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The scattering theory for the nonlinear wave equation with small data

Dedicated to Professor Sigeru Mizohata on his sixtieth birthday

By

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1. Introduction and Results

In this paper we consider a small data scattering problem for the nonlinear wave equation

$$\partial_t^2 w - \Delta w + f(w) = 0, \quad (x, t) \in \boldsymbol{R}^n \times \boldsymbol{R}. \tag{1.1}$$

Here $2 \le n \le 5$, $\Delta = \sum_{j=1}^{n} \partial_{x_j}^2$ and $f(w)$ represents a nonlinearity which satisfies the following conditions:

(A1) $\quad f \in C^1(\boldsymbol{R}),\ \ f(0)=0,\ \ f'(0)=0;$

(A2) $\quad |f'(s_1) - f'(s_2)| \le C(|s_1|^{\rho-2} + |s_2|^{\rho-2})|s_1 - s_2| \quad$ for $\ s_1, s_2 \in \boldsymbol{R}$.

In (A2) we have to choose $\rho \ge 2$. Moreover, in the following we require a more stringent condition

$$\text{(A3)} \quad \begin{cases} \dfrac{n^2+3n-2+\sqrt{(n^2+3n-2)^2-8(n^2-n)}}{2(n^2-n)} < \rho \le \dfrac{n+3}{n-1} & \text{for } \ n=2, 3, 4 \\ \rho = 2 & \text{for } \ n=5. \end{cases}$$

Scattering theory compares the asymptotic behaviors for $t \to \pm\infty$ of solutions of (1.1) with those of the free wave equation

$$\partial_t^2 w - \Delta w = 0, \quad (x, t) \in \boldsymbol{R}^n \times \boldsymbol{R}. \tag{1.2}$$

The comparison will be done in the energy space. For $s \in \boldsymbol{R}$ and $1 \le p \le \infty$, let $H^{s,p} = H^{s,p}(\boldsymbol{R}^n)$ and $\dot{H}^{s,p} = \dot{H}^{s,p}(\boldsymbol{R}^n)$ be the Sobolev spaces which are the completion of $C_0^\infty(\boldsymbol{R}^n)$ with norms

$$\|u\|_{H^{s,p}} = \|\mathscr{F}^{-1}[(1+|\xi|^2)^{\frac{s}{2}}\hat{u}(\xi)]\|_{L^p}$$

and

$$\|u\|_{\dot{H}^{s,p}} = \|\mathscr{F}^{-1}[|\xi|^s \hat{u}(\xi)]\|_{L^p},$$

respectively. Here $\hat{}$ denotes the Fourier transformation and $\mathscr{F}^{-1}$ is its inverse. By $\|u\|_e$ we mean the energy norm:

$$\|u\|_e^2 = \frac{1}{2}[\|\partial_t u\|_{L^2}^2 + \|u\|_{\dot{H}^{1,2}}^2].$$

Now let $H = \sqrt{-\Delta}$ in L^2, $F(s) = \int_0^s f(\lambda)d\lambda$ and

$$V = \{u;\ \|u\|_V = \sup_{t \in \boldsymbol{R}} (1+|t|)^d \|u(t)\|_{H^{1,r}} < \infty\}$$

with

$$r = \frac{2n\rho - 2}{n+1} \quad \text{and} \quad d = (n-1)\left(\frac{1}{2} - \frac{1}{r}\right). \tag{1.3}$$

Then the main results of this paper are the following

Theorem. *Assume that f satisfies* (A1)$\sim$(A3).

(a) *There exists a $\delta > 0$ with the following property: If*

$$(\phi^-, \psi^-) \in \bigcap_{j=0}^{1} \{\dot{H}^{\frac{n-1}{2} - \frac{n+1}{r} + j + 1, \frac{r}{r-1}} \times \dot{H}^{\frac{n-1}{2} - \frac{n+1}{r} + j, \frac{r}{r-1}}\} \cap \{H^{\frac{n}{n+1}+1,2} \times H^{\frac{n}{n+1},2}\}$$

and

$$\sum_{j=0}^{1} \{\|\phi^-\|_{\dot{H}^{\frac{n-1}{2} - \frac{n+1}{r} + j + 1, \frac{r}{r-1}}} + \|\psi^-\|_{\dot{H}^{\frac{n-1}{2} - \frac{n+1}{r} + j, \frac{r}{r-1}}}\} + \|\phi^-\|_{H^{\frac{n}{n+1}+1,2}} + \|\psi^-\|_{H^{\frac{n}{n+1},2}} \le \delta, \tag{1.4}$$

then there exists a unique solution $w(t)$ of the integral equation

$$w(t) = w^-(t) - \int_{-\infty}^{t} H^{-1} \sin\{H(t-\tau)\} f(w(\tau)) d\tau \qquad \textit{for} \quad t \in \boldsymbol{R} \tag{1.5}$$

such that $w \in V$, $\|w\|_V \le \frac{4}{3}\|w^-\|_V$ and

$$\|w(t) - w^-(t)\|_e \longrightarrow 0 \quad \textit{as} \quad t \longrightarrow -\infty. \tag{1.6}$$

Here $w^-(t)$ is the solution of (1.2) *with initial data*

$$(w^-(0), \partial_t w^-(0)) = (\phi^-, \psi^-).$$

(b) *Furthermore, there exists a unique solution $w^+(t)$ of* (1.2) *such that $w^+(t) \in H^{1,2}$ and*

$$\|w(t) - w^+(t)\|_e \longrightarrow 0 \quad \textit{as} \quad t \longrightarrow +\infty. \tag{1.7}$$

Thus, we can define the scattering operator

$$S: (w^-(0), \partial_t w^-(0)) \longrightarrow (w^+(0), \partial_t w^+(0)).$$

(c) *If in addition* $(\phi^-, \psi^-) \in H^{2,2} \times H^{1,2}$, *then the energy conservation law holds and the energy of* w, w^- *and* w^+ *are equal*:

$$\frac{1}{2}\|w(t)\|_e^2 + \int F(w(t))dx = \frac{1}{2}\|w^-(0)\|_e^2 = \frac{1}{2}\|w^+(0)\|_e^2. \tag{1.8}$$

For f satisfying (A1), (A2) with $\rho \geq 2$, the left side of (1.8) does not in general give a definite energy. So we can not always expect (1.1) has a global solution in time. For power nonlineality $f(w) = \lambda|w|^{\rho-1}w$ with $\lambda < 0$, it is conjectured by several authors that

$$\rho(n) = \frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)}$$

should be a "critical" power. Indeed, in case $n=3$, John [4] has already proved that most solutions of (1.1) blow up in finite time if $1 < \rho < \rho(3) = 1+\sqrt{2}$, on the other hand, global solution exists if $\rho > \rho(3)$. The same results for $n=2$ has been shown in the recent work [3] of Glassey. In this sense, to develop the scattering theory for (1.1) it is necessary to assume $\rho > \rho(n)$. Note that in Strauss [10] is developed the theory in case

$$\frac{n+2+\sqrt{n^2+8n}}{2(n-1)} < \rho \leq \frac{n+3}{n-1} \qquad (n \geq 2),$$

and in Klainerman [5] is treated the case

$$\rho > \frac{n+\sqrt{2n-1}}{n-1} \qquad (n \geq 2).$$

Compared with these works, our assumption (A3) slightly weakens the restriction on ρ, especially for $n=3$ and 4, though it still remains some open space between $\rho(n)$ and the lower bound in (A3).

2. Decay Estimates for the Linear Wave Equation

In this section we present a short proof of a well-known result on the decay of solutions of the linear wave equation. Similar results can be found in Brenner [2] and Pecher [7] [8], and our proof is essentially due to [2].

We begin with defining the homogeneous Besov space $\dot{B}_q^{s,p} = \dot{B}_q^{s,p}(\boldsymbol{R}^n)$ with $s \in \boldsymbol{R}$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. Let $\phi_j(\xi) = \phi(2^{-j}\xi)$ $(-\infty < j < \infty)$, where $\phi \in C_0^\infty(\boldsymbol{R}^n)$, $\phi \geq 0$ and $\operatorname{supp}\phi \subset \left\{\xi; \frac{1}{2} < |\xi| < 2\right\}$, be such that

$$\sum_{j=-\infty}^{\infty} \phi_j(\xi) \equiv 1 \qquad (\xi \neq 0). \tag{2.1}$$

Then $\dot{B}_q^{s,p}$ is the completion of $C_0^\infty(\boldsymbol{R}^n)$ in the semi-norm

$$\|u\|_{\dot{B}_q^{s,p}} = \{\sum_{j=-\infty}^{\infty} (2^{js}\|\mathscr{F}^{-1}(\phi_j\hat{u}\|_{L^p})^q\}^{\frac{1}{q}}.$$

Lemma 2.1. *Let $2\le p<\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$. Then we have*

$$\dot{B}_2^{0,p} \subsetneq L^p \quad and \quad \dot{H}^{s,p'} \subsetneq \dot{B}_2^{s,p'}, \tag{2.2}$$

where $A\subsetneq B$ denotes that A is continuously embedded in B.

Proof. See Bergh-Löfström [1], §6.4. q.e.d.

The following lemma can be proved by use of the stationary phase method (see, e.g., Littman [6]).

Lemma 2.2. *Let $P(\xi)$ be real, C^∞ in a neighborhood of the support of $v\in C_0^\infty(\boldsymbol{R}^n)$. Assume that the rank of $H_P(\xi)=(\partial^2 P(\xi)/\partial\xi_j\partial\xi_k)$ is at least μ on the support of v. Then for some $M\in \boldsymbol{N}$ (natural number),*

$$\|\mathscr{F}^{-1}[\exp(itP)v]\|_{L^\infty}\le C(1+|t|)^{-\frac{1}{2}\mu}\sum_{|\alpha|\le M}\|\partial_\xi^\alpha v\|_{L^1}. \tag{2.3}$$

Here C depends on bounds of the derivatives of P on supp v *and on a lower bound of the maximum of the absolute values of the minors of order μ of H_P on* supp v, *and on* supp v.

Now let us prove the following

Proposition 2.3. *Let $\gamma\ge 0$, $m\in\boldsymbol{N}$. Then for any $\psi\in C_0^\infty(\boldsymbol{R}^n)$ the following estimate holds:*

$$\left\|\mathscr{F}^{-1}\left[\frac{\exp(it|\xi|^m)}{|\xi|^{2m\gamma}}\hat{\psi}(\xi)\right]\right\|_{L^p}\le C|t|^{-\mu\left(\frac{1}{2}-\frac{1}{p}\right)}\|\psi\|_{\dot{H}^{s,p'}}, \tag{2.4}$$

where $1\le p'\le 2\le p\le\infty$, $\frac{1}{p}+\frac{1}{p'}=1$, $\mu=n-1$ (if $m=1$) and $=n$ (if $m\ge 2$) and $s=(2n-m\mu)\left(\frac{1}{2}-\frac{1}{p}\right)-2m\gamma$.

Proof. We only prove (2.4) for $t=1$. (2.4) for general $t>0$ then easily follows from a change of variables $\xi t^{\frac{1}{m}}\to\xi$.

Let $\phi_j(\xi)$ be as in (2.1). Putting $2^{-j}\xi=\eta$, we have

$$\left\|\mathscr{F}^{-1}\left[\frac{\exp(i|\xi|^m)}{|\xi|^{2m\gamma}}\phi_j(\xi)\right]\right\|_{L^\infty}=2^{j(n-2m\gamma)}\left\|\mathscr{F}^{-1}\left[\frac{\exp(i2^{jm}|\eta|^m)}{|\eta|^{2m\gamma}}\phi(\eta)\right](2^j x)\right\|_{L^\infty}.$$

Since $|\eta|^{-2m\gamma}\phi(\eta)\in C_0^\infty(\boldsymbol{R}^n)$, it follows from Lemma 2.2 that

$$\left\|\mathscr{F}^{-1}\left[\frac{\exp(i2^{jm}|\eta|^m)}{|\eta|^{2m\gamma}}\phi(\eta)\right](2^j x)\right\|_{L^\infty}$$

$$\le C(1+2^{jm})^{-\frac{1}{2}\mu}\sum_{|\alpha|\le M}\|\partial_\xi^\alpha\phi\|_{L^1}\le C2^{-\frac{1}{2}\mu jm},$$

where

$$\mu=\operatorname{rank}\left(\frac{\partial^2|\eta|^m}{\partial\eta_j\partial\eta_k}\right)=\begin{cases} n-1 & (\text{if } m=1) \\ n & (\text{if } m\geq 2). \end{cases}$$

We then have, noting the Hausdorff-Young inequality,

$$\left\|\mathscr{F}^{-1}\left[\frac{\exp(i|\xi|^m)}{|\xi|^{2m\gamma}}\phi_j(\xi)\hat{\psi}(\xi)\right]\right\|_{L^\infty} \tag{2.5}$$
$$\leq\left\|\mathscr{F}^{-1}\left[\frac{\exp(i|\xi|^m)}{|\xi|^{2m\gamma}}\phi_j(\xi)\right]\right\|_{L^\infty}\|\psi\|_{L^1}$$
$$\leq C2^{j(n-2m\gamma-m\mu/2)}\|\psi\|_{L^1}.$$

On the other hand, since

$$\left\|\frac{\exp(i|\xi|^m)}{|\xi|^{2m\gamma}}\phi_j(\xi)\right\|_{L^\infty}=\sup_{1/2<\eta<2}\left|\frac{\exp(i|2^j\eta|^m)}{|2^j\eta|^{2m\gamma}}\phi(\eta)\right|\leq C2^{-2jm\gamma},$$

we obtain by the Parseval equality

$$\left\|\mathscr{F}^{-1}\left[\frac{\exp(i|\xi|^m)}{|\xi|^{2m\gamma}}\phi_j(\xi)\hat{\psi}(\xi)\right]\right\|_{L^2}\leq C2^{-2mj\gamma}\|\psi\|_{L^2}. \tag{2.6}$$

An interpolation between (2.5) and (2.6) then gives

$$\left\|\mathscr{F}^{-1}\left[\frac{\exp(i|\xi|^m)}{|\xi|^{2m\gamma}}\phi_j(\xi)\hat{\psi}(\xi)\right]\right\|_{L^p}\leq C2^{j(2n\delta-m\mu\delta-2m\gamma)}\|\psi\|_{L^{p'}}, \tag{2.7}$$

where $\delta=\frac{1}{2}-\frac{1}{p}$, $1\leq p'\leq 2\leq p\leq\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$.

Put $\hat{\psi}_j=(\phi_{j-1}+\phi_j+\phi_{j+1})\hat{\psi}$. Since $\phi_j\hat{\psi}_j=\phi_j\hat{\psi}$, we can replace $\hat{\psi}$ in (2.7) by this $\hat{\psi}_j$. Next raise to the power q on both sides of (2.7) and sum over j. Then we obtain

$$\left\|\mathscr{F}^{-1}\left[\frac{\exp(i|\xi|^m)}{|\xi|^{2m\gamma}}\hat{\psi}(\xi)\right]\right\|_{\dot{B}^{0,p}_q}\leq C\|\psi\|_{\dot{B}^{s,p'}_q} \tag{2.8}$$

where $s=(2n-m\mu)\delta-2m\gamma$, $q\geq 1$ and $\frac{1}{p}+\frac{1}{p'}=1$. This and Lemma 2.1 imply

$$\left\|\mathscr{F}^{-1}\left[\frac{\exp(i|\xi|^m)}{|\xi|^{2m\gamma}}\hat{\psi}(\xi)\right]\right\|_{L^p}\leq C\|\psi\|_{\dot{H}^{s,p'}}, \tag{2.9}$$

and (2.4) is proved for $t=1$. q.e.d.

3. Proof of Theorem

Let r and d be defined by (1.3), and ρ satisfy (A3).

Lemma 3.1. *The following relations holds.*

$$\frac{\rho}{r}=\frac{r-1}{r}-\frac{\frac{n-1}{2}-\frac{n+1}{r}}{n}, \tag{3.1}$$

$$r>2 \quad and \quad \frac{n-1}{2}-\frac{n+1}{r}\leq 0, \tag{3.2}$$

$$0<d<1 \quad and \quad d\rho>1. \tag{3.3}$$

Proof. Since $\rho=\frac{(n+1)r+2}{2n}$, we see that (3.1) holds and (3.2) is equivalent to

$$\frac{n+2}{n}<\rho\leq\frac{n+3}{n-1}. \tag{3.2$'$}$$

Further, $d=\frac{(n^2-n)\rho-(n^2+n-2)}{2n\rho-2}$ and (3.3) is rewritten as

$$\frac{n+2}{n}<\rho, \quad (n^2-3n)\rho<n^2+n-4 \quad \text{and} \tag{3.3$'$}$$

$$(n^2-n)\rho^2-(n^2+n-2)\rho>2n\rho-2.$$

Thus, our problem reduces to verify (3.2)′ and (3.3)′, which is easy if we remember (A3).

Lemma 3.2. *The following inequalities hold.*

$$\frac{1}{r}\geq\frac{1}{2}-\frac{1}{n}, \tag{3.4}$$

$$\frac{1}{r}\geq\frac{1}{\rho+1}>\frac{1}{2\rho}>\frac{r-2}{2r(\rho-1)}\geq\frac{1}{r}-\frac{1}{n}. \tag{3.5}$$

Proof. (3.4) is obvious from (3.2). The first three inequalities of (3.5) easily follow from (A3) and (1.3), since we have

$$\frac{1}{r}-\frac{1}{\rho+1}=\frac{-(n-1)\rho+n+3}{(2n\rho-2)(\rho+1)} \quad \text{and} \quad \frac{1}{2\rho}-\frac{r-2}{2r(\rho-1)}=\frac{2\rho-r}{2r\rho(\rho-1)}.$$

Further, the last inequality is equivalent to

$$\rho\geq\frac{3n+2+\sqrt{(3n+2)^2+8n(n-2)}}{4n}.$$

The right side being not greater than 2, we conclude this to hold. q.e.d.

Lemma 3.3. *Let* $\psi\in\dot{H}^{\frac{n-1}{2}-\frac{n+1}{r},\frac{r}{r-1}}$. *Then we have for* $|t|>1$,

$$\|H^{-1}\exp(iHt)\psi\|_{L^r}\leq C|t|^{-d}\|\psi\|_{\dot{H}^{\frac{n-1}{2}-\frac{n+1}{r},\frac{r}{r-1}}}. \tag{3.6}$$

Proof. Since $r>2$, we can choose $p=r$, $m=1$ and $\gamma=1/2$ in Proposition 2.3 to obtain (3.6). q.e.d.

Corollary 3.4. *Let* $\psi\in L^{\frac{r}{\rho}}$. *Then we have for* $|t|>1$,

$$\|H^{-1}\sin(Ht)\psi\|_{L^r}\leq C|t|^{-d}\|\psi\|_{L^{\frac{r}{\rho}}}. \tag{3.7}$$

Proof. (3.1) and (3.2) imply the embedding $L^{\frac{r}{\rho}} \hookrightarrow \dot{H}^{\frac{n-1}{2}-\frac{n+1}{r},\frac{r}{r-1}}$. Then (3.7) follows directly from (3.6). q.e.d.

The proof of Theorem will be based on the following two propositions.

Proposition 3.5. *There exists a $\delta_1 > 0$ with the following property: Let $w^-(t) \in V$ and $\|w^-\|_V \leq \frac{1}{4}\delta_1$. Then the integral equation (1.5) has a unique solution $w(t) \in V$, which also satisfies $\|w\|_V \leq \frac{4}{3}\|w^-\|_V$.*

Proof. Put

$$\Phi u(t) = w^-(t) - \int_{-\infty}^{t} H^{-1} \sin\{H(t-\tau)\} f(u(\tau)) d\tau. \tag{3.8}$$

Then we can apply Corollary 3.4 and the Hölder inequality to obtain

$$\begin{aligned}
\|\Phi u(t) - \Phi v(t)\|_{H^{1,r}} &\leq C\int_{-\infty}^{t} |t-\tau|^{-d} \| f(u(\tau)-f(v(\tau))\|_{H^{1,\frac{r}{\rho}}} d\tau \\
&\leq C\int_{-\infty}^{t} |t-\tau|^{-d} (\|u(\tau)\|_{H^{1,r}}^{\rho-1} + \|v(\tau)\|_{H^{1,r}}^{\rho-1}) \|u(\tau)-v(\tau)\|_{H^{1,r}} d\tau \\
&\leq C\int_{-\infty}^{t} |t-\tau|^{-d} (1+|\tau|)^{-d\rho} d\tau (\|u\|_V^{\rho-1} + \|v\|_V^{\rho-1}) \|u-v\|_V \\
&\leq C(1+|t|)^{-d} (\|u\|_V^{\rho-1} + \|v\|_V^{\rho-1}) \|u-v\|_V .
\end{aligned}$$

Here both (A1) and (A2) have been used for the second inequality, and (3.3) for the last inequality. Thus, we have for $u, v \in V$

$$\|\Phi u - \Phi v\|_V \leq C(\|u\|_V^{\rho-1} + \|v\|_V^{\rho-1}) \|u-v\|_V, \tag{3.9}$$

and similarly

$$\|\Phi u\|_V \leq \|w^-\|_V + C\|u\|_V^{\rho}. \tag{3.10}$$

Now choose δ_1 to satisfy $2C\delta_1^{\rho-1} \leq \frac{1}{2}$, and put $B(\delta_1) = \{u \in V;\ \|u\|_V \leq \delta_1\}$. Then (3.9) and (3.10) show that Φ is a contraction map from $B(\delta_1)$ to itself, and hence it has a unique fixed point $w \in B(\delta_1)$, which is a solution of (1.5). Furthermore, we have from (3.8)

$$\|w\|_V \leq \|w^-\|_V + C\delta_1^{\rho-1}\|w\|_V \leq \|w^-\|_V + \frac{1}{4}\|w\|_V .$$

This implies $\|w\|_V \leq \frac{4}{3}\|w^-\|_V$ and the proof is completed. q.e.d.

Proposition 3.6. *Let $w^-(t)$ and $w(t)$ be as in the above Proposition. For any $s \in \mathbf{R}$ the integral equation*

$$u_s(t) = w^-(t) - \int_{s}^{t} H^{-1} \sin\{H(t-\tau)\} f(u_s(\tau)) d\tau \tag{3.11}$$

has a unique solution $u_s(t) \in V$, *which also satisfies* $\|u_s\|_V \leq \frac{4}{3}\|w^-\|_V$ *and*

$$\|u_s - w\|_V \longrightarrow 0 \quad as \quad s \longrightarrow -\infty. \tag{3.12}$$

If in addition $(w^-(0), \partial_t w^-(0)) \in H^{2,2} \times H^{1,2}$, *we have*

$$u_s(t) \in C_t^2(\boldsymbol{R}; L^2) \cap C_t^1(\boldsymbol{R}; H^{1,2}) \cap C_t(\boldsymbol{R}; H^{2,2}), \tag{3.13}$$

and $u_s(t)$ *becomes a unique strong solution of* (1.1).

Proof. The unique existence of $u_s(t)$ can be verified by the same reasoning as in the proof of Proposition 3.5. To derive (3.12), we subtract the equations (3.11) and (1.5). Then

$$\begin{aligned} u_s(t) - w(t) = &-\int_s^t H^{-1} \sin\{H(t-\tau)\}\{f(u_s(\tau)) - f(w(\tau))\} d\tau \\ &+ \int_{-\infty}^s H^{-1} \sin\{H(t-\tau)\} f(w(\tau)) d\tau, \end{aligned}$$

and we have

$$\begin{aligned} \|u_s(t) - w(t)\|_{H^{1,r}} &\leq C \int_s^t |t-\tau|^{-d}(1+|\tau|)^{-d\rho} d\tau (\|w\|_V^{\rho-1} + \|u_s\|_V^{\rho-1}) \|w - u_s\|_V \\ &\quad + C \int_{-\infty}^s |t-\tau|^{-d}(1+|\tau|)^{-d\rho} d\tau \|w\|_V^\rho \\ &\leq C(1+|t|)^{-d}(\|w\|_V^{\rho-1} + \|u_s\|_V^{\rho-1}) \|w - u_s\|_V \\ &\quad + C_\varepsilon (1+|t|)^{-d}(1+|s|)^{-\varepsilon} \|w\|_V^\rho \end{aligned}$$

for ε satisfying $0 < \varepsilon < d\rho - 1$. Since $C(\|w\|_V^{\rho-1} + \|u_s\|_V^{\rho-1}) \leq 2C\delta_1^{\rho-1} < \frac{1}{2}$, from this it follows that

$$\|u_s - w\|_V \leq 2C_\varepsilon (1+|s|)^{-\varepsilon} \|w\|_V^\rho \longrightarrow 0 \quad \text{as} \quad s \longrightarrow -\infty,$$

and (3.12) holds.

Next we show (3.13). For this purpose it is enough to verify

$$f(u_s(t)) \in C_t(\boldsymbol{R}; H^{1,2}) \tag{3.14}$$

since $u_s(t)$ satisfies (3.11). By use of (3.5) we have $H^{1,r} \subsetneq L^{\frac{2r(2-1}{r-\rho}}$ and $H^{1,r} \subsetneq L^{2\rho}$. Thus,

$$\begin{aligned} \|f(u_s(t))\|_{H^{1,2}} &\leq \|f(u_s(t))\|_{L^2} + \sum_{|\alpha|=1} \|f'(u_s(t)) \partial_x^\alpha u_s(t)\|_{L^2} \\ &\leq C\|u_s(t)\|_{L^{2\rho}}^\rho + C\|u_s(t)\|_{L^{\frac{2r(\rho-1)}{r-2}}}^{\rho-1} \|u_s(t)\|_{H^{1,r}} \\ &\leq 2C' \|u_s(t)\|_{H^{1,r}}^\rho, \end{aligned}$$

and it follows that $f(u_s(t)) \in H^{1,2}$. Note that $w^-(t) \in C_t(\boldsymbol{R}; H^{2,2})$ follows from the condition $(w^-(0), \partial_t w^-(0)) \in H^{2,2} \times H^{1,2}$. Then

$$
\begin{aligned}
&\|f(u_s(t))-f(u_s(t'))\|_{H^{1,2}}\\
&\le C(\|u_s(t)\|^{\rho-1}_{H^{1,r}}+\|u_s(t')\|^{\rho-1}_{H^{1,r}})\|u_s(t)-u_s(t')\|_{H^{1,r}}\\
&\le C'(u_s)\Big\{\|w^-(t)-w^-(t')\|_{H^{1,r}}+\int_{t'}^{t}\|H^{-1}\sin\{H(t'-\tau)\}f(u_s(\tau))\|_{H^{1,r}}\,d\tau\\
&\quad+\int_s^t\|H^{-1}[\sin\{H(t-\tau)\}-\sin\{H(t'-\tau)\}]f(u_s(\tau))\|_{H^{1,r}}d\tau\Big\}\\
&\le C'(u_s)\Big\{\|w^-(t)-w^-(t')\|_{H^{2,2}}+\int_{t'}^{t}|t'-\tau|^{-d}(1+|\tau|)^{-d\rho}d\tau\|u_s\|_V^{\rho}\\
&\quad+\int_s^t\|H^{-1}[\sin\{H(t-\tau)\}-\sin\{H(t'-\tau)\}]f(u_s(\tau))\|_{H^{2,2}}d\tau\Big\}\longrightarrow 0\quad(t'\longrightarrow t).
\end{aligned}
$$

Here we have used (3.3) and the embedding $H^{2,2}\subsetneq H^{1,r}$ which follows from (3.4). Summarizing these results, we see (3.14). q. e. d.

Proof of Theorem. (a) We can write $w^-(t)$ as

$$w^-(t)=\cos\{Ht\}\phi^-+H^{-1}\sin\{Ht\}\psi^-. \tag{3.15}$$

Then applying Lemma 3.3, we obtain for $|t|>1$,

$$
\begin{aligned}
\|w^-(t)\|_{H^{1,r}}\le C|t|^{-d}\sum_{j=0}^{1}\{&\|\phi^-\|_{\dot H^{\frac{n-1}{2}-\frac{n+1}{r}+j+1,\frac{r}{r-1}}}\\
&+\|\psi^-\|_{\dot H^{\frac{n-1}{2}-\frac{n+1}{r}+j,\frac{r}{r-1}}}\}.
\end{aligned}
\tag{3.16}
$$

On the other hand, by (3.2) we have $H^{\frac{n}{n+1}+1,2}\subsetneq H^{1,r}$. Thus, it follows from (3.15) that for $|t|\le 1$,

$$
\begin{aligned}
\|w^-(t)\|_{H^{1,r}}&\le C\{\|\cos\{Ht\}\phi^-\|_{H^{\frac{n}{n+1}+1,2}}+\|H^{-1}\sin\{Ht\}\psi^-\|_{H^{\frac{n}{n+1}+1,2}}\}\\
&\le C\{\|\phi^-\|_{L^2}+\|\phi^-\|_{\dot H^{\frac{n}{n+1}+1,2}}\\
&\qquad+|t|\,\|\psi^-\|_{L^2}+\|H^{-1}\psi^-\|_{\dot H^{\frac{n}{n+1}+1,2}}\}\\
&\le C\{\|\phi^-\|_{H^{\frac{n}{n+1}+1,2}}+\|\psi^-\|_{H^{\frac{n}{n+1},2}}\}.
\end{aligned}
\tag{3.16'}
$$

Now choose δ in (a) to satisfy $C\delta<\frac{1}{4}\delta_1$, where δ_1 is the constant given in Proposition 3.5. Then combining (1.4), (3.16) and (3.16)′, we obtain

$$w^-(t)\in V\quad\text{and}\quad\|w^-\|_V\le C\delta\le\frac{1}{4}\delta_1.$$

This and Proposition 3.5 show the unique existence of the solution $w(t)\in V$ of (1.5) satisfying $\|w\|_V\le\frac{4}{3}\|w^-\|_V$. This $w(t)$ also satisfies (1.6). In fact, noting $H^{1,r}\subsetneq L^{2\rho}$ and $d\rho>1$, we have

$$\|w(t)-w^-(t)\|_e \leq C\int_{-\infty}^{t} \|f(w(\tau))\|_{L^2} d\tau \leq C\int_{-\infty}^{t} \|w(\tau)\|_{L^{2\rho}}^{\rho} d\tau$$

$$\leq C\int_{-\infty}^{t} \|w(\tau)\|_{H^{1,r}}^{\rho} d\tau \leq C\int_{-\infty}^{t} (1+|t|)^{-d\rho} d\tau \|w\|_V^{\rho} \longrightarrow 0$$

as $t\to-\infty$. Thus, (a) is proved.

(b) We define

$$w^+(t)=w^-(t)-\int_{-\infty}^{+\infty} H^{-1}\sin\{H(t-\tau)\}f(w(\tau))d\tau \tag{3.17}$$

Here $w^-(t)\in H^{1,2}$ since we have assumed $(\phi^-, \psi^-)\in H^{\frac{n}{n+1}+1,2}\times H^{\frac{n}{n+1},2} \subsetneq H^{1,2}\times L^2$. On the other hand, we obtain as above

$$\|w^+(t)-w^-(t)\|_e \leq \int_{-\infty}^{+\infty} (1-|\tau|)^{-d\rho} d\tau \|w\|_V^{\rho} < \infty.$$

These imply $w^+(t)\in \dot{H}^{1,2}$. That $w^+(t)$ solves (1.2) is now obvious from (3.17). Finally, (1.7) easily follows from the relation

$$w^+(t)-w(t)=-\int_{t}^{+\infty} H^{-1}\sin\{H(t-\tau)\}f(w(\tau))d\tau,$$

and (b) is proved.

(c) First we shall show that the equality

$$\frac{1}{2}\|w(t)\|_e^2+\int F(w(t))dx=\frac{1}{2}\|w^-(0)\|_e^2 \tag{3.18}$$

holds for any $t\in \boldsymbol{R}$. For this aim we apply Proposition 3.6. Recalling (3.5) once more, we see that $H^{1,r}\subsetneq L^{\rho+1}$ holds. By use of this and (3.12), we have

$$\left|\int [F(w(t))-F(u_s(t))]dx\right| \tag{3.19}$$

$$\leq C(\|w(t)\|_{L^{\rho+1}}^{\rho}+\|u_s(t)\|_{L^{\rho+1}}^{\rho})\|w(t)-u_s(t)\|_{L^{\rho+1}}$$

$$\leq C(\|w\|_V^{\rho}+\|u_s\|_V^{\rho})\|w-u_s\|_V \longrightarrow 0 \quad \text{as} \quad s\longrightarrow -\infty.$$

On the other hand, by use of the embedding $H^{1,r}\subsetneq L^{2\rho}$, we have

$$\|w(t)-u_s(t)\|_e \leq \int_{s}^{t} \|f(w(\tau))-f(u_s(\tau))\|_{L^2} d\tau+\int_{-\infty}^{s} \|f(w(\tau))\|_{L^2} d\tau \tag{3.20}$$

$$\leq \int_{s}^{t} (1+|\tau|)^{-d\rho} d\tau (\|w\|_V^{\rho-1}+\|u_s\|_V^{\rho-1})\|w-u_s\|_V$$

$$+\int_{-\infty}^{s} (1+|\tau|)^{-d\rho} d\tau \|w\|_V^{\rho} \longrightarrow 0$$

as $s\to-\infty$. Moreover, $u_s(t)$ being a strong solution of (1.1), we see that it conserves the total energy:

$$\frac{1}{2}\|u_s(t)\|_e^2+\int F(u_s(t))dx=\frac{1}{2}\|w^-(s)\|_e^2+\int F(w^-(s))dx. \tag{3.21}$$

Here

$$\|w^-(s)\|_e^2 = \|w^-(0)\|_e^2$$

and

$$\left|\int F(w^-(s))dx\right| \leq C\|w^-(s)\|_{L^{\rho+1}}^{\rho+1} \leq C(1+|s|)^{-d(\rho+1)}\|w^-\|_V^{\rho+1} \longrightarrow 0$$

as $s \to -\infty$. (3.19), (3.20) and (3.21) show (3.18) as follows:

$$\frac{1}{2}\|w(t)\|_e^2 + \int F(w(t))dx = \lim_{s\to-\infty}\left[\frac{1}{2}\|u_s(t)\|_e^2 + \int F(u_s(t))dx\right]$$

$$= \lim_{s\to-\infty}\left[\frac{1}{2}\|w^-(s)\|_e^2 + \int F(w^-(s))dx\right] = \frac{1}{2}\|w^-(0)\|_e^2.$$

Now to complete the proof it is enough to show that

$$\frac{1}{2}\|w(t)\|_e^2 + \int F(w(t))dx \longrightarrow \frac{1}{2}\|w^+(0)\|_e^2 \quad \text{as} \quad t \longrightarrow +\infty, \tag{3.22}$$

which is obvious from (1.7) since we have

$$\int F(w(t))dx \longrightarrow 0 \quad \text{as} \quad t \longrightarrow +\infty.$$

Thus, (c) is proved. q.e.d.

4. Final Remarks

A. In our theorem it is assumed that the initial data (ϕ^-, ψ^-) lies in a neighborhood of 0 in the space

$$\bigcap_{j=0}^{1} \{\dot{H}^{\frac{n-1}{2}-\frac{n+1}{r}+j+1,\frac{r}{r-1}} \times \dot{H}^{\frac{n-1}{2}-\frac{n+1}{r}+j,\frac{r}{r-1}}\} \cap \{H^{\frac{n}{n+1}+1,2} \times H^{\frac{n}{n+1},2}\}. \tag{4.1}$$

Note that the embedding $L^{\frac{r}{\rho}} \subsetneq \dot{H}^{\frac{n-1}{2}-\frac{n+1}{r},\frac{r}{r-1}}$ holds by (3.1) and (3.2). Then (4.1) can be replaced by a narrower but simpler space

$$\{H^{2,\frac{r}{\rho}} \times H^{1,\frac{r}{\rho}}\} \cap \{H^{\frac{n}{n+1}+1,2} \times H^{\frac{n}{n+1},2}\}. \tag{4.2}$$

The condition $(\phi^-, \psi^-) \in H^{\frac{n}{n+1}+1,2} \times H^{\frac{n}{n+1},2}$ is used to give an estimation of $\|w^-(t)\|_{H^{1,r}}$ for $|t| \leq 1$ (see (3.16)), and to guarantee that $w^-(t)$ has a finite energy.

A similar estimate can be obtained for

$$(\phi^-, \psi^-) \in H^{2,r} \times H^{1,r} \tag{4.3}$$

if we use the following $L^p - L^p$ estimate due to Peral [9].

Proposition 4.1. *Let $w^-(t)$ be given by (3.15). Then we have for $|t| \leq 1$,*

$$\|w^{-}(t)\|_{L^p} \leq C(\|\phi^{-}\|_{H^{1,p}} + \|\psi^{-}\|_{L^p}), \tag{4.4}$$

where $p \geq 1$ *is a constant satisfying* $\left|\frac{1}{p}-\frac{1}{2}\right| \leq \frac{1}{n-1}$.

Now put $p=r$. Then the above condition is valid by (3.4). Hence the space $H^{\frac{n}{n+1}+1,2} \times H^{\frac{n}{n+1},2}$ can be replaced by $\{H^{2,r} \times H^{1,r}\} \cap \{\dot{H}^{1,2} \times L^2\}$ in (4.1). Combining this and (4.2), we finally see that (4.1) can be replaced by the space

$$\{H^{2,\frac{r}{\rho}} \times H^{1,\frac{r}{\rho}}\} \cap \{H^{2,r} \times H^{1,r}\} \cap \{\dot{H}^{1,2} \times L^2\}, \tag{4.5}$$

to obtain all the assertions of our theorem.

B. The method given in this paper can be applied to the scattering problem for the nonlinear Schrödinger equation

$$i\partial_t w - \Delta w + f(w) = 0 \quad (x, t) \in \boldsymbol{R}^n \times \boldsymbol{R} \qquad (1 \leq n \leq 5)$$

with f satisfying (A1), (A2) and

$$\frac{1}{\rho} < \frac{n(\rho-1)}{2(\rho+1)} < 1 \qquad \text{for} \quad 1 \leq n \leq 5. \tag{A3$'$}$$

Let

$$V = \{u;\ \|u\|_V = \sup_{t \in \boldsymbol{R}} (1+|t|)^d \|u(t)\|_{H^{1,\rho+1}} < \infty\};\ d = n\left(\frac{1}{2} - \frac{1}{\rho+1}\right).$$

Then we obtain the following

Theorem 4.2. *For sufficiently small* $\delta > 0$, *let*

$$\|\phi^{-}\|_{H^{1,\frac{\rho+1}{\rho}}} + \|\phi^{-}\|_{H^{\mu,2}} \leq \delta, \tag{4.6}$$

where $\mu = \min\left\{\frac{n+2}{2}, 2\right\}$. *Then the same conclusions hold as in Theorem* (a), (b) *and* (c), *if we understand*

$$\|u(t)\|_e^2 = \frac{1}{2}\|u(t)\|_{\dot{H}^{1,2}}^2.$$

Note that a more general results has been proved in Strauss [10]. Our above theorem corresponds to his Theorem 9 in case $1 \leq n \leq 3$.

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Added in Proof. The assertions a) and b) of our Theorem can be extended to all $n \geq 2$ and f satisfying

$$\begin{cases} f \in C^1(\boldsymbol{R}), f(0)=0, \ |f'(s)| \leq C|s|^{\rho-1} \text{ for all } s \in \boldsymbol{R}; \\ \dfrac{n^2+3n-2+\sqrt{(n^2+3n-2)^2-8(n^2-n)}}{2(n^2-n)} < \rho \leq \dfrac{n+3}{n-1}. \end{cases}$$

For this aim we introduce the function space

$$V = \{u(t) \in C_t(\boldsymbol{R}; H^{s,r}); \ \|u\|_V = \sup_{t \in \boldsymbol{R}} (1+|t|)^d \|u(t)\|_{H^{s,r}} < \infty\}$$

with $s = \dfrac{n+1}{r} - \dfrac{n-1}{2}$, and r and d as given in (1.3). If we consider solutions of (1.1) in this space, an approximate energy method is applicable to obtain a suitable conservation property of energy, and we can follow the proof of Strauss [10, Theorem 5] to obtain the desired results. Details of the proof will be published elswhere.