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An inverse problem in potential theory and the inverse scattering problem

By

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0. Introduction

Let us start with the following result ([8], Theorem 5.3): Let $S(k)$, $k>0$, be the S-matrix associated with the Schrödinger operator $H=-\Delta+Q$ in $L^2(R^3)$ with a short-range potential $Q(y)$. Then we have an asymptotic formula

$$(0.1)\qquad \lim_{K\to\infty} k^2(F(k)x_{k,z}, x_{k,z})_{S^2} = -2\pi\int_{R^3}|z-y|^{-2}Q(y)dy,$$

where

$$(0.2)\qquad \begin{cases} F(k)=-2\pi ik^{-1}(S(k)-I) & (k>0), \\ x_{k,z}(\omega)=e^{-ikz\omega} & (\omega\in S^2,\ z\in R^3), \end{cases}$$

I is the identity operator on $L^2(S^2)$ and $(\ ,\ )_{S^2}$ denotes the inner product of $L^2(S^2)$. (0.1) was used in [8] to show the uniqueness of the inverse scattering problem for general short-range potential $Q(y)=O(|y|^{-\mu})$ with $\mu>1$ ([8], Theorem 5.4).

In this work we shall discuss the integral equation

$$(0.3)\qquad g(z)=\lambda\int_{R^3}|z-y|^{-\alpha}Q(y)dy$$

in the following two sections. Here λ and α are constants such that λ is a complex number with $\lambda\neq 0$ and $0<\alpha<3$. When a function $g(z)$ is given, we seek the solution $Q(y)$ which satisfies

$$(0.4)\qquad |Q(y)|\leqq C(1+|y|)^{-\mu}\qquad (y\in R^3)$$

with $C>0$ and $\mu>3-\alpha$. In § 1 we shall show a necessary and sufficient condition on $g(z)$ that the equation (0.3)–(0.4) has a unique solution $Q(y)$. A sufficient condition for the solvability of (0.3)–(0.4) will be given in § 2. For instance we shall show the following (Theorem 2.4): Let $1<\alpha<3$ and let

$$\begin{cases} |g(z)| \leqq C(1+|z|)^{-\varepsilon} \\ |\Delta g(z)| \leqq C(1+|z|)^{-\varepsilon'} \end{cases} \tag{0.5}$$

for all $z \in R^3$ with $C>0$, $\varepsilon>0$, $\varepsilon'>2$. Then there exists a unique solution $Q(y)$ of the equation (0.3)–(0.4) and we have

$$Q(y) = \text{const.} \int_{R^3} |y-z|^{\alpha-4} (\Delta g)(z) dz. \tag{0.6}$$

The results obtained in § 1 and § 2 will be applied to the inverse scattering problem in § 3. We shall discuss uniqueness and reconstruction of the potential $Q(y)$. Some characterization of the asymptotic behavior of the S-matrix will be given, too.

1. The equation $g = \lambda(|y|^{-\alpha} * Q)$

Let us first introduce some notations. Let μ be a real number. Then a function space A_μ is defined by

$$A_\mu = \{ f \in C(R^3) / f(y) = O(|y|^{-\mu}) \quad \text{as} \quad |y| \longrightarrow \infty \}, \tag{1.1}$$

where $C(R^3)$ is all continuous functions on R^3. Accordingly the estimate

$$|f(y)| \leqq C(1+|y|)^{-\mu} \qquad (y \in R^3) \tag{1.2}$$

holds for any $f \in A_\mu$ with a constant $C>0$. Let $\mathscr{S} = \mathscr{S}(R^3)$ be all rapidly decreasing functions on R^3, and let $\mathscr{S}' = \mathscr{S}'(R^3)$ be all linear continuous functionals on $\mathscr{S}$. The pairing between $\mathscr{S}$ and $\mathscr{S}'$ will be denoted by $\langle\ ,\ \rangle$. Further we set

$$\mathscr{S}_0 = \mathscr{S}_0(R^3) = \{ \varphi \in \mathscr{S} / \varphi = 0 \quad \text{in a neighborhood of } y=0 \}. \tag{1.3}$$

The Fourier transform $\mathscr{F}$, $\bar{\mathscr{F}}$, $\mathscr{F}^*$, $\bar{\mathscr{F}}^*$ are defined by

$$\begin{cases} (\mathscr{F} f)(\xi) = (2\pi)^{-3/2} \int_{R^3} e^{-i\xi y} f(y) dy, \\ (\bar{\mathscr{F}} f)(\xi) = (\mathscr{F} f)(-\xi), \\ (\mathscr{F}^* F)(y) = (2\pi)^{-3/2} \int_{R^3} e^{i\xi y} F(\xi) d\xi, \\ (\bar{\mathscr{F}}^* F)(y) = (\mathscr{F}^* F)(-y). \end{cases} \tag{1.4}$$

Here $\xi y = \xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3$ for $\xi = (\xi_1, \xi_2, \xi_3)$ and $y = (y_1, y_2, y_3)$.

Let α and λ be real and complex numbers, respectively, such that

$$0 < \alpha < 3 \quad \text{and} \quad \lambda \neq 0, \tag{1.5}$$

and let us consider the integral equation

$$\begin{cases} g(z) = \lambda \int_{R^3} |z-y|^{-\alpha} Q(y) dy \quad (= \lambda(|y|^{-\alpha} * Q)(z)), \\ Q \in A_\mu \end{cases} \tag{1.6}$$

with $3-\alpha<\mu<3$[1]. Here $f*h$ means the vonvolution. If $Q\in A_\mu$ with $3-\alpha<\mu<3$ and $g(z)$ is defined by (1.6), then we have

$$g\in A_{\mu-(3-\alpha)}, \tag{1.7}$$

which is a consequence of the following well-known theorem.

Lemma 1.1. *Let* $0<a<3$, $0<b<3$ *and* $a+b>3$. *Then*

$$|y|^{-a}*f\in A_{a+b-3} \tag{1.8}$$

for any $f\in A_b$.

In order to solve the equation (1.6), let us introduce a linear functional $\Lambda^s g\in\mathscr{S}'_\xi=\mathscr{S}'(R^3_\xi)$ for $s>0$ and $g\in A_\varepsilon$ with $\varepsilon>0$.

Definition 1.2. Let $g\in A_\varepsilon$ with $\varepsilon>0$ and let $s>0$. Then a linear functional $\Lambda^s g$ on $\mathscr{S}_\xi=\mathscr{S}(R^3_\xi)$ is defined by

$$\langle\Lambda^s g, G\rangle=\int_{R^3} g(y)\{\mathscr{F}^*(|\xi|^s G)\}(y)dy \qquad (G\in\mathscr{S}_\xi). \tag{1.9}$$

$\Lambda^s g$ is well-defined as the element of $\mathscr{S}'_\xi$ as will be shown in the next proposition. Let us introduce the norm $|\ |_s$, $s>0$, by

$$|G|_s=\sum_{|\beta|\leqq 3}\int_{R^3}|\xi|^{s-3+|\beta|}|D^\beta G(\xi)|d\xi+\int_{R^3}|\xi|^s|G(\xi)|d\xi. \tag{1.10}$$

Here $\beta=(\beta_1,\beta_2,\beta_3)$ is a multi-index with $|\beta|=\beta_1+\beta_2+\beta_3$ and

$$D^\beta=(\partial/\partial\xi_1)^{\beta_1}(\partial/\partial\xi_2{}^\beta)^2(\partial/\partial\xi_3)^{\beta_3}. \tag{1.11}$$

Obviously the topology induced in $\mathscr{S}_\xi$ by the norm $|\ |_s$ is weaker than the proper topology of $\mathscr{S}_\xi$.

Proposition 1.3. *Let* $g\in A_\varepsilon$ *with* $\varepsilon>0$ *and let* $s>0$. *Then* $\Lambda^s g$ *defined by Definition* 1.2 *is an element of* $\mathscr{S}'_\xi$ *and the estimate*

$$|\langle\Lambda^s g, G\rangle|\leqq C_s\|(1+|y|)^{-3}g\|_{L_1}|G|_s \tag{1.12}$$

holds for any $G\in\mathscr{S}_\xi$, *where* $\|\ \|_{L^1}$ *is the norm of* $L^1(R^3)$ *and* C_s *is a constant depending only on* $s>0$.

Proof. Repeating partial integration, we get

$$(\mathscr{F}^*(|\xi|^s G))(y)=i|y|^{-2}y_l^{-1}(\mathscr{F}^*H_{j,s})(y), \tag{1.13}$$

where

$$\begin{aligned} H_{j,s}(\xi,G)=&\,s(s+1)(s+2)|\xi|^{s-4}\xi_j G(\xi)+s(s+3)|\xi|^{s-2}D_j G\\ &+2s(s-2)|\xi|^{s-4}\xi_j\xi\cdot\nabla G(\xi)+2s|\xi|^{s-2}\xi\cdot\nabla(D_j G)\end{aligned} \tag{1.14}$$

1) Here and in the sequel we assume $\mu<3$. We are naturally interested in the case that μ is larger than $3-\alpha$ but is close to $3-\alpha$.

$$+s|\xi|^{s-2}\xi_j\Delta G(\xi)+|\xi|^s\Delta(D_jG).$$

Here $D_j=\partial/\partial\xi_j$, ∇ is the gradient and Δ is the Laplacian. It can be easily seen that

$$|(\bar{\mathscr{F}}^*H_{j,s})(y)|\leqq C_s|G|_s \qquad (y\in R^3) \tag{1.15}$$

holds with a constant C_s depending only on s. Take $\phi_j\in C(R^3)$, $j=0, 1, 2, 3$, such that $0\leqq\phi_j(y)\leqq 1$, $\sum_j \phi_j(y)=1$, support of $\phi_0(y)$ is contained in $\{y/|y|\leqq 2\}$ and the support of $\phi_j(y)$ is contained in $\{y/2|y_j|\geqq|y|, |y|\geqq 1\}$ for each $j=1, 2, 3$. Thus we obtain

$$\langle A^s g, G\rangle=\langle\phi_0 g, \bar{\mathscr{F}}^*(|\xi|^s G)\rangle+\sum_{j=1}^3\langle\phi_j g, i|y|^{-2}y_j^{-1}\bar{\mathscr{F}}^*H_{j,s}\rangle. \tag{1.16}$$

(1.12) follows from (1.15) and (1.16). Q. E. D.

Let $\mathscr{S}_{0\xi}=\mathscr{S}_0(R^3_\xi)$ be as in (1.3). It will be shown that $\mathscr{S}_{0\xi}$ is dence in $\mathscr{S}_\xi$ with respect to the norm $|\ |_s$. Let $\rho(\xi)\in C^\infty(R^3_\xi)$ such that $0\leqq\rho(\xi)\leqq 1$ and $\rho(\xi)=0$ $(|\xi|\leqq 1/2)$, $=1$ $(|\xi|\geqq 1)$, and set

$$\rho_m(\xi)=\rho(m\xi) \qquad (m=1, 2, 3,\dots). \tag{1.17}$$

Lemma 1.4. *Let $s>0$ and let $|\ |_s$ be as in* (1.10). *Then we have*

$$\lim_{m\to\infty}|G-\rho_m G|_s=0 \tag{1.18}$$

for any $G\in\mathscr{S}_\xi$.

Proof. Noting that $\rho_m(\xi)=1$ for $|\xi|\geqq m^{-1}$, we can see that

$$\int_{R^3}|\xi|^{s-3}|G-\rho_m G|d\xi\leqq\int_0^{m^{-1}}r^{s-1}dr\int_{S_2}|G(r\omega)|d\omega \longrightarrow 0 \tag{1.19}$$

as $m\to\infty$. As for the first derivatives we get

$$\begin{aligned}&\int_{R^3}|\xi|^{s-2}|D_j(G-\rho_m G)|d\xi\\ &\leqq c_j\int_{|\xi|\leqq m^{-1}}|\xi|^{s-3}|G|d\xi+\int_{|\xi|\leqq m^{-1}}|\xi|^{s-2}|D_jG|d\xi\end{aligned} \tag{1.20}$$

for $j=1, 2, 3$, where $c_j=\max_\xi|D_j\rho(\xi)|$ and we have used the fact that $m\leqq|\xi|^{-1}$ on the support of $D_j\rho_m(\xi)$. Thus we get

$$\int_{R^3}|\xi|^{s-2}|D_j(G-\rho_m G)|d\xi \longrightarrow 0 \qquad (m\longrightarrow\infty). \tag{1.21}$$

Similarly it can be seen that

$$\begin{cases}\displaystyle\int_{R^3}|\xi|^{s-3+|\beta|}|D^\beta(G-\rho_m G)|d\xi \longrightarrow 0,\\ \displaystyle\int_{R^3}|\xi|^s|G-\rho_m G|d\xi \longrightarrow 0\end{cases} \tag{1.22}$$

as $m\to\infty$ for any multi-index β with $0\leqq|\beta|\leqq 3$, which implies (1.18). Q. E. D.

Let us now give an inversion formula for the equation (1.6)

Proposition 1.5. *Let $0<\alpha<3$ and let $Q\in A_\mu$ with $3-a<\mu<3$. Let $g(z)$ be defined by (1.6). Then we have $g\in A_{\mu-(3-\alpha)}$ and*

$$Q=(c_\alpha\lambda)^{-1}\mathscr{F}^*\Lambda^{3-\alpha}g \quad \text{in} \quad \mathscr{S}', \tag{1.23}$$

i.e.,

$$\langle Q, \varphi\rangle=(c_\alpha\lambda)^{-1}\langle\mathscr{F}^*\Lambda^{3-\alpha}g, \varphi\rangle \qquad (\varphi\in\mathscr{S}), \tag{1.24}$$

with a constant

$$c_\alpha=2^{3-\alpha}\pi^{3/2}\Gamma((3-\alpha)/2)\Gamma(2/\alpha)^{-1}, \tag{1.25}$$

where $\Gamma(t)$ is the Γ-function.

Proof. Let $\{Q_n\}$ be a sequence such that

$$\begin{cases} Q_n\in C_0^\infty=C_0^\infty(R^3) & (n=1, 2,\dots), \\ |Q_n(y)|\leqq C(1+|y|)^{-\mu} & (y\in R^3, n=1, 2,\dots), \\ Q_n(y)\longrightarrow Q(y) & (y\in R^3, n\longrightarrow\infty), \end{cases} \tag{1.26}$$

where $C>0$ is independent of $n=1, 2,\dots$. Such an approximate sequence can be constructed by making use of the Friedrichs molifier. If we set

$$g_n(z)=\lambda\int_{R_3}|z-y|^{-\alpha}Q_n(y)dy, \tag{1.27}$$

then $g_n\in A_{\mu-(3-\alpha)}$ by Lemma 1.1 and g_n satisfies

$$\begin{cases} |g_n(z)|\leqq C(1+|z|)^{-(\mu+\alpha-3)} & (z\in R^3, n=1, 2,\dots), \\ g_n(z)\longrightarrow g(z) & (z\in R^3, n\longrightarrow\infty), \end{cases} \tag{1.28}$$

C being independent of $n=1, 2,\dots$. Let $G\in\mathscr{S}_\xi$. Then we have for $n=1, 2,\dots$.

$$\begin{aligned} \langle\mathscr{F}g_n, G\rangle&=\langle g_n, \bar{\mathscr{F}}^*G\rangle=\lambda\langle|y|^{-\alpha}*Q_n, \bar{\mathscr{F}}^*G\rangle \\ &=\lambda\langle|y|^{-\alpha}, \check{Q}_n*(\bar{\mathscr{F}}^*G)\rangle \qquad (\check{Q}_n(y)=Q_n(-y)). \end{aligned} \tag{1.29}$$

Noting that

$$\begin{aligned} \mathscr{F}^*(|\xi|^{-\gamma})(y)&=2^{-\gamma+(3/2)}\Gamma((3-\gamma)/2)\Gamma(2/\gamma)^{-1}|y|^{-3+\gamma} \\ &=b_\gamma|y|^{-3+\gamma} \qquad \text{in} \quad \mathscr{S}' \end{aligned} \tag{1.30}$$

for $0<\gamma<3$[2], and setting $\gamma=3-\alpha$ in (1.30), we get from (1.29)

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2) See, e.g., Gel'fand-Shilov [4], p. 194, though the definition of the Fouier transforms in [4] is a little bit different from the ones used here.

$$(1.31)\qquad \langle \mathscr{F} g_n, G\rangle = \lambda b_{3-\alpha}^{-1}\langle \mathscr{F}^*(|\xi|^{\alpha-3}), \check{Q}_n*(\bar{\mathscr{F}}^*G)\rangle = (2\pi)^{3/2}\lambda b_{3-\alpha}^{-1}\langle |\xi|^{\alpha-3}(\mathscr{F}Q_n), G\rangle,$$

where we have used the relations

$$(1.32)\qquad \bar{\mathscr{F}}(f*g)(\xi) = (2\pi)^{3/2}(\bar{\mathscr{F}}f)(\xi)\cdot(\bar{\mathscr{F}}g)(\xi) \qquad (f, g\in\mathscr{S})$$

and

$$(1.33)\qquad (\bar{\mathscr{F}}\check{Q}_n)(\xi) = (\mathscr{F}Q_n)(\xi).$$

Therefore we have

$$(1.34)\qquad \mathscr{F} g_n = c_\alpha\lambda|\xi|^{\alpha-3}(\mathscr{F}Q_n) \quad \text{in} \quad \mathscr{S}'_\xi$$

with c_α given in (1.25). Let ρ_m be as in (1.17). Since $|\xi|^{3-\alpha}\rho_m G\in\mathscr{S}_{0\xi}\subset\mathscr{S}_\xi$ for any $G\in\mathscr{S}_\xi$, we have from (1.34)

$$(1.35)\qquad (c_\alpha\lambda)^{-1}\langle \mathscr{F} g_n, |\xi|^{3-\alpha}\rho_m G\rangle = \langle |\xi|^{\alpha-3}\mathscr{F}Q_n, |\xi|^{3-\alpha}\rho_m G\rangle \qquad (G\in\mathscr{S}_\xi),$$

whence follows that

$$(1.36)\qquad (c_\alpha\lambda)^{-1}\langle \Lambda^{3-\alpha}g_n, \rho_m G\rangle = \langle \mathscr{F}Q_n, \rho_m G\rangle.$$

Letting $m\to\infty$ in (1.36) and taking account of Proposition 1.3 and Lemma 1.4, we can see that

$$(1.37)\qquad (c_\alpha\lambda)^{-1}\langle \Lambda^{3-\alpha}g_n, G\rangle = \langle \mathscr{F}Q_n, G\rangle$$

for any $G\in\mathscr{S}_\xi$. Further we let $n\to\infty$ in (1.37) and make use of (1.26) and (1.28) to get

$$(1.38)\qquad (c_\alpha\lambda)^{-1}\Lambda^{3-\alpha}g = \mathscr{F}Q \quad \text{in} \quad \mathscr{S}'_\xi$$

whence (1.24) directly follows. Q. E. D.

The converse of Proposition 1.5 will be shown in the next proposition.

Proposition 1.6. *Let $0<\alpha<3$ and let $3-\alpha<\mu<3$. Let $g\in A_{\mu-(3-\alpha)}$ with*

$$(1.39)\qquad \mathscr{F}^*\Lambda^{3-\alpha}g\in A_\mu,$$

i.e., there exists $h\in A_\mu$ such that $h=\mathscr{F}^*\Lambda^{3-\alpha}g$ in $\mathscr{S}'$. Then

$$(1.40)\qquad Q = (c_\alpha\lambda)^{-1}\mathscr{F}^*\Lambda^{3-\alpha}g\,(=(c_\alpha\lambda)^{-1}h)$$

is a solution of the equation (1.6). c_α is given in (1.25).

Proof. Let $Q(y)$ be defined by (1.40). We have for $G\in\mathscr{S}_{0\xi}$

$$(1.41)\qquad \langle |y|^{-\alpha}*Q, \bar{\mathscr{F}}^*G\rangle = \langle Q, |y|^{-\alpha}*(\bar{\mathscr{F}}^*G)\rangle.$$

Here we should note that $|y|^{-\alpha}*(\bar{\mathscr{F}}^*G)\in\mathscr{S}$, because, noting that (1.32) is valid in $\mathscr{S}'$ for $f=|y|^{-\alpha}$ and $g=\bar{\mathscr{F}}^*G$, we have from (1.30)

(1.42) $$\bar{\mathscr{F}}^*(|y|^{-\alpha}*(\bar{\mathscr{F}}^*G))=(2\pi)^{3/2}b_\alpha|\xi|^{\alpha-3}G(\xi)\in\mathscr{S}_{0\xi}.$$

From (1.42) and (1.42) we see that

(1.43) $$\begin{aligned}\langle|y|^{-\alpha}*Q,\ \bar{\mathscr{F}}^*G\rangle &= (c_\alpha\lambda)^{-1}\langle\mathscr{F}^*\Lambda^{3-\alpha}g,\ |y|^{-\alpha}*(\bar{\mathscr{F}}^*G)\rangle\\ &= (c_\alpha\lambda)^{-1}(2\pi)^{3/2}b_\alpha\langle\Lambda^{3-\alpha}g,\ |\xi|^{\alpha-3}G\rangle\\ &= \lambda^{-1}\langle g,\ \bar{\mathscr{F}}^*\{|\xi|^{3-\alpha}|\xi|^{\alpha-3}G\}\rangle\\ &= \lambda^{-1}\langle g,\ \bar{\mathscr{F}}^*G\rangle,\end{aligned}$$

where it should be noted that $(2\pi)^{3/2}c_\alpha^{-1}b_\alpha=b_\alpha b_{3-\alpha}=1$. Thus we get

(1.44) $$\langle\mathscr{F}\{g-\lambda(|y|^{-\alpha}*Q)\},\ G\rangle=0$$

for any $G\in\mathscr{S}_{0\xi}$, which implies that the support of $\mathscr{F}\{g-\lambda^{-1}(|y|^{-\alpha}*Q\}$ is contained in the origin $\{0\}$. Therefore there exists a polynomial $P(y)=P(y_1, y_2, y_3)$ such that

(1.45) $$\mathscr{F}\{g-\lambda^{-1}(|y|^{-\alpha}*Q)\}=P(D)\delta,$$

where δ is the Dirac δ-function and $D=(-D_1, -iD_2, -iD_3)$. For the proof of (1.45) see, e.g., Schwartz [9], p. 100. Since $\mathscr{F}^*P(D)\delta=P(y)$, it can be seen from (1.45) that

(1.46) $$g(y)-\lambda(|y|^{-\alpha}*Q)(y)=P(y).$$

Here the left-hand side of (1.46) is $o(1)$ at infinity by Lemma 1.1. and hence we have $P(y)\equiv 0$. Thus it has been shown htat Q defined by (1.40) is a solution of the equation (1.6). Q. E. D.

The main result of this section directly follows from Propositions 1.5 and 1.6.

Theorem 1.7. *Let* $0<\alpha<3$. *Then the integral equation* (1.6) *has a unique solution* $Q\in A_\mu$ *with* $3-\alpha<\mu<3$ *if and only if*

(1.47) $$\begin{cases} g\in A_{\mu-(3-\alpha)},\\ \mathscr{F}^*\Lambda^{3-\alpha}g\in A_\mu.\end{cases}$$

Then the solution $Q(y)$ *has the form* (1.40). $Q(y)$ *is real-valued if* λ *is real and* $g(z)$ *is real-valued.*

Proof. Now that Propositions 1.5 and 1.6 have been shown, we have only to show the final statement of the theorem. Let λ be real and let $g(z)$ be real-valued. Then, taking the conjugate of (1.6), we can see that the conjugate $\overline{Q(y)}$ of $Q(y)$ is a solution of the equation (1.6), and hence the uniqueness of the solution of (1.6) can be applied to get $\overline{Q(y)}=Q(y)$, which completes the proof. Q. E. D.

It can be seen from Theorem 1.7 that the equation (1.6) is solvable if $g(z)$ and its derivatives decrecase sufficiently rapidly at infinity.

Example 1.8. (i) Let g satisfy

$$\left\{\begin{array}{l} g\in A_{\mu-1}\cap D((-\triangle)^{1/2}), \\ (-\triangle)^{1/2}g\in A_\mu \end{array}\right. \tag{1.48}$$

with $1<\mu<3$, where $D((-\Delta)^{1/2})$ is the domain of the self-adjoint operator $(-\Delta)^{1/2}$ in $L^2(R^3)$. Then the equation (1.6) with $\alpha=2$ has a unique solution $Q=(2\pi^2\lambda)^{-1}\cdot(-\Delta)^{1/2}g$.

(ii) Let $g(z)$ be a smooth function such that $|\mathscr{F}g|_{3-\alpha}<\infty$ with $0<\alpha<3$. Then the equation (1.6) has a unique solution $Q=(c_\alpha\lambda)^{-1}\mathscr{F}^*(|\xi|^{3-\alpha}g)$. The proof is easy by the use of (1.15) in the proof of Proposition 1.3.

(iii) Let us consider the case that $\alpha=1$. Then the quation (1.6) has a unique solution if and only if $\mathscr{F}^*\Lambda^2 g\in A_\mu$ and $g\in A_{\mu-2}$ with $\mu>2$. Since

$$\langle\mathscr{F}^*\Lambda^2 g,\varphi\rangle=\langle g,-\Delta\varphi\rangle\qquad(\varphi\in\mathscr{S}) \tag{1.49}$$

we can say that (1.6) with $\alpha=1$ has a unique solution $Q(y)=-(2\pi^{\frac{3}{2}}\lambda)^{-1}\Delta g$ if and only if $\Delta g\in A_\mu$ and $g\in A_{\mu-2}$ with $\mu>2$, where Δg is defined in the sense of distributions.

2. A sufficient condition for the solvability

The purpose of this section is to show the following theorem which gives a sufficient condition on $g(z)$ that the equation (1.6) is solvable.

Theorem 2.1. *Let $0<\alpha<3$. Suppose that $g\in A_\varepsilon$ with $\varepsilon>0$ and there exits a positive number a such that $3-\alpha<s<3$ and*

$$\mathscr{F}^*\Lambda^s g=g_s\in A_\mu \tag{2.1}$$

with $s<\mu<3$. Then the integral equation (1.6) has a unique solution

$$Q(y)=(c_{\alpha,s}\lambda)^{-1}\int_{R^3}|y-z|^{\alpha+s-6}g_s(z)dz\in A_{3-\alpha+\varepsilon}, \tag{2.2}$$

where Λ^s is as in Definition 1.2 and

$$\left\{\begin{array}{l} \delta=\min(\mu-s,\varepsilon), \\ c_{\alpha,s}=2^s\pi^3\Gamma((3-\alpha)/2)\Gamma((\alpha+s-3)/2)\{\Gamma(\alpha/2)\Gamma((6-\alpha-s)/2)\}^{-1}. \end{array}\right. \tag{2.3}$$

We need some preparations before giving the proof of this theore.

Let $g_s\in A_\mu$ as in Theorem 2.1 and take a sequence $\{g_{s,n}\}$ which satisfies

$$\left\{\begin{array}{ll} g_{s,n}\in C_0^\infty, & \\ g_{s,n}(z)=C(1+|z|)^{-\mu} & (z\in R^3), \\ g_{s,n}(z)\longrightarrow g_s(z) & (z\in R^3,\ n\longrightarrow\infty), \end{array}\right. \tag{2.4}$$

where the constant C is independent of $n=1,2,\dots$. Set

$$F_{s,n}(y)=(\mathscr{F}^*|\xi|^{3-\alpha-s}\mathscr{F}g_{s,n})(y) \tag{2.5}$$

for each $n=1,2,\dots$. Since $3-\alpha-s>-3$, we have $|\xi|^{3-\alpha-s}(\mathscr{F}g_{s,n})(\xi)\in L^1(R^3_\xi)$, which implies that $F_{s,n}(y)$ is well-defined as a continuous function on R^3.

Proposition 2.2. *Let $F_{s,n}(y)$ be as above. Then we have*

$$(2.6)\qquad \begin{cases} F_{s,n}(y) = d_{\alpha,s}\int_{R^3} |y-z|^{\alpha+s-6} g_{s,n}(z)\,dy \in A_{3-\alpha+\mu-s}, \\ |F_{s,n}(y)| \leqq C(1+|y|)^{-(3-\alpha+\mu-s)} \end{cases}$$

with a constant C independent of $n=1, 2,\ldots$ and $d_{\alpha,s}=(2\pi)^{-3/2}b_{\alpha+s-3}$, where b_γ is given in (1.30). *Further,*

$$(2.7)\qquad \lim_{n\to\infty} F_{s,n}(y) = d_{\alpha,s}\int_{R^3} |y-z|^{\alpha+s-6} g_s(z)\,dz$$

holds for any $y \in R^3$.

Proof. First let us note that $0<6-\alpha-s<3$. Then we can apply (1.30) with $\gamma=6-\alpha-s$ to show

$$(2.8)\qquad \begin{aligned} \langle F_{s,n}, \varphi\rangle &= \langle |\xi|^{3-\alpha-s}\mathscr{F} g_{s,n}, \varphi\rangle \\ &= b_{\alpha+s-3}\langle \bar{\mathscr{F}} |y|^{\alpha+s-6}, (\mathscr{F} g_{s,n})(\bar{\mathscr{F}}\varphi)\rangle. \end{aligned}$$

Thus, using (1.32), we have

$$(2.9)\qquad \begin{aligned} \langle F_{s,n}, \varphi\rangle &= (2\pi)^{-3/2}\langle |y|^{\alpha+s-6}, g_{s,n}*(\mathscr{F}^*\bar{\mathscr{F}}\varphi) \\ &= d_{\alpha,s}\langle |y|^{\alpha+s-6}, g_{s,n}*\check{\varphi}\rangle \qquad (\check{\varphi}(y)=\varphi(-y)) \\ &= d_{\alpha,s}\langle |y|^{\alpha+s-6}*g_{s,n}, \varphi\rangle, \end{aligned}$$

which is combined with (2.4) to give (2.6). By letting $n\to\infty$ in (2.6) and noting (2.4), (2.7) can be easily derived. Q.E.D.

Proposition 2.3. *Let $F_{s,n}(y)$ be as above. Then*

$$(2.10)\qquad \lim_{n\to\infty} F_{s,n} = \mathscr{F}^* \Lambda^{3-\alpha} g \quad \text{in } \mathscr{S}'.$$

Proof. It sufficies to show

$$(2.11)\qquad \lim_{n\to\infty} \langle F_{s,n}, \varphi\rangle = \langle \mathscr{F}^*\Lambda^{3-\alpha}g, \varphi\rangle$$

for $\varphi\in\mathscr{S}$. Let us first consider the case that $\varphi=\bar{\mathscr{F}}^*G$ with $G\in\mathscr{S}_{0\xi}$. Then $\bar{\mathscr{F}}^*(|\xi|^t\varphi)\in\mathscr{S}$ for any real t. Using

$$(2.12)\qquad \langle F_{s,n}, \varphi\rangle = \langle g_s, \bar{\mathscr{F}}^*(|\xi|^{3-\alpha-s}G)\rangle$$

and recalling (2.4) and the definition of Λ^s, we have

$$(2.13)\qquad \begin{aligned} \lim_{n\to\infty} \langle F_{s,n}, \varphi\rangle &= \langle g_s, \bar{\mathscr{F}}^*(|\xi|^{3-\alpha}G) \\ &= \langle \mathscr{F}^*\Lambda^s g, \bar{\mathscr{F}}^*(|\xi|^{3-\alpha-s}G)\rangle \\ &= \langle g, \bar{\mathscr{F}}^*(|\xi|^s|\xi|^{3-\alpha-s}G)\rangle \end{aligned}$$

$$=\langle g, \bar{\mathscr{F}}^*(|\xi|^{3-\alpha}G)\rangle$$

$$=\langle \Lambda^{3-\alpha}g, G\rangle=\langle \mathscr{F}^*\Lambda^{3-\alpha}g, \varphi\rangle.$$

Let us next consider the general case, i.e., let us show (2.11) for $\varphi\in\mathscr{S}$. Let ρ_m be as in (1.17) and set

$$\varphi_m=\bar{\mathscr{F}}^*(\rho_m\bar{\mathscr{F}}\varphi) \qquad (m=1, 2,\dots). \tag{2.14}$$

Then $\varphi_m\in\mathscr{S}_{0\xi}$ and it follows from Lemma 1.4 that

$$|\varphi-\varphi_m|_t \longrightarrow 0 \tag{2.15}$$

as $m\to\infty$ for each $t>0$. In the relation

$$\langle \mathscr{F}^*\Lambda^{3-\alpha}g-F_{s,n}, \varphi\rangle=\langle \mathscr{F}^*\Lambda^{3-\alpha}g-F_{s,n}, \varphi_m\rangle+\langle \mathscr{F}^*\Lambda^{3-\alpha}g, \varphi-\varphi_m\rangle \\ -\langle F_{s,n}, \varphi-\varphi_m\rangle \tag{2.16}$$

the first term of the right-hand side tends to 0 as $n\to\infty$ for each m because of (2.12). By the use of Proposition 1.3 and (2.15) with $t=3-\alpha$ the second term of the right-hand side is estimated as

$$|\langle \mathscr{F}^*\Lambda^{3-\alpha}g, \varphi-\varphi_m\rangle|\leqq c_{3-\alpha}\|(1+|y|)^{-3}g\|_{L^1}|\varphi-\varphi_m|_{3-\alpha} \longrightarrow 0 \tag{2.17}$$

as $m\to\infty$. Thus, in order to show (2.11), it is sufficient to prove

$$|\langle F_{s,n}, \varphi\rangle|\leqq C|\bar{\mathscr{F}}\varphi|_{3-\alpha} \qquad (\varphi\in\mathscr{S}) \tag{2.18}$$

with a constant C independent of $n=1, 2,\dots$. Let us now show (2.18). Since we can see that

$$\begin{cases} \mathscr{F}^*(|\xi|^{-s}\mathscr{F}g_{s,n})(z)=(2\pi)^{3/2}b_{3-s}^{-1}\int_{R^3}|z-y|^{s-3}g_{s,n}(y)dy \\ \qquad\qquad \equiv h_{s,n}(z)\in A_{\mu-s}, \\ |h_{s,n}(z)|\leqq C(1+|z|)^{s-\mu} \qquad (z\in R^3,\ n=1, 2,\dots) \end{cases} \tag{2.19}$$

in quite a similar way to the one used in the proof of (2.6) in Proposition 2.2, $\langle F_{s,n}, \varphi\rangle$ is calculated as

$$\begin{aligned} \langle F_{s,n}, \varphi\rangle &=\langle |\xi|^{3-\alpha-s}\mathscr{F}g_{s,n}, \bar{\mathscr{F}}\varphi\rangle \\ &=\langle |\xi|^{-s}\mathscr{F}g_{s,n}, |\xi|^{3-\alpha}\bar{\mathscr{F}}\varphi\rangle \\ &=\langle h_{s,n}, \bar{\mathscr{F}}^*(|\xi|^{3-\alpha}\bar{\mathscr{F}}\varphi)\rangle=\langle \Lambda^{3-\alpha}h_{s,n}, \bar{\mathscr{F}}\varphi\rangle. \end{aligned} \tag{2.20}$$

(2.18) follows from (2.19), (2.20) and Proposition 1.3. Q.E.D.

Proof of Theorem 2.1. From Propositions 2.2 and 2.3 we have

$$\mathscr{F}^*\Lambda^{3-\alpha}g=d_{\alpha,s}(|y|^{\alpha+s-6}*g_s)\in A_{3-\alpha+\mu-s} \tag{2.21}$$

in $\mathscr{S}'$. On the other hand g is assumed to satisfy

(2.22) $$g \in A_\varepsilon$$

with $\varepsilon>0$. Therefore, setting $\delta=\min(\mu-s, \varepsilon)$, we can apply Theorem 1.7 to see that the equation (1.6) has a unique solution $Q=(c_\alpha\lambda)^{-1}\mathscr{F}^*\Lambda^{3-\alpha}g=(c_\alpha\lambda)^{-1}d_{\alpha,s}|y|^{\alpha+s-6}*g_s$. Now we have only to note that $c_\alpha/d_{\alpha,s}$ is equal to $c_{\alpha,s}$ defined by (2.3). Q.E.D.

The next theorem is an application of Theorem 2.1.

Theorem 2.4. *Let* $1<\alpha<3$. *Assume that* $g \in A_\varepsilon$ *with* $\varepsilon>0$ *and* $\Delta g \in Q_\mu$ *with* $\mu>2$, *where* Δg *is defined in the sense of distributions. Then the equation* (1.6) *has a unique solution* $Q \in A_{3-\alpha+\delta}$ *with* $\delta=\min(\mu-2, \varepsilon)$ *which has the form*

(2.23) $$Q(y)=-(c_{\alpha,2}\lambda)^{-1}\int_{R^3}|y-z|^{\alpha-4}(\Delta g)(z)dz,$$

where

(2.24) $$c_{\alpha,2}=4\pi^3\Gamma((3-\alpha)/2)\Gamma((\alpha-1)/2)\{\Gamma(\alpha/2)\Gamma((4-\alpha)/2)\}^{-1}.$$

Proof. Since $\mathscr{F}^*(|\xi|^2\mathscr{F}\varphi)=-\Delta\varphi$ for $\varphi \in \mathscr{S}$, we have

(2.25) $$\langle\mathscr{F}^*\Lambda^2 g, \varphi\rangle=\langle-\Delta g, \varphi\rangle \qquad (\varphi \in \mathscr{S}),$$

i.e.,

(2.26) $$\mathscr{F}^*\Lambda^2 g=-\Delta g \in A_\mu \quad \text{in} \quad \mathscr{S}'.$$

Therefore Theorem 2.1 with $s=2$ can be applied. Q.E.D.

Remark 2.5. Let $\lambda>0$ in the equation (1.6). Then, under the assumptions of Theorem 2.4, the solution $Q(y)\leqq 0$ (or $Q(y)\geqq 0$) if $g(z)$ is subharmonic (or superharmonic).

3. Some remarks on the inverse scattering problem

1° Let us consider the Schrödinger operator

(3.1) $$H=-\Delta+Q$$

in R^3, where Q is a multiplication operator by a real-valued function $Q(y) \in A_\mu$ with $\mu>1$, i.e., $Q(y)$ is a short-range potential. As is well-known, the restriction of the differential operator H to C_0^∞ is essentailly self-adjoint in $L^2(R^3)$. The (unique) self-adjoint extension will be denoted again by H.

2° Let us give a brief sketch of scattering theory for H. First the wave operators

(3.2) $$W_\pm=\underset{t\to\pm\infty}{\text{s-lim}}\, e^{itH}e^{-itH_0}$$

are defined as partially isometric operators in $L^2(R^3)$ (Kuroda [5]). Let $E(\lambda)$ be the spectral measure associated with H. Then the ranges of $W_\pm$ are known to be $E((0, \infty))L^2(R^3)$ which is also the absolutely continuous subspace of $L^2(R^3)$ with respect to H (Agmon [1], Saitō [7]). By the use of $W_{\pm}$ the scattering operator S is defined by

$$S = W_{+}^{*} W_{-}. \tag{3.3}$$

S is a unitary operator on $L^2(R^3)$. Further it can be shown that there exists a family of unitary operators $S(k)$, $k>0$, on $L^2(S^2)$ such that

$$\{\mathscr{F} S \mathscr{F}^{*}) G\}(\xi) = \{S(|\xi|) G(|\xi| \cdot)\}(\tilde{\xi}) \qquad (G \in C_0^{\infty}(R_{\xi}^3),\ \tilde{\xi} = \xi/|\xi|). \tag{3.4}$$

Here $\mathscr{F}$ is the Fourier transform defined by the first relation of (1.4). $S(k)$, $k>0$, called the S-matrix associated with H. Set

$$F(k) = -2\pi i k^{-1}(S(k) - I), \tag{3.5}$$

I being the identity operator on $L^2(R^3)$. Then $F(k)$ is a compact operator for each $k>0$ (Agmon [1]). If $Q \in A_{\mu}$ with $\mu>2$, then $F(k)$ is a Hilbert-Schmidt operator on $L^2(S^2)$ with its Hilbert Schmidt kernel $F(k, \omega, \omega')$ ($k>0$, $\omega, \omega' \in S^2$) (Amerein et al. [2], Saitō [8]), i.e.,

$$\begin{cases} (F(k)x)(\omega) = \displaystyle\int_{S^2} F(k, \omega, \omega') x(\omega') d\omega', \\ \displaystyle\int_{S^2}\int_{S^2} |F(k, \omega, \omega')|^2 d\omega d\omega' < \infty \end{cases} \tag{3.6}$$

for $x \in L^2(S^2)$. $F(k, \omega, \omega')$ is called the scattering amplitude.

3^0 Usually the inverse scattering problem consists of the attempt to reconstruct the potential $Q(y)$, which is often called the underlying potential, from the scattering amplitude $F(k, \omega, \omega')$. Let us now replace the scattering amplitude $F(k, \omega, \omega')$ by the S-matrix $S(k)$ in the above definition of the inverse scattering problem, because we treat general short-range potentials $Q(y) \in A_{\mu}$ with $\mu>1$ and $F(k)$ has no Hilbert-Schmidt kernel in general for such a potential. As has been mentioned in Newton [6], the inverse scattering problem has the following three separate aspects:

(a) Reconstruction of the underlying potential.

(b) Uniqueness of the underlying potential.

(c) Existence of an underlying potential, or characterization of the class of S-matricies associated with potentials (in a given class).

4^0 Let us first consider the problem (a). Newton [6] showed that the underlying potential $Q(y)$ is reconstructed by solving an integral equation which can be regarded as an extension of the Marchenko equation in the one-dimensional case. Here $Q(y)$ has to fulfil some pretty restrictive conditions which are satisfied by, e.g., $|Q(y)| \leqq C(1+|y|)^{-3-\varepsilon}$ and $|\nabla Q(y)| \leqq C(1+|y|)^{-2-\varepsilon}$ with $C, \varepsilon>0$. Now, using the results obtained in § 1 and § 2 and the formula (0.1), we shall show another reconstruction formula for the underlying potential $Q(y) \in A_{\mu}$ with $\mu>1$. Set

$$g(z, k) = k^2 (F(k) x_{k,z}, x_{k,z})_{S^2} \tag{3.7}$$

for $k>0$, $z\in R^3$, where $x_{k,z}(\omega)=e^{-ikz\omega}$ and $(\ ,\ )_{S^2}$ is the inner product of $L^2(S^2)$.

Theorem 3.1. (i) *Let $S(k)$, $k>0$, be the S-matrix for the Schrödinger operator $H=-\Delta+Q$ with real $Q\in A_\mu(1<\mu<3)$ and let $F(k)$ be as in* (3.5). *Then the limit*

$$g(z)=\lim_{k\to\infty} g(z,k)=-2\pi\int_{R^3}|z-y|^{-2}Q(y)dy \tag{3.8}$$

exists for each $z\in R^3$ and $g(z)$ satisfies

$$\begin{cases} g\in A_{\mu-1}, \\ \mathscr{F}^*\Lambda^1 g\in A_\mu, \end{cases} \tag{3.9}$$

Λ^1 being as in the Definition 1.2 *with $s=1$. We have a reconstruction formula*

$$Q(y)=-(4\pi^3)^{-1}\mathscr{F}^*\Lambda^1 g(y). \tag{3.10}$$

(ii) *We have another expression for $Q(y)$.*

$$Q=-(4\pi^3)^{-1}\lim_{k\to\infty}\mathscr{F}^*\Lambda^1 g(\cdot,k) \quad \text{in } \mathscr{S}'. \tag{3.11}$$

Proof. The existence of limit (3.8) is shown in [8] (Theorem 5.3). (3.9) and (3.10) follows from Theorem 1.7 with $\alpha=2$ and $\lambda=-2\pi$. Since it can be seen from the proof of Theorem 5.3 of [8] that the estimate

$$|g(z,k)|\leqq C(1+|z|)^{-(\mu-1)} \quad (z\in R^3,\ k\geqq 1) \tag{3.12}$$

holds with a constant $C>0$ independent of $k\geqq 1$. (3.12) is combined with (3.8) to give

$$\lim_{k\to\infty}\int_{R^3} g(z,k)\mathscr{F}^*(|\xi|G)(z)dz=\int_{R^3} g(z)\mathscr{F}^*(|\xi|G)(z)dz \quad (G\in\mathscr{S}), \tag{3.13}$$

which completes the proof of (3.11). Q.E.D.

We can use the results in §2 to get another reconstruction formula for $Q(y)$. The following results corresponds to Theorem 2.4.

Theorem 3.2. *Let $S(k)$ be as in Theorem* 3.1 *and $g(z)$ be as in* (3.8). *Further, let $\Delta g\in A_\mu$ with $\lambda>2$, where Δg is defined in the sense of distributions. Then we have*

$$Q(y)=(8\pi^5)^{-1}\int_{R^3}|z-y|^{-2}(\Delta g)(z)dz. \tag{3.14}$$

Proof. We have only to apply Theorem 2.4 with $\alpha=2$ and $\lambda=-2\pi$, and note that $c_{2,2}=4\pi^4$ which follows from (2.24) with $\alpha=s=2$. Q.E.D.

5⁰ Let us next consider the problem (b); uniqueness of the underlying potential. Faddeev [3] showed the uniquness of the underlying potential for the potential $Q\in A_\mu$ with $\mu>3$. This result was extended to the case that Q belongs to the Robin class R (Newton [6], §3). Saitō [8] showed that the uniquness holds for $Q\in A_\mu$

with $\mu>1$. Now it can be obtained directly from Theorem 1.7 with $\alpha=2$, $\lambda=-2\pi$.

Theorem 3.3. *Let $Q_1(y)$ and $Q_2(y)$ belong to A_μ with $\mu>1$ and let $S_1(k)$ and $S_2(k)$ be the S-matricies $H_1=-\Delta+Q_1$ and $H_2=-\Delta+Q_2$, respectively. If $S_1(k)=S_2(k)$ for all $k>0$ (or more exactly, $S_1(k_n)=S_2(k_n)$ for a sequence $\{k_n\}$ such that $k_n\uparrow\infty$ as $n\to\infty$), then $Q_1(y)=Q_2(y)$ for all $y\in R^3$.*

6^0 As for the existence problem (c), we can say very little though the existence of the limit (3.8) gives a necessary condition for the S-matrix. Let us now show one more result concerning the problem (c). Let us take two families $\{S_j(k)/k>0\}$, $j=1, 2$, of unitary operators on $L^2(S^2)$. We shall say that $S_1(k)$ is asymptotically equal to $S_2(k)$ with respect to $x_{k,z}(\omega)=e^{-ikz\omega}$ if

$$\lim_{k\to\infty} k(\{S_2(k)-S_1(k)\}x_{k,z}, x_{k,z})_{S^2}=0. \tag{3.15}$$

Theorem 3.4. *Let $\{S(k)/k>0\}$ be a family of unitary opertos on $L^2(S^2)$. Then there exists $Q\in A_\mu$ with $\mu>1$ such that $S(k)$ is asymptotically equal to the S-matrix $S_0(k)$ associated with the Schrödinger operator $H=-\Delta+Q$ with respect to $x_{k,z}$ if and only if there exists the limit*

$$g(z)=\lim_{k\to\infty} k^2(F(k)x_{k,z}, x_{k,z})_{S^2} \tag{3.16}$$

for each $z\in R^3$ and $g(z)$ satisfies

$$\begin{cases} g\in A_{\mu-1}, \\ \mathscr{F}^*\Lambda^1 g\in A_\mu, \end{cases} \tag{3,17}$$

where $F(k)=-2\pi ik^{-1}(S(k)-I)$. Then the potential $Q(y)$ is determined uniquely by

$$Q(y)=-(4\pi^3)^{-1}(\mathscr{F}^*\Lambda^1 g)(y). \tag{3.18}$$

Proof. Let us first suppose that $S(k)$ is asymptotically equal to the S-matrix $S_0(k)$ associated with the Schrödinger operator $H=-\Delta+Q$ with a potential $Q(y)\in A_\mu$, $\mu>1$. Then it follows from

$$\lim_{k\to\infty} k(\{S(k)-S_0(k)\}x_{k,z}, x_{k,z})_{S^2}=0 \tag{3.19}$$

that

$$\lim_{k\to\infty} k^2(\{F(k)-F_0(k)\}x_{k,z}, x_{k,z})_{S^2}=0 \tag{3.20}$$

with $F_0(k)=-2\pi ik^{-1}(S_0(k)-I)$. On the other hand, as is shown in Theorem 3.1, we have

$$\lim_{k\to\infty} k^2(F_0(k)x_{k,z}, x_{k,z})_{S^2}=-2\pi\int_{R^3}|z-y|^{-2}Q(y)dy\in A_{\mu-1}. \tag{3.21}$$

Thus it can be seen from (3.20) and (3.21) that the limit (3.16) exists and is equal to the right-hand side of (3.21). Therefore Theorem 1.7 with $\alpha=2$ and $\lambda=-2\pi$ can be applied to get (3.17).

Let us next assume that the limit (3.16) exists and (3.17) holds. Then it follows from Theorem 1.7 with $\alpha=2$, $\lambda=-2\pi$ that there exists $Q\in A_{\mu}$ such that

$$g(z)=\lim_{k\to\infty} k^2(F(k)x_{k,z},\ x_{k,z})_{S^2}=-2\pi\int_{R^3}|z-y|^{-2}Q(y)dy \tag{3.22}$$

holds for each $z\in R^3$. Let $S_0(k)$, $k>0$, be the S-matrix associated with the Schrodinger $H=-\Delta+Q$. Then (3.21) holds with $F_0(k)=-2\pi ik^{-1}(S_0(k)-I)$, which, together with (3.22), yields (3.20). (3.19) directly follows from (3.20). Q.E.D.

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