad \tau\geqq 0\}.$$

and let $K^\pm$ be a compact set in $\Pi^\pm_\delta$. Then there exists an $R_1=R_1(K^\pm)>R_0$ and a complex-valued function $k(x,\zeta)=k(x,\lambda\pm i\tau)$ which is continuous in $(x,\zeta)\in B(R_1)\times K^\pm$ and satisfies the following conditions: for any $(x,\zeta)\in B(R_1)\times K^\pm$

(A2-1) $$|V(x)-\zeta+\partial_r k(x,\zeta)+\frac{n-1}{r}k(x,\zeta)-k(x,\zeta)^2|\leqq C_1 r^{-1-\delta},$$

(A2-2) $$|k(x,\zeta)|\leqq C_2,$$

(A2-3) $$\mp\mathrm{Im}\, k(x,\lambda\pm i\tau)\geqq C_3,$$

(A2-4) $$\mathrm{Re}\, k(x,\zeta)-\frac{n-1-\beta}{2r}\geqq C_4 r^{-1}.$$

(A2-5) $$|(\nabla-\tilde{x}\partial_r)k(x,\zeta)|\leqq C_5 r^{-1-\delta} \ (\tilde{x}=x/|x|).$$

Here $C_j=C_j(K^\pm)>0\,(j=1\sim 5)$ and $\beta=\beta(K^\pm)>0$ is chosen as follows:

(A2-6) $$\gamma(\lambda)/2<\beta<2\delta \quad \text{for any} \quad \zeta=\lambda\pm i\tau\in K^\pm \quad \text{and} \quad \beta\leqq 1.$$

We shall show that the above assumptions can be verified for a class of potentials consisting of the sum of several "oscillating" long-range potentials.

The main results of this paper will be summarized in § 1 in three theorems. Theorem 1 which asserts the growth property of solutions of (0.4) is a consequence of [1]. Theorem 2 summarizes results concerning the principle of limiting absorption. We shall prove it in § 2. Theorem 3 summarizes results concerning spectral representations for the selfadjoint realization of $-\Delta+V(x)$ in the Hilbert space $L^2(\Omega)$. An outline of the proof will be given in § 3 (we can follow the same line of proof of [3]). Finally, in §4 we shall give several examples.

§1. Conditions and results

Let Ω be an infinite domain in $\boldsymbol{R}^n$ with smooth compact boundary $\partial\Omega$ lying inside some sphere $S(R_0)=\{x\,;\,|x|=R_0\}$. We consider in Ω the Schrödinger operator $-\Delta+V(x)$, where Δ is the Laplacian and $V(x)$ is a potential function of the form

$$V(x)=V_1(x)+V_s(x)=\sum_{j=1}^{m} V_{1j}(x)+V_s(x)\,. \tag{1.1}$$

$V_s(x)$ is a short-range potential and the $V_{1j}(x)$ are "oscillating" long-range potentials. More precisely, we assume:

(V1) $V_1(x)$ is a real-valued function belonging to a Stummel class $Q_\mu(\mu>0)$ and $V_s(x)$ is a real-valued bounded measurable function in Ω. Moreover, the unique continuation property holds for both $-\Delta+V(x)$ and $-\Delta+V_1(x)$.

(V2) For some $0<\delta_l\leqq 1(l=0,\,1,\,2)$ and $0<\varepsilon_j\leqq 1(j=1,\,\cdots,\,m)$,

(i) $$V_{1j}(x)=0(r^{-\varepsilon_j})\,,$$

(ii) $$\partial_r V_{1j}(x)=0(r^{-1})\,,$$

(iii) $$\partial_r^2 V_{1j}(x)+a_j(r)V_{1j}(x)=0(r^{-1-\delta_1})$$

(iv) $$(\nabla-\tilde{x}\partial_r)V_{1j}(x)=0(r^{-1-\delta_2})\,,$$

(v) $$(\nabla-\tilde{x}\partial_r)\partial_r V_{1j}(x)=0(r^{-1-\delta_1})\,,$$

(vi) $$(\nabla-\tilde{x}\partial_r)\cdot(\nabla-\tilde{x}\partial_r)V_{1j}(x)=0(r^{-1-2\delta_2})\,,$$

(vii) $$V_s(x)=0(r^{-1-\delta_0})$$

as $r=|x|\to\infty$, where the $a_j(r)(j=1,\,\cdots,\,m)$ are non-negative functions of $r>R_0$ satisfying

$$a_1(r)\geqq a_2(r)\geqq\cdots\geqq a_m(r)\geqq 0\,, \tag{1.2}$$

$$a_j(r)=0(r^{-\tau_j}),\quad a_j'(r)=\frac{d}{dr}a_j(r)=0(r^{-\mu_j}) \tag{1.3}$$

and
$$a_j''(r)=\frac{d^2}{dr^2}a_j(r)=0(r^{-1-\delta_1+\varepsilon_j})$$

with $\tau_j=\max\{0,\,1+\delta_1-\varepsilon_j-\min\{\varepsilon_j\}\}$ and μ_j such that $\mu_j\geqq\delta_1$ and $\mu_j>1-\varepsilon_j$.

Remark 1.1. (V2-i) is stronger than the corresponding condition required in [2] and [3] (cf., (0.1)). (V2-vi) is used only to show (e) of Theorem 3 stated below (cf., [3]).

Lemma 1.1. *If* δ_1 *and* $\{\varepsilon_j\}$ *satisfy* $\max\{\varepsilon_j\}-\min\{\varepsilon_j\}\leqq 1-\delta_1,\ 0<\delta_1,\ \varepsilon_j\leqq 1,$ *and*

$$a_j(r)=0(r^{-2+2\varepsilon_j}),\quad a_j'(r)=0(r^{-1})\quad \textit{and}\quad a_j''(r)=0(r^{-2})\,,$$

$a_j(r)$ satisfies the condition (1.3).

Proof. Obvious from $2-2\varepsilon_j \geqq 1+\delta_1-\varepsilon_j-\min\{\varepsilon_j\}$, $1\geqq\delta_1$, $1>1-\varepsilon_j$ and $2>1+\delta_1-\varepsilon_j$. q. e. d.

Lemma 1.2. (cf., [2]; Remark 8.5). *If δ_1 and $\{\varepsilon_j\}$ are as above, and*

$$a_j(r)=0(r^{-2+2\varepsilon_j}) \quad \textit{and} \quad a_j'(r),\ a_j''(r)=0(r^{-1-\delta_1+\varepsilon_j}),$$

$a_j(r)$ satisfies the condition (1.3).

Proof. Obvious from $2-2\varepsilon_j\geqq 1+\delta_1-\varepsilon_j-\min\{\varepsilon_j\}$, $1+\delta_1-\varepsilon_j\geqq\delta_1$ and $1+\delta_1-\varepsilon_j>1-\varepsilon_j$. q. e. d.

In the following we put $\delta=\min\{\delta_0, \delta_1, \delta_2\}$ and

$$a_j^*=\limsup_{r\to\infty} a_j(r)\ (\geqq 0).$$

Then we have by (1.2)

$$a_1^*\geqq a_2^*\geqq \cdots \geqq a_m^*\geqq 0.$$

For $V_{1j}(x)$ satisfying (V2-i) and (V2-ii) we put

$$E(\gamma)=\frac{1}{\gamma}\limsup_{r\to\infty}\{r\partial_r V_1(x)+\gamma V_1(x)\}=\frac{1}{\gamma}\limsup_{r\to\infty}\{r\partial_r V_1(x)\}\ (\gamma>0). \tag{1.4}$$

Then obviously (cf., [2]; Lemma 8.2) $E(\gamma)\geqq \lim_{r\to\infty} V_1(x)=0$ and we have for $0<\gamma\leqq 2$,

$$0\leqq E(2)\leqq E(\gamma)\leqq\frac{1}{\gamma}\max_{1\leq l\leq m}\ \limsup_{r\to\infty}\{\sum_{j=1}^{l} r\partial_r V_{1j}(x)\} \tag{1.5}$$

$$\leqq\frac{1}{\gamma}\sum_{j=1}^{m}\limsup_{r\to\infty}\{r\partial_r V_{1j}(x)\}<\infty.$$

The following theorem is already proved in [1] (Theorems 1, 2 and Remark 2).

Theorem 1. *Suppose that $V(x)$ satisfies* (V1), (V2-i), (V2-ii) *and* (V2-vii). *Then we have*:

(a) *Let $\lambda>E(2)$ and $u\in H^2_{\mathrm{loc}}(\bar{\Omega})$ be a not identically vanishing solution of $\{-\Delta+V(x)-\lambda\}u=0$ in Ω. Then for any $\tilde{\gamma}$ such that*

$$0<\tilde{\gamma}<2 \quad \textit{and} \quad E(2)\leqq E(\tilde{\gamma})<\lambda,$$

we have

$$\lim_{R\to\infty} R^{-1+\tilde{\gamma}/2}\int_{R_0<|x|<R}|u(x)|^2dx=\infty.$$

(b) *Any selfadjoint realization of $-\Delta+V(x)$ in $L^2(\Omega)$ has no eigenvalue in* $(E(2), \infty)$.

We put for $\sigma>0$

$$\Lambda_\sigma=\frac{1}{\min\{4\sigma, 2\}}\max_{1\leq l\leq m}\ \limsup_{r\to\infty}\left\{\sum_{j=1}^{l} r\partial_r V_{1j}(x)\right\}+\frac{1}{4}a_1^* \tag{1.6}$$

and

$$\Pi_\delta^\pm=\left\{\zeta=\lambda\pm i\tau\ ;\ \lambda>\Lambda_\delta \text{ and } \frac{1}{2}\left(\lambda-\frac{1}{4}a_1^*\right)\geqq\tau\geqq 0\right\}.$$

For $\zeta\in\Pi_\delta^\pm$ let

$$\eta_j(r)=\frac{4\zeta}{4\zeta-a_j(r)} \quad \text{and} \tag{1.7}$$

$$\eta(r)\cdot V_1(x)=\sum_{j=1}^m \eta_j(r)V_{1j}(x). \tag{1.8}$$

Lemma 1.3. *Let $K^\pm$ be any compact set of $\Pi_\delta^\pm$. Then there exist $R_2=R_2(K^\pm)>R_0$ and $C=C(K^\pm)>1$ such that*

$$0\leqq\pm\mathrm{Im}\{\zeta-\eta(r)\cdot V_1(x)\}\leqq C,$$

$$C^{-1}\leqq\mathrm{Re}\{\zeta-\eta(r)\cdot V_1(x)\}\leqq C$$

for any $(x, \zeta)\in B(R_2)\times K^\pm$.

Proof. It follows from (1.7) and (1.8) that

$$\mathrm{Im}\{\zeta-\eta(r)\cdot V_1(x)\}=\mathrm{Im}\,\zeta\left\{1+\sum_{j=1}^m\frac{4a_j(r)V_{1j}(x)}{|4\zeta-a_j(r)|^2}\right\},$$

$$\mathrm{Re}\{\zeta-\eta(r)\cdot V_1(x)\}=\mathrm{Re}\,\zeta-V_1(x)-\sum_{j=1}^m\mathrm{Re}\frac{a_j(r)V_{1j}(x)}{4\zeta-a_j(r)}.$$

Here by (V2-i), (1.5) and (1.6),

$$\lim_{r\to\infty}V_{1j}(x)=0 \quad\text{and}\quad \mathrm{Re}\,\zeta>\frac{1}{4}a_1^*\geqq\frac{1}{4}\limsup_{r\to\infty}a_j(r) \quad \text{for any } j. \tag{1.9}$$

Thus, noting the boundedness of $a_j(r)$, we have the assertion of the lemma.

q. e. d.

Let $\gamma(\lambda)$, $\lambda>\Lambda_\delta$, be defined by

$$\gamma(\lambda)=\begin{cases}\delta & \text{if } \Lambda_\delta=0\\ \dfrac{2\Lambda_\delta}{\Lambda_\delta+\lambda}\min\{4\delta, 2\} & \text{if } \Lambda_\delta>0.\end{cases} \tag{1.10}$$

Lemma 1.4. *We have for any $\lambda>\Lambda_\delta$*

$$0<\gamma(\lambda)<\min\{4\delta, 2\},$$

$$E(\gamma(\lambda))\leqq\frac{1}{\gamma(\lambda)}\max_{1\leq l\leq m}\limsup_{r\to\infty}\left\{\sum_{j=1}^l r\partial_r V_{1j}(x)\right\}+\frac{1}{4}a_1^*$$

$$\leqq\frac{1}{2}(\Lambda_\delta+\lambda)<\lambda.$$

Proof. The first assertion is obvious from the definition of $\gamma(\lambda)$. If $\Lambda_\delta=0$, the second assertion is also obvious since $a_1^*=0$ and $\limsup_{r\to\infty}\left\{\sum_{j=1}^l r\partial_r V_{1j}(x)\right\}=0$ $(l=1, \cdots, m)$. On the other hand, if $\Lambda_\delta>0$, we have noting (1.6)

$$\frac{1}{\gamma(\lambda)}\max_{1\leq l\leq m}\limsup_{r\to\infty}\Big\{\sum_{j=1}^{l} r\partial_r V_{1j}\Big\}+\frac{1}{4}a_1^*$$

$$=\frac{\Lambda_\delta+\lambda}{2\Lambda_\delta}\Big(\Lambda_\delta-\frac{1}{4}a_1^*\Big)+\frac{1}{4}a_1^*\leqq\frac{1}{2}(\Lambda_\delta+\lambda).$$

The lemma is proved. q. e. d.

For $\mu\in\boldsymbol{R}$ and $G\subset\Omega$, let $L_\mu^2(G)$ denote the space of all functions $f(x)$ such that

$$\|f\|_{\mu,G}^2=\int_G(1+r)^{2\mu}|f(x)|^2dx<\infty.$$

If $\mu=0$ or $G=\Omega$, the subscript μ or G will be omitted. Let $k(x,\zeta)$, $(x,\zeta)\in B(R_2)\times K^{\pm}$, be defined by

$$k(x,\zeta)=-i\sqrt{\zeta-\eta(r)\cdot V_1(x)}+\frac{n-1}{2r}+\frac{-\partial_r\{\eta(r)\cdot V_1(x)\}}{4\{\zeta-\eta(r)\cdot V_1(x)\}},\tag{1.11}$$

where $R_2=R_2(\zeta)$ and we take the branch $\mathrm{Im}\sqrt{}\geqq0$.

Definition. For solutions $u\in H_{\mathrm{loc}}^2(\bar{\Omega})$ of (0.2) with $\zeta\in\Pi_\delta^{\pm}$, the outgoing (+) [or incoming (−)] radiation condition at infinity is defined by

$$u\in L_{(-1-\alpha)/2}^2(\Omega)\quad\text{and}\quad\partial_r u+k(x,\zeta)u\in L_{(-1+\beta)/2}^2(B(R_2)),\tag{1.12$_\pm$}$$

where $\alpha=\alpha(\zeta)$, $\beta=\beta(\zeta)$ is a pair of positive constants such that

$$0<\alpha\leqq\beta\leqq1,\ \frac{1}{2}\gamma(\mathrm{Re}\,\zeta)<\beta<2\delta\quad\text{and}\quad\alpha\leqq2\delta-\beta.\tag{1.13}$$

A solution u of (0.2) which also satisfies the radiation condition $(1.12)_+$ [or $(1.12)_-$] is called an outgoing [incoming] solution.

We are now ready to state two theorems concerning the principle of limiting absorption and spectral representations for the Schrödinger operator $-\Delta+V(x)$.

Theorem 2. *Suppose that* $V(x)$ *satisfies* (V1), (V2-i)∼(V2-v) *and* (V2-vii). *Then we have*:

(a) *Let* $K^{\pm}$ *be a compact set of* $\Pi_\delta^{\pm}$, *let* $\gamma(K^{\pm})=\max_{\zeta\in K^{\pm}}\gamma(\lambda)$ $(\lambda=\mathrm{Re}\,\zeta)$, *and let* $\alpha=\alpha(K^{\pm})$, $\beta=\beta(K^{\pm})$ *be a pair satisfying* (1.13) *with* $\gamma(\lambda)$ *replaced by* $\gamma(K^{\pm})$. *Then for any* $\zeta=\lambda\pm i\tau\in K^{\pm}$ *and* $f\in L_{(1+\beta)/2}^2(\Omega)$, (0.2) *has a unique outgoing [incoming] solution* $u=u(x,\lambda\pm i\tau)=R_{\lambda\pm i\tau}f$, *which also satisfies the inequalities*

$$\|u\|_{(-1-\alpha)/2}\leqq C\|f\|_{(1+\beta)/2},$$

$$\|\Gamma u+\tilde{x}k(x,\lambda\pm i\tau)u\|_{(-1+\beta)/2,B(R_3)}\leqq C\|f\|_{(1+\beta)/2},$$

where $C=C(K^{\pm})>0$ *and* $R_3=R_3(K^{\pm})\geqq R_2(K^{\pm})$ *are independent of* f.

(b) *Let* $R_\zeta^*:L_{(1+\alpha)/2}^2(\Omega)\to L_{(-1-\beta)/2}^2(\Omega)$ *be the adjoint of* R_ζ. *Then*

$$R_{\lambda\pm i\tau}^*f=R_{\lambda\mp i\tau}f\quad\textit{for}\quad f\in L_{(1+\beta)/2}^2(\Omega).$$

(c) $u=R_\zeta f$ *is continuous in* $L_{(-1-\alpha)/2}^2(\Omega)$ *with respect to* $(\zeta,f)\in K^{\pm}\times L_{(1+\beta)/2}^2(\Omega)$.

(d) *Let L be the selfadjoint operator in $L^2(\Omega)$ defined by*

$$\begin{cases} \mathcal{D}(L)=\{u\in H^2(\Omega);\ Bu|_{\partial\Omega}=0\} \\ Lu=-\Delta u+V(x)u \quad \text{for} \quad u\in\mathcal{D}(L), \end{cases}$$

and let $\{\mathcal{E}(\lambda);\ \lambda\in\boldsymbol{R}\}$ be its spectral measure. Then for any Borel set $e\Subset(\Lambda_\delta, \infty)$, $f\in L^2_{(1+\beta)/2}(\Omega)$ and $g\in L^2_{(1+\alpha)/2}(\Omega)$ (α, β is chosen as above with $K^\pm=\bar{e}$) we have

$$(\mathcal{E}(e)f,\ g)=\frac{1}{2\pi i}\int_e(\{R_{\lambda+i0}-R_{\lambda-i0}\}f,\ g)d\lambda,$$

where (,) denotes the inner product in $L^2(\Omega)$, or more generally, the duality between $L^2_{(-1-\alpha)/2}(\Omega)$ and $L^2_{(1+\alpha)/2}(\Omega)$. Namely, the part of L in $\mathcal{E}((\Lambda_\delta, \infty))L^2(\Omega)$ is absolutely continuous with respect to the Lebesgue measure.

Theorem 3. *Suppose that $V(x)$ satisfies* (V1) *and* (V2) *with $\delta_l>1/2(l=1, 2)$. Then we have:*

(a) *For any $\varepsilon>0$ and $\lambda\geqq\Lambda_\delta+\varepsilon$ there exist bounded linear operators $\mathcal{F}_\pm(\lambda)$: $L^2_{(1+2\delta)/2}(\Omega)\to L^2(S^{n-1})(S^{n-1}=\{x;\ |x|=1\})$ such that each $\mathcal{F}_\pm(\lambda)f\in L^2(S^{n-1})$ depends continuously on $(\lambda, f)\in(\Lambda_\delta+\varepsilon, \infty)\times L^2_{(1+2\delta)/2}(\Omega)$, and the following relations hold:*

$$(\mathcal{F}_\pm(\lambda)f,\ \mathcal{F}_\pm(\lambda)g)_{L^2(S^{n-1})}=\frac{1}{2\pi i}(R_{\lambda+i0}f-R_{\lambda-i0}f,\ g),$$

$$(\mathcal{E}(e)f,\ g)=\int_e(\mathcal{F}_\pm(\lambda)f,\ \mathcal{F}_\pm(\lambda)g)_{L^2(S^{n-1})}d\lambda,\ e\Subset(\Lambda_\delta+\varepsilon,\ \infty).$$

(b) *The operators $\mathcal{F}_\pm$: $L^2_{(1+2\delta)/2}(\Omega)\to L^2((\Lambda_\delta+\varepsilon, \infty)\times S^{n-1})$ defined by $[\mathcal{F}_\pm f](\lambda, \tilde{x})=[\mathcal{F}_\pm(\lambda)f](\tilde{x})$ can be uniquely extended by continuity to partial isometric operators from $L^2(\Omega)$ into $L^2((\Lambda_\delta+\varepsilon, \infty)\times S^{n-1})$ with initial set $\mathcal{E}((\Lambda_\delta+\varepsilon, \infty))L^2(\Omega)$.*

(c) *For any bounded Borel function $b(\lambda)$ on $(\Lambda_\delta+\varepsilon, \infty)$, the following relation holds:*

$$\mathcal{F}_\pm b(L)=b(\lambda)\mathcal{F}_\pm.$$

(d) *Let $\mathcal{F}^*_\pm$: $L^2((\Lambda_\delta+\varepsilon, \infty)\times S^{n-1})\to L^2(\Omega)$ be the adjoint operators of $\mathcal{F}_\pm$. Then each $\mathcal{F}^*_\pm$ admits the representation*

$$\mathcal{F}^*_\pm\hat{f}=\text{strong}\lim_{N\to\infty}\int_{\Lambda_\delta+\varepsilon}^N\mathcal{F}_\pm(\lambda)^*\hat{f}(\lambda)d\lambda \quad \text{in} \quad L^2(\Omega)$$

for any $\hat{f}\in L^2((\Lambda_\delta+\varepsilon, \infty)\times S^{n-1})$, where $\mathcal{F}_\pm(\lambda)^*$: $L^2(S^{n-1})\to L^2_{(-1-2\delta)/2}(\Omega)$ is the adjoint of $\mathcal{F}_\pm(\lambda)$.

(e) *Let $\tilde{\delta}=\min\{\delta, 2\delta_2-1\}$. Then each $\mathcal{F}_\pm$ maps $\mathcal{E}((\Lambda_{\tilde{\delta}}+\varepsilon, \infty))L^2(\Omega)$ onto $L^2((\Lambda_{\tilde{\delta}}+\varepsilon, \infty)\times S^{n-1})$, that is, $\mathcal{F}_\pm$ restricted on $\mathcal{E}((\Lambda_{\tilde{\delta}}+\varepsilon, \infty))L^2(\Omega)$ is a unitary operator.*

Remark 1.2. In [3] we neglect the fact that the construction of $\mathcal{F}_\pm(\lambda)$ depends in general on ε. So the above is a correction of [3]. Note that for two ε, $\varepsilon'>0$ and $\lambda>\Lambda_\delta+\max\{\varepsilon, \varepsilon'\}$, the corresponding operators $\mathcal{F}_\pm(\lambda)$ and $\mathcal{F}'_\pm(\lambda)$ are unitary equivalent to each other in $L^2(S^{n-1})$.

§ 2. Proof of Theorem 2

As we see in [2] (Theorems 1~5), all the assertions of Theorem 2 hold true under Assumptions 1 ([2]) and 2 ([2]) stated in the introduction of this article. So to complete the proof, we have only to check that these assumptions are satisfied by Λ_δ, $\gamma(\lambda)$ and $k(x, \zeta)$ given in the previous section.

Assumption 1 ([2]) directly follows from Theorem 1 and the definition of Λ_δ and $\gamma(\lambda)$. In fact, let $\lambda>\Lambda_\delta$ and u satisfy (0.4) and (0.5) for some $\beta>\gamma(\lambda)/2$ (without loss of generality we can assume $\beta\leqq 1$). Then we have

$$R^{-1+\gamma(\lambda)/2}\int_{R_0<|x|<R}|u(x)|^2dx\leqq R^{-\beta+\gamma(\lambda)/2}\int_{R_0<|x|<R}r^{-1+\beta}|u(x)|^2dx$$
$$\rightarrow 0 \quad \text{as} \quad R\rightarrow\infty.$$

Since $\tilde{\gamma}=\gamma(\lambda)$ satisfies the condition of Theorem 1 (a) by Lemma 1.4, this and Theorem 1 (a) lead $u=0$.

Next let us verify Assumption 2 ([2]) for $k(x, \zeta)$ restricted in $(x, \zeta)\in B(R_2)\times K^{\pm}$, where $K^{\pm}$ is any compact set in $\Pi_\delta^{\pm}$, and $R_2=R_2(K^{\pm})$ is what is given in Lemma 1.3. We choose $\gamma(K^{\pm})$ and $\beta(K^{\pm})$ as in (a) of Theorem 2. Then (A2-6) is obviously satisfied by this $\beta=\beta(K^{\pm})$. Further, the continuity of $k(x, \zeta)$ in $(x, \zeta)\in B(R_2)\times K^{\pm}$ is easily known by Lemma 1.3. So it remains only to verify (A2-1)~(A2-5).

For this purpose we prepare a lemma.

Lemma 2.1. *There exist some constant $C>0$ such that for any $(x, \zeta)\in B(R_2)\times K^{\pm}$ we have*

$$|\eta_j(r)V_{1j}(x)\{1-\eta_k(r)\}V_{1k}(x)|\leqq Cr^{-1-\delta}\ (j, k=1, \cdots, m),$$
$$|\partial_r\{\eta(r)\cdot V_1(x)\}|\leqq Cr^{-1},$$
$$|\partial_r^2\{\eta_j(r)V_{1j}(x)\}+\eta_j(r)a_j(r)V_{1j}(x)|\leqq Cr^{-1-\delta},$$
$$|(\nabla-\tilde{x}\partial_r)\{\eta(r)\cdot V_1(x)\}|\leqq Cr^{-1-\delta},$$
$$|(\nabla-\tilde{x}\partial_r)\partial_r\{\eta(r)\cdot V_1(x)\}|\leqq Cr^{-1-\delta}.$$

Proof. The above inequalities respectively follow from (V2-i)~(V2-v) if we note that

$$\eta_j(r)=0(1),\ 1-\eta_j(r)=0(r^{-\tau_j}),\ \eta_j'(r)=0(r^{-\mu_j}) \quad \text{and}$$
$$\eta_j''(r)=0(r^{-1-\delta_1+\varepsilon_j})+0(r^{-2\mu_j})=0(r^{-1-\delta+\varepsilon_j}). \qquad \text{q. e. d.}$$

Now, (A2-2) and (A2-3) easily follow from Lemma 1.3 and Lemma 2.1 since we have

$$|k(x, \zeta)|\leqq\left|\sqrt{\zeta-\eta(r)\cdot V_1(x)}\right|+\frac{n-1}{2r}+\left|\frac{-\partial_r\{\eta(r)\cdot V_1(x)\}}{4\{\zeta-\eta(r)\cdot V_1(x)\}}\right|,$$

$$\mp\operatorname{Im}k(x, \zeta)=\pm\operatorname{Re}\sqrt{\zeta-\eta(r)\cdot V_1(x)}\mp\operatorname{Im}\left[\frac{-\partial_r\{\eta(r)\cdot V_1(x)\}}{4\{\zeta-\eta(r)\cdot V_1(x)\}}\right].$$

Note that

$$(\nabla-\tilde{x}\partial_r)k(x,\zeta)=\frac{i(\nabla-\tilde{x}\partial_r)(\eta\cdot V_1)}{2\sqrt{\zeta-\eta\cdot V_1}}+\frac{-(\nabla-\tilde{x}\partial_r)\partial_r(\eta\cdot V_1)}{4(\zeta-\eta\cdot V_1)}$$
$$-\frac{\partial_r(\eta\cdot V_1)(\nabla-\tilde{x}\partial_r)(\eta\cdot V_1)}{4(\zeta-\eta\cdot V_1)^2}.$$

Then (A2-5) also follows from Lemma 2.1.

Next we prove (A2-1). It follows from (1.11) that (cf., Appendix of [2])

$$V(x)-\zeta+\partial_r k+\frac{n-1}{r}k-k^2$$
$$=V_1-\eta\cdot V_1+\frac{-\partial_r^2(\eta\cdot V_1)}{4(\zeta-\eta\cdot V_1)}+\frac{(n-1)(n-3)}{4r^2}-\frac{5\{\partial_r(\eta\cdot V_1)\}^2}{16(\zeta-\eta\cdot V_1)^2}+V_s.$$

Since we have

$$V_1-\eta\cdot V_1+\frac{-\partial_r^2(\eta\cdot V_1)}{4(\zeta-\eta\cdot V_1)}=\frac{1}{4(\zeta-\eta\cdot V_1)}\Big[\sum_{j=1}^m\{4\zeta(1-\eta_j)V_{1j}-\partial_r^2(\eta_jV_{1j})\}$$
$$-\sum_{j,k=1}^m 4(1-\eta_j)\eta_k V_{1j}V_{1k}\Big]$$

and

$$4\zeta(1-\eta_j)V_{1j}-\partial_r^2(\eta_jV_{1j})=-\{\partial_r^2(\eta_jV_{1j})+\eta_j a_j(r)V_{1j}\},$$

(A2-1) follows from Lemma 2.1 and (V2-vii).

Lastly, we prove (A2-4).

$$\operatorname{Re}k(x,\zeta)-\frac{n-1-\beta}{2r}=\frac{1}{4}\Big\{\operatorname{Re}\Big[\frac{-\partial_r(\eta\cdot V_1)}{\zeta-\eta\cdot V_1}\Big]+\frac{\gamma}{r}\Big\}+\operatorname{Im}\sqrt{\zeta-\eta\cdot V_1}+\frac{2\beta-\gamma}{4r}.$$

Since $2\beta>\gamma$ by definition and $\operatorname{Im}\sqrt{\zeta-\eta\cdot V_1}\geqq 0$, we have only to show that there exists an $R_4=R_4(K^\pm)\geqq R_2$ such that

$$\operatorname{Re}\Big[\frac{-\partial_r(\eta\cdot V_1)}{\zeta-\eta\cdot V_1}\Big]+\frac{\gamma}{r}\geqq 0 \quad \text{for any} \quad (x,\zeta)\in B(R_4)\times K^\pm. \tag{2.1}$$

We have

$$\frac{r|\zeta-\eta\cdot V_1|^2}{\gamma|\zeta|^2}\Big\{\operatorname{Re}\Big[\frac{-\partial_r(\eta\cdot V_1)}{\zeta-\eta\cdot V_1}\Big]+\frac{\gamma}{r}\Big\}$$
$$=\frac{|\zeta-\eta\cdot V_1|^2}{|\zeta|^2}\operatorname{Re}\Big[\frac{\zeta-(r/\gamma)\partial_r(\eta\cdot V_1)-\eta\cdot V_1}{\zeta-\eta\cdot V_1}\Big]$$
$$=1-\frac{1}{\gamma}\operatorname{Re}\Big\{\sum_{j=1}^m\frac{r\partial_r V_{1j}(x)}{\zeta-(1/4)a_j(r)}\Big\}+|\zeta|^{-2}I;$$

$$I=\operatorname{Re}\Big\{-\frac{\bar{\zeta}}{\gamma}\sum_{j=1}^m r\eta_j' V_{1j}+\frac{r}{\gamma}\partial_r(\eta\cdot V_1)\overline{\eta\cdot V_1}-2\operatorname{Re}[\bar{\zeta}\eta\cdot V_1]+|\eta\cdot V_1|^2\Big\}.$$

Noting $K^\pm\Subset\Pi_\delta^\pm$, we can choose a constant $\varepsilon>0$ to satisfy

(2.2) $$\varepsilon(3+\varepsilon)(1+\varepsilon)^{-1} \leqq \min_{\zeta \in K^{\pm}} \frac{\lambda - \Lambda_{\delta}}{2\lambda - (1/2)a_1^*} \quad (\lambda = \mathrm{Re}\,\zeta).$$

We fix such an ε. Since $\zeta = \lambda \pm i\tau \in K^{\pm} \Subset \Pi_{\delta}^{\pm}$ satisfies $(1/2)(\lambda - (1/4)a_1^*) \geqq \tau \geqq 0$, there exists an $R_5 = R_5(K^{\pm}) \geqq R_4$ such that for any $(x, \zeta) \in B(R_5) \times K^{\pm}$,

(2.3) $$0 < \mathrm{Re}\frac{1}{\zeta - (1/4)a_m(r)} \leqq \mathrm{Re}\frac{1}{\zeta - (1/4)a_{m-1}(r)} \leqq \cdots$$

$$\cdots \leqq \mathrm{Re}\frac{1}{\zeta - (1/4)a_1(r)} \leqq \frac{1+\varepsilon}{\lambda - (1/4)a_1^*},$$

(2.4) $$|\zeta|^{-2}|I| < \varepsilon,$$

where in the last inequality (2.4) we have used (1.7), Lemmas 1.3, 2.1 and the fact that $\mu_j > 1 - \varepsilon_j$ in (1.3). Note here

$$\limsup_{r\to\infty} \max_{1\leq l\leq m} \sum_{j=1}^{l} r\partial_r V_{1j}(x) = \max_{1\leq l\leq m} \limsup_{r\to\infty} \sum_{j=1}^{l} r\partial_r V_{1j}(x).$$

Then we see that there exists an $R_6 = R_6(K^{\pm}) \geqq R_5$ such that for any $x \in B(R_6)$

(2.5) $$\max_{1\leq l\leq m} \sum_{j=1}^{l} r\partial_r V_{1j}(x) \leqq \max_{1\leq l\leq m} \limsup_{r\to\infty} \sum_{j=1}^{l} r\partial_r V_{1j}(x) + \gamma\varepsilon \min_{\zeta\in K^{\pm}} (\lambda - (1/4)a_1^*).$$

(2.3), (2.5) and Abel's theorem imply that (cf., also (1.5))

(2.6) $$\frac{1}{\gamma}\mathrm{Re}\Big\{\sum_{j=1}^{m} \frac{r\partial_r V_{1j}(x)}{\zeta - (1/4)a_j(r)}\Big\} \leqq \mathrm{Re}\frac{1}{\zeta - (1/4)a_1(r)} \frac{1}{\gamma} \max_{1\leq l\leq m} \sum_{j=1}^{l} r\partial_r V_{1j}(x)$$

$$\leqq \frac{1+\varepsilon}{(\lambda - (1/4)a_1^*)\gamma} \Big\{\max_{1\leq l\leq m} \limsup_{r\to\infty} \sum_{j=1}^{l} r\partial_r V_{1j}(x) + \gamma\varepsilon \min_{\zeta\in K^{\pm}} \Big(\lambda - \frac{1}{4}a_1^*\Big)\Big\}.$$

Since

$$\frac{1}{\gamma} \max_{1\leq l\leq m} \limsup_{r\to\infty} \sum_{j=1}^{l} r\partial_r V_{1j}(x) \leqq \frac{1}{2}\Big(\lambda + \Lambda_{\delta} - \frac{1}{2}a_1^*\Big)$$

by Lemma 1.4, we have from (2.4), (2.6) and (2.2)

$$1 - \frac{1}{\gamma}\mathrm{Re}\Big\{\sum_{j=1}^{m} \frac{r\partial_r V_{1j}(x)}{\zeta - (1/4)a_j(r)}\Big\} + |\zeta|^{-2} I$$

$$\geqq \Big\{1 - \frac{\lambda + \Lambda_{\delta} - (1/2)a_1^*}{2\lambda - (1/2)a_1^*}\Big\}(1+\varepsilon) - \varepsilon(3+\varepsilon) \geqq 0$$

for any $(x, \zeta) \in B(R_6) \times K^{\pm}$. This proves (2.1), and we have (A2-4).

Remark 2.1. The condition (1.2) on $\{a_j(r)\}$ is used only to show (2.6) (Abel's theorem). Note that Assumptions 1 and 2 can be verified without (1.2) if we replace Λ_{δ}(see (1.6)) by the following

$$\tilde{\Lambda}_{\delta} = \frac{1}{\min\{4\delta, 2\}} \sum_{j=1}^{m} \limsup_{r\to\infty} \{r\partial_r V_{1j}(x)\} + \frac{1}{4}\max\{a_j^*\}.$$

Here, in general, $\Lambda_{\delta} \leqq \tilde{\Lambda}_{\delta}$.

§ 3. Sketch of proof of Theorem 3

On the bases of the principle of limiting absorption (Theorem 2), we can prove Theorem 3 by the same argument as in [3]. Here, in this section, we shall sketch an outline of the proof.

First we restrict ourselves to the case $V_s(x)=0$. Then $\Lambda_\delta=\Lambda_{1/2}$ since we have assumed $\delta_l>1/2\,(l=1, 2)$ in (V2) (we can choose $\delta_0=1$ in this case). For $f\in L_1^2(\Omega)$ and $\lambda>\Lambda_{1/2}+\varepsilon(\varepsilon>0)$, let $R_{1,\lambda\pm i0}f$ be the outgoing [incoming] solution of (0.2) with $V(x)=V_1(x)$ and $\zeta=\lambda\pm i0$. We choose $R_7=R_7(\varepsilon)>R_0$ so large that

$$\lambda-\eta(r)\cdot V_1(x)\geqq C>0 \quad \text{for} \quad (x, \lambda)\in B(R_7)\times(\Lambda_{1/2}+\varepsilon, \infty),$$

and define the function $\rho(x, \lambda\pm i0)=\rho(x, \lambda\pm i0\,;\,\varepsilon)$ as follows:

$$(3.1)\qquad \rho(x, \lambda\pm i0)=\int^r k(s\tilde{x}, \lambda\pm i0)ds$$

$$=\mp i\int_{R_7}^r \sqrt{\lambda-\eta(s)\cdot V_1(s\tilde{x})}ds+\frac{n-1}{2}\log r+\frac{1}{4}\log\{\lambda-\eta(r)\cdot V_1(x)\}.$$

Then as we see in Propositions 1.2 and 2.1 of [3], there exists a sequence $r_p=r_p(\lambda, f)\to\infty$ (as $p\to\infty$) such that

$$\left\{\frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot,\lambda\pm i0)}[R_{1,\lambda\pm i0}f](r_p\cdot)\right\}$$

strongly converges in $L^2(S^{n-1})$. We define the operator $\mathcal{F}_{1,\pm}(\lambda)=\mathcal{F}_{1,\pm}(\lambda, \varepsilon)$: $L_1^2(\Omega)\to L^2(S^{n-1})$ as follws:

$$(3.2)\qquad \mathcal{F}_{1,\pm}(\lambda)f=\text{strong}\lim_{p\to\infty}\frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot,\lambda\pm i0)}[R_{1,\lambda\pm i0}f](r_p\cdot) \quad \text{in} \quad L^2(S^{n-1}).$$

Then $\mathcal{F}_{1,\pm}(\lambda)$ is independent of the choice of $\{r_p\}$, and becomes a bounded operator from $L_1^2(\Omega)$ to $L^2(S^{n-1})$ which depends continuously on $\lambda>\Lambda_{1/2}+\varepsilon$ ([3], Lemma 2.4). Further, we have for $f\in L_1^2(\Omega)$ and $\lambda>\Lambda_{1/2}$ ([3], Proposition 1.3)

$$(3.3)\qquad \|\mathcal{F}_{1,\pm}(\lambda)f\|_{L^2(S^{n-1})}^2=\frac{1}{2\pi i}(R_{1,\lambda+i0}f-R_{1,\lambda-i0}f, f).$$

By use of this $\mathcal{F}_{1,\pm}(\lambda)$, the operator $\mathcal{F}_\pm(\lambda)$: $L_{(1+2\delta)/2}^2(\Omega)\to L^2(S^{n-1})$ ($\delta=\min\{\delta_0, \delta_1, \delta_2\}$) will be defined by

$$(3.4)\qquad \mathcal{F}_\pm(\lambda)=\mathcal{F}_{1,\pm}(\lambda)\{1-V_sR_{\lambda\pm i0}\}, \quad \lambda>\Lambda_\delta+\varepsilon(\geqq\Lambda_{1/2}+\varepsilon).$$

In order to verify that (3.4) is well defined as a bounded operator from $L_{(1+2\delta)/2}^2(\Omega)$ to $L^2(S^{n-1})$, we have to show that for any $\lambda>\Lambda_\delta+\varepsilon$, $\mathcal{F}_{1,\pm}(\lambda)$ can be extended to a bounded operator from $L_{(1+\beta)/2}^2(\Omega)$ to $L^2(S^{n-1})$, where β is any constant satisfying (1.13). This is possible by (3.3) and Theorem 2 (a) with $V(x)=V_1(x)$.

Now, as is proved in Theorems 2.1 and 3.1 of [3], we can have the assertions (a)$\sim$(d) of Theorem 3 with this $\mathcal{F}_\pm(\lambda)$.

To establish the assertion (e), we follow the argument of the proof of Theorem 4.1 of [3]. Namely, if $\hat{f}\in L^2((\Lambda_\delta+\varepsilon, \infty)\times S^{n-1})$ is orthogonal to the range of $\mathcal{F}_{\pm}$, we can have for any smooth $\phi\in L^2(S^{n-1})$,

$$(\hat{f}(\lambda, \cdot), \phi)_{L^2(S^{n-1})}=0 \quad \text{a.e.} \quad \lambda>\Lambda_{\tilde{\delta}}+\varepsilon, \tag{3.5}$$

where $\tilde{\delta}=\min\{\delta, 4\delta_2-2\}$. Note that to obtain (3.5) we have used the following relation satisfied by any $f\in L^2_{(1+\beta)/2}(\Omega)$, $\phi\in L^2(S^{n-1})$ and $\lambda>\Lambda_\delta+\varepsilon$:

$$(\mathcal{F}_{1,\pm}(\lambda)f, \phi)_{L^2(S^{n-1})}=\lim_{p\to\infty}\Big(\frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot,\lambda\pm i0)}[R_{1,\lambda\pm i0}f](r_p\cdot), \phi\Big)_{L^2(S^{n-1})} \tag{3.6}$$

(cf., [3]; Lemma 3.2 and Proposition 1.4).

§ 4. Examples

I. We consider potentials of the form

$$V(x)=\sum_{j=1}^{m}\frac{c_j\sin b_j r^{\varepsilon_j}}{r^{\varepsilon_j}}+0(r^{-1-\delta_0}) \quad \text{near infinity,} \tag{4.1}$$

where b_j, c_j are non-zero real, $0<\varepsilon_m\leqq\varepsilon_{m-1}\leqq\cdots\leqq\varepsilon_1\leqq 1$ and $0<\delta_0\leqq 1$. If $\varepsilon_k=\varepsilon_{k+1}=\cdots=\varepsilon_{k+p}$, the order of summation is chosen like $|b_k|\geqq|b_{k+1}|\geqq\cdots\geqq|b_{k+p}|$. We put

$$V_{1j}(x)=V_{1j}(r)=\frac{c_j\sin b_j r^{\varepsilon_j}}{r^{\varepsilon_j}}, \quad a_j(r)=\varepsilon_j^2 b_j^2 r^{-2+2\varepsilon_j}.$$

Then it follows that

$$V_{1j}(r)=0(r^{-\varepsilon_j}),$$

$$V'_{1j}(r)=\frac{\varepsilon_j b_j c_j\cos b_j r^{\varepsilon_j}}{r}+0(r^{-1-\varepsilon_j})=0(r^{-1}),$$

$$V''_{1j}(r)=\frac{-\varepsilon_j^2 b_j^2 c_j\sin b_j r^{\varepsilon_j}}{r^{2-\varepsilon_j}}+0(r^{-2})=-a_j(r)V_{1j}(r)+0(r^{-2}).$$

Thus, choosing $\delta_1=1-\varepsilon_1+\varepsilon_m$ and $\delta_2=1$, we see that $V_{1j}(r)$ satisfies (V2-i)$\sim$(V2-iii) and $a_j(r)$ satisfies (1.2) and (1.3) (see Lemma 1.1). Note that in this case, (V2-iv)$\sim$(V2-vi) are trivially satisfied by $V_{1j}(r)$. Since $\delta=\tilde{\delta}=\min\{\delta_0, 1-\varepsilon_1+\varepsilon_m\}$ and

$$\limsup_{r\to\infty} rV'_{1j}(r)=\varepsilon_j|b_j c_j|,$$

it follows from (1.4), (1.5) and (1.6) that

$$E(2)\leqq\frac{1}{2}\sum_{j=1}^{m}\varepsilon_j|b_j c_j|, \tag{4.2}$$

$$\Lambda_\delta=\Lambda_{\tilde{\delta}}\leqq\frac{1}{\min\{4\delta, 2\}}\sum_{j=1}^{m}\varepsilon_j|b_j c_j|+\frac{1}{4}\varepsilon_1^2 b_1^2\lim_{r\to\infty}r^{-2+2\varepsilon_1}. \tag{4.3}$$

Namely, we have the following results for the potential (4.1): $(E(2), \infty)$ is contained in the continuous spectrum of $L=-\Delta+V(x)$ (Theorem 1). (Λ_δ, ∞) is contained in the absolutely continuous spectrum of L (Theorem 2). If $1/2>\varepsilon_1-\varepsilon_m$, for any $\varepsilon>0$ there exists a unitary operator $\mathcal{F}_{\pm}$(depending on ε) from $\mathcal{E}((\Lambda_\delta+\varepsilon, \infty))L^2(\Omega)$ onto $L^2((\Lambda_\delta+\varepsilon, \infty)\times S^{n-1})$ which diagonalizes L (Theorem 3).

In (4.3) we have used the fact that $\varepsilon_1=\max\{\varepsilon_j\}$ and $|b_1|=\max\{|b_k|\,;\,\varepsilon_k=\varepsilon_1\}$.

Remark 4.1. If $\varepsilon_1=1$, then we have

$$\Lambda_\delta=\Lambda_{\tilde{\delta}}\leqq\frac{1}{\min\{4\delta,\,2\}}\sum_{j=1}^{m}\varepsilon_j|b_jc_j|+\frac{1}{4}b_1^2$$

(cf., [2]; Example II-1). On the other hand, if $\varepsilon_j<1$ for any j, we have

$$\Lambda_\delta=\Lambda_{\tilde{\delta}}\leqq\frac{1}{\min\{4\delta,\,2\}}\sum_{j=1}^{m}\varepsilon_j|b_jc_j|$$

(cf., [2]; Example III).

II. We consider a more general case:

$$V(x)=\sum_{j=1}^{m}c_j(x)\sin b_jr^{\varepsilon_j}+0(r^{-1-\delta_0}) \tag{4.4}$$

near infinity, where b_j, ε_j and δ_0 are as given above and $c_j(x)$ is a real-valued function such that

$$\nabla^l c_j(x)=0(r^{-l-\varepsilon_j})\ (l=0,\,1,\,2)\,. \tag{4.5}$$

We put

$$V_{1j}(x)=c_j(x)\sin b_jr^{\varepsilon_j}\,,\quad a_j(r)=\varepsilon_j^2b_j^2r^{-2+2\varepsilon_j}\,.$$

Then, choosing $\delta_1=1-\varepsilon_1+\varepsilon_m$ and $\delta_2=\varepsilon_m$, we see that $V_{1j}(x)$ satisfies (V2-i)~(V2-vi) and $a_j(r)$ satisfies (1.2) and (1.3). In this case, we have $\delta=\min\{\delta_0,\,\varepsilon_m\}$ and

$$E(2)\leqq\frac{1}{2}\sum_{j=1}^{m}\varepsilon_j|b_j|c_j^*\,;\ c_j^*=\limsup_{r\to\infty}|r^{\varepsilon_j}c_j(x)|\,, \tag{4.6}$$

$$\Lambda_\delta\leqq\frac{1}{\min\{4\delta,\,2\}}\sum_{j=1}^{m}\varepsilon_j|b_j|c_j^*+\frac{1}{4}\varepsilon_1^2b_1^2\lim_{r\to\infty}r^{-2+2\varepsilon_1}\,. \tag{4.7}$$

Namely, Theorems 1 and 2 hold with the above $E(2)$ and Λ_δ, respectively. In order to apply Theorem 3 we have to assume

$$\delta_2=\varepsilon_m>1/2\,.$$

Then we see that for any $\varepsilon>0$ there exists a partial isometric operator $\mathcal{F}_\pm$ (depending on ε) from $\mathcal{E}((\Lambda_\delta+\varepsilon,\,\infty))L^2(\Omega)$ to $L^2((\Lambda_\delta+\varepsilon,\,\infty)\times S^{n-1})$ which diagonalizes L, and maps $\mathcal{E}((\Lambda_{\tilde{\delta}}+\varepsilon,\,\infty))L^2(\Omega)$ onto $L^2((\Lambda_{\tilde{\delta}}+\varepsilon,\,\infty)\times S^{n-1})$, where

$$\Lambda_{\tilde{\delta}}\leqq\frac{1}{\min\{4\tilde{\delta},\,2\}}\sum_{j=1}^{m}\varepsilon_j|b_j|c_j^*+\frac{1}{4}\varepsilon_1^2b_1^2\lim_{r\to\infty}r^{-2+2\varepsilon_1} \tag{4.8}$$

with $\tilde{\delta}=\min\{\delta,\,2\delta_2-1\}=\min\{\delta_0,\,2\varepsilon_m-1\}$.

Remark 4.2. In general $\Lambda_{\tilde{\delta}}\geqq\Lambda_\delta$. However, if $\delta_2=\varepsilon_m\geqq3/4$, we have $\Lambda_{\tilde{\delta}}=\Lambda_\delta$ (cf., [3]; Corollary 5.1).

III. The above results can be applied to potentials of the form

$$V(x)=c(x)\sin^pbr^\varepsilon+0(r^{-1-\delta_0})\ (b\neq0,\ 0<\delta_0\leqq1,\ 0<\varepsilon\leqq1) \tag{4.9}$$

near infinity, where $c(x)$ is a real-valued function satisfying

(4.10) $$\nabla^{l}c(x)=0(r^{-l-\varepsilon})\ (l=0,1,2).$$

In fact,

$$\sin^{p} br^{\varepsilon}=c_{0}+\sum_{k=1}^{p}\{c_{k}\sin kbr^{\varepsilon}+d_{k}\cos kbr^{\varepsilon}\}$$

for suitable constants c_0, c_k and $d_k(k=1, \cdots, p)$. So, if we put

$$\begin{cases} V_{1j}(x)=c(x)\{c_{p-j+1}\sin(p-j+1)br^{\varepsilon}+d_{p-j+1}\cos(p-j+1)br^{\varepsilon}\}\,(j=1,\cdots,p), \\ V_{1(p+1)}(x)=c_{0}c(x), \end{cases}$$

$$\begin{cases} a_{j}(r)=\varepsilon^{2}(p-j+1)^{2}b^{2}r^{-2+2\varepsilon}(j=1,\cdots,p), \\ a_{p+1}(r)=0, \end{cases}$$

$V_{1j}(x)\,(j=1, \cdots, p+1)$ satisfies (V2-i)~(V2-vi) and $a_j(r)\,(j=1, \cdots, p+1)$ satisfies (1.2) and (1.3).

Remark 4.3. Potentials of the form

$$V(x)=c_{1}\sin(\log r)+c_{2}r^{-\varepsilon}\sin br^{\varepsilon}+0(r^{-1-\delta_{0}})$$

$(bc_1c_2\neq 0, 0<\varepsilon\leqq 1)$ are not covered by our theory (see V2-i), though each potential $\sin(\log r)$ or $r^{-\varepsilon}\sin br^{\varepsilon}$ is in the framework of our "oscillating" long-range potentials ([2], [3]).

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