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Ample vector bundles of small c_1-sectional genera

By

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Introduction

Let $\mathscr{E}$ be a vector bundle of rank r on a compact complex manifold M of dimension n. Let $P = \mathbf{P}(\mathscr{E})$ be the associated $\mathbf{P}^{r-1}$-bundle and let $H = H(\mathscr{E})$ be the tautological line bundle $\mathcal{O}(1)$ on P. $\mathscr{E}$ is said to be ample if so is H on P. In this case $A = \det(\mathscr{E})$ is also ample on M. The c_1-sectional genus g of $\mathscr{E}$ is defined to be $g(M, A)$, which is determined by the formula $2g(M, A) - 2 = (K + (n-1)A)A^{n-1}$, where K is the canonical bundle of M. Then $g(M, A)$ is a non-negative integer by [F5]. In this paper we establish a classification theory of the case $g(M, A) \leqq 2$. The case $r = 1$ was treated in [F6] and we study here the case $r > 1$. In §1, we study the case $g = 0$ or 1. The case $g = 2$ is studied in §2. The main theorem is in (2.25). In §3, we give a classification according to the sectional genus of (P, H).

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We employ similar notation to that in our previous papers on polarized manifolds.

§1. The case $g \leqq 1$

(1.1) Throughout this paper let $\mathscr{E}$, M, P, H, A and K be as in the introduction. We further assume that $n \geqq 2$ and $r \geqq 2$.

(1.2) The canonical bundle K^P of P is $\pi^*(K + A) - rH$, where π is the projection $P \to M$. So $2g(P, H) - 2 = (K^P + (n + r - 2)H)H^{n+r-2} = (n-2)H^{n+r-1} + H^{n+r-2}\pi^*(K + A) = (n-2)s_n(\mathscr{E}) + (K + A)s_{n-1}(\mathscr{E})$, where $s_j(\mathscr{E})$ is the j-th Segre class of $\mathscr{E}$. The total Segre class $s(\mathscr{E})$ is related to the Chern class by the formula $s(\mathscr{E})c(\mathscr{E}^{\vee}) = 1$. Thus, in particular, we have $2g(P, H) - 2 = (K + A)A = 2g(M, A) - 2$ and $g(P, H) = g(M, A)$ in case $n = 2$.

Remark. If M were a curve, both $g(P, H)$ and $g(M, A)$ would be equal to the genus of M. However, if $n \geqq 3$, $g(P, H) \neq g(M, A)$ in general.

(1.3) Lemma. *$AC \geqq r$ for any rational curve C in M.*

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Proof. Let $f: \mathbf{P}^1 \to C$ be the normalization. Then $f^*\mathscr{E}_C$ is ample and is a direct sum of line bundles. Hence $AC = \deg(f^*\mathscr{E}_C) \geqq r$.

Remark. If $AC = r$, then $f^*\mathscr{E}_C$ is a direct sum of $\mathcal{O}(1)$'s.

(1.4) Theorem. *If $g(M, A) = 0$, then $M \simeq \mathbf{P}_\alpha^2$, $\mathscr{E} = H_\alpha \oplus H_\alpha$, $P \simeq \mathbf{P}_\alpha^2 \times \mathbf{P}_\beta^1$ and $H = H_\alpha + H_\beta$.*

Proof. We have $\Delta(M, A) = 0$ by [F5]. Using (1.3) and [F1] we infer $(M, A) \simeq (\mathbf{P}_\alpha^2, 2H_\alpha)$. Moreover $\mathscr{E}_\ell \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$ for any line ℓ in M. This implies $\mathscr{E} \simeq H_\alpha \oplus H_\alpha$ by [V].

Alternately, one can argue as follows: By (1.2) we have $g(P, H) = 0$. So $\Delta(P, H) = 0$. Obviously (P, H) is a scroll over $\mathbf{P}^2$. By the theory in [F1], this implies $(P, H) \simeq (\mathbf{P}_\alpha^2 \times \mathbf{P}_\beta^1, H_\alpha + H_\beta)$.

(1.5) Theorem. *If $g(M, A) = 1$, then $(M, \mathscr{E})$ is one of the following:*

1) *$M \simeq \mathbf{P}_\alpha^2$ and $\mathscr{E} \simeq 2H_\alpha \oplus H_\alpha$.*
2) *$M \simeq \mathbf{P}^2$ and $\mathscr{E}$ is the tangent bundle of $\mathbf{P}^2$.*
3) *$M \simeq \mathbf{P}_\alpha^2$ and $\mathscr{E} \simeq H_\alpha \oplus H_\alpha \oplus H_\alpha$.*
4) *$M \simeq \mathbf{P}_\alpha^1 \times \mathbf{P}_\beta^1$ and $\mathscr{E} \simeq [H_\alpha + H_\beta] \oplus [H_\alpha + H_\beta]$.*
5) *$M \simeq \mathbf{P}_\alpha^3$ and $\mathscr{E} \simeq H_\alpha \oplus H_\alpha$.*

Proof. By (1.3), (M, A) cannot be a scroll. So (M, A) is a Del Pezzo manifold. Moreover, by (1.3), the two-dimensional rung S of (M, A) contains no exceptional curve. Hence $S \simeq \mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$. This implies $(M, A) \simeq (\mathbf{P}_\alpha^2, 3H_\alpha)$, $(\mathbf{P}_\alpha^1 \times \mathbf{P}_\beta^1, 2H_\alpha + 2H_\beta)$ or $(\mathbf{P}_\alpha^3, 2H_\alpha)$ by [F2].

If $n = 2$, we have $g(P, H) = 1$ by (1.2). So (P, H) is also a Del Pezzo manifold. Since $b_2(P) \geqq 2$ and $\dim P \geqq 3$, using the theory in [F2] we infer that P is either

1) the blowing-up of $\mathbf{P}^3$ at a point,
2) a general member of $|H_\alpha + H_\beta|$ on $\mathbf{P}_\alpha^2 \times \mathbf{P}_\beta^2$,
3) $\mathbf{P}_\alpha^2 \times \mathbf{P}_\beta^2$ or
4) $\mathbf{P}_\alpha^1 \times \mathbf{P}_\beta^1 \times \mathbf{P}_\tau^1$.

In these cases we easily see that $(M, \mathscr{E})$ is of the corresponding type in the statement of the theorem.

If $(M, A) \simeq (\mathbf{P}_\alpha^3, 2H_\alpha)$, then $\mathscr{E}_\ell \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$ for any line ℓ in M. So $\mathscr{E} \simeq H_\alpha \oplus H_\alpha$. Thus we are in case 5).

Remark. In case 5), (P, H) is the Segre product of $(\mathbf{P}_\alpha^3, H_\alpha)$ and $(\mathbf{P}_\beta^1, H_\beta)$. So $g(P, H) = 0$.

§2. The case $g = 2$

(2.1) In this section we assume $g(M, A) = 2$. First we study the case $n = 2$.

(2.2) $d(P, H) = H^{r+1} = s_2(\mathscr{E}) = c_1(\mathscr{E})^2 - c_2(\mathscr{E})$. Since $c_2 > 0$ by [BG], we have $A^2 = d(P, H) + c_2 \geqq 2$.

(2.3) In view of (1.3), (2.2) and the classification theory of polarized surfaces of sectional genus two (see [F6; §4]), we infer that (M, A) is one of the following types:

1) $K \sim 0$ and $A^2 = 2$.

2) M is a $\mathbf{P}^1$-bundle over an elliptic curve C, $AF = 2$ for any fiber F and $A^2 = 4$.

3) M is as above and $AF = A^2 = 3$.

4) $-K$ is ample, $A = -2K$ and $K^2 = 1$. $A^2 = 4$.

5_0) $M \simeq \mathbf{P}^1_\alpha \times \mathbf{P}^1_\beta$ and $A = 2H_\alpha + 3H_\beta$. $A^2 = 12$.

5_1) M is the blowing-up Σ_1 of $\mathbf{P}^2_\alpha$ at a point and $A = 4H_\alpha - 2E$, where E is the exceptional curve. $A^2 = 12$.

The case 1) will be studied after the others. See (2.11).

(2.4) Suppose that (M, A) is of the type (2.3; 2). Then $M \simeq \mathbf{P}_C(\mathscr{F})$ for some vector bundle $\mathscr{F}$ of rank two over C with $f = c_1(\mathscr{F}) = 0$ or 1. Moreover, by [F6; (4.5)], $A = 2H(\mathscr{F}) + \rho^*B$ for some $B \in \mathrm{Pic}(C)$ with $b = \deg B = 1 - f$, where ρ is the map $M \to C$ and $H(\mathscr{F})$ is the tautological line bundle on $\mathbf{P}(\mathscr{F})$.

Since $AF = 2$, we have $\mathscr{E}_X \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$ for every fiber X of ρ. Hence $\mathscr{G} = \rho_*(\mathscr{E} \otimes [-H(\mathscr{F})])$ is a locally free sheaf of rank two on C and $\mathscr{E} \simeq \rho^*\mathscr{G} \otimes H(\mathscr{F})$. Clearly $c_1(\mathscr{G}) = B$, $c_2(\mathscr{E}) = c_1(\mathscr{G})H(\mathscr{F}) + H(\mathscr{F})^2 = 1$ and $d(P, H) = H^3 = s_2(\mathscr{E}) = 3$.

Since $\mathscr{E} \simeq \rho^*\mathscr{G} \otimes H(\mathscr{F})$, (P, H) can be viewed as a scroll over $N = \mathbf{P}_C(\mathscr{G})$. More precisely, $P \simeq \mathbf{P}_N(\mathscr{F}_N \otimes H(\mathscr{G}))$ and H is the tautological line bundle on it. Furthermore, P is the fiber product of M and N over C.

We claim that $\mathscr{F}$ and $\mathscr{G}$ are semistable. By the above symmetry it suffices to consider $\mathscr{F}$ only. Let Q be a quotient line bundle of $\mathscr{F}$ and set $q = c_1(Q)$. Then ρ has a section Z with $H(\mathscr{F})Z = q$. Since $0 < AZ = 2q + 1 - f$, we have $q > 0$ if $f = 1$, and $q \geqq 0$ if $f = 0$. This implies the claim.

(2.5) Conversely, let $\mathscr{F}$, $\mathscr{G}$ be semistable vector bundles of rank two over an elliptic curve C with $(c_1(\mathscr{F}), c_1(\mathscr{G})) = (1, 0)$ or $(0, 1)$. Then $\mathscr{E} = \rho^*\mathscr{G} \otimes H(\mathscr{F})$ is an ample vector bundle on $M = \mathbf{P}_C(\mathscr{F})$ of the type (2.3; 2).

This is clear except the ampleness of $\mathscr{E}$. We should show the ampleness of $H(\mathscr{E})$ on $P = \mathbf{P}_M(\mathscr{E})$. We may assume $c_1(\mathscr{F}) = 1$ and $c_1(\mathscr{G}) = 0$ by symmetry. Then $H(\mathscr{F})$ is ample on M and $H(\mathscr{G})$ is nef on $N = \mathbf{P}(\mathscr{G})$ by the theory in [M; §3]. So we easily see that $H(\mathscr{F}) + H(\mathscr{G}) = H(\mathscr{E})$ is ample on P. Thus we complete the proof.

(2.6) Suppose that (M, A) is of the type (2.3; 3). By [F6; (4.5)], $M \simeq \mathbf{P}_C(\mathscr{F})$ for some indecomposable vector bundle $\mathscr{F}$ of rank two over C with $c_1(\mathscr{F}) = 1$ and $A = 3H(\mathscr{F}) - F$ *for some fiber* F of $\rho: M \to C$. We have $r \leqq AF = 3$ by (1.3). Note that $\mathscr{F}$ is stable.

When $r = 3$, similarly as in (2.4), we see $\mathscr{E} \simeq \rho^*\mathscr{G} \otimes H(\mathscr{F})$ for some vector bundle $\mathscr{G}$ of rank three on C. So $c_1(\rho^*\mathscr{G}) = -F$, $c_2(\mathscr{E}) = 2c_1(\mathscr{G})H(\mathscr{F}) + 3H(\mathscr{F})^2 = 1$ and $d(P, H) = H^4 = s_2(\mathscr{E}) = 2$.

Let Q be any quotient bundle of $\mathscr{G}$ of rank one. Then $Q + H(\mathscr{F})$ is an ample line bundle since it is a quotient of $\mathscr{E}$. Restricting to a section Z of ρ with $H(\mathscr{F})Z = 1$, we get $\deg(Q) \geqq 0$. We have $t = \deg(T) < 0$ for any subbundle T of $\mathscr{G}$ of rank one. Indeed, $\mathscr{E}' = \rho^*(\mathscr{G}/T) \otimes H(\mathscr{F})$ is ample since it is a quotient of $\mathscr{E}$. Hence $0 < c_1(\mathscr{E}')^2 = (2H(\mathscr{F}) - (t+1)F)^2 = -4t$. Combining these observations we conclude that $\mathscr{G}$ is stable.

Conversely, for any stable vector bundle $\mathscr{G}$ of rank three with $c_1(\mathscr{G}) = -1$ on C, $\mathscr{E} = \rho^*\mathscr{G} \otimes H(\mathscr{F})$ is an ample vector bundle of the type considered here. This is obvious except the ampleness. To see the ampleness, note that $\mathbf{P} = \mathbf{P}(\mathscr{E})$ is the fiber product of M and $N = \mathbf{P}(\mathscr{G})$ over C, and that $H(\mathscr{E}) = H(\mathscr{F})_P + H(\mathscr{G})_P$. By the theory in [$M$; §3], $2H(\mathscr{F}) - F$ is nef on M and $3H(\mathscr{G}) + F'$ is nef on N, where F' is a fiber of $N \to C$. Hence $6H(\mathscr{E}) - F = 3(2H(\mathscr{F}) - F)_P + 2(3H(\mathscr{G}) + F')_P$ is nef on P. So $H(\mathscr{E})$ is ample.

(2.7) Next we consider the case $r = 2$. Then $\mathscr{E}_X \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)$ for any fiber X of ρ. So $\mathscr{G} = \rho_*(\mathscr{E} \otimes [-2H(\mathscr{F})])$ is an invertible sheaf on C. We have an exact sequence $0 \to \rho^*\mathscr{G} \otimes [2H(\mathscr{F})] \to \mathscr{E} \to \mathcal{O}[Q] \to 0$ for some line bundle Q. Then $Q = \det(\mathscr{E}) - \rho^*\mathscr{G} - 2H(\mathscr{F}) = H(\mathscr{F}) + \rho^*T$ for some $T \in \mathrm{Pic}(C)$ with $\deg(\mathscr{G}) + \deg(T) = -1$. We have $\deg(T) \geqq 0$ since Q is ample. On the other hand $c_2(\mathscr{E}) = (\rho^*\mathscr{G} + 2H(\mathscr{F}))(H(\mathscr{F}) + \rho^*T) = 2 + 2\deg(T) + \deg(\mathscr{G}) = 1 + \deg(T)$ and $0 < s_2(\mathscr{E}) = 3 - c_2(\mathscr{E})$. So $\deg(T) = 0$ or 1. Thus:

1) $\deg(T) = 1$, $\deg(\mathscr{G}) = -2$, $c_2(\mathscr{E}) = 2$ and $H^3 = s_2(\mathscr{E}) = 1$, or
2) $\deg(T) = 0$, $\deg(\mathscr{G}) = -1$, $c_2(\mathscr{E}) = 1$ and $H^3 = 2$.

Remark. Both types seem to exist really. But we have troubles in showing the ampleness of $\mathscr{E}$.

(2.8) Suppose that (M, A) is of the type (2.3; 4). By (1.3), $\mathrm{rank}(\mathscr{E}) = 2$ since $AE = 2$ for any exceptional curve E on M. We easily see that $\mathscr{E}$ is A-semistable. We have $c_2(\mathscr{E}) = 1$, 2 or 3 since $4 = A^2 = c_2(\mathscr{E}) + s_2(\mathscr{E})$.

(2.8.1) In case $c_2(\mathscr{E}) = 1$, we have $c_1(\mathscr{F}) = 0 = c_2(\mathscr{F})$ for $\mathscr{F} = \mathscr{E} \otimes [K]$. So $\chi(\mathscr{F}) = 2$ by the Riemann–Roch theorem. We have $h^2(\mathscr{F}) = h^0(\mathscr{E}^\vee) = h^0(\mathscr{E} \otimes [2K])$ and $h^0(M, -K) = 2$. Therefore $2 \leqq h^0(\mathscr{F}) = h^0(P, H + \pi^*K)$. For any member D of $|H + \pi^*K|$, we have $H^2D = s_2(\mathscr{E}) + s_1(\mathscr{E})K = 1$. Hence D is irreducible and reduced since H is ample. So $\dim(D \cap D') < 2$ for any other member D' of $|H + \pi^*K|$. We have $HDD' = H(H + \pi^*K)^2 = 0$. Hence $D \cap D' = \varnothing$ since H is ample. So D and D' are disjoint sections of π. This implies $\mathscr{F} \simeq \mathcal{O} \oplus \mathcal{O}$ and $\mathscr{E} \simeq \mathcal{O}[-K] \oplus \mathcal{O}[-K]$.

(2.8.2) In case $c_2(\mathscr{E}) = 2$, we have $c_1(\mathscr{F}) = 0$ and $c_2(\mathscr{F}) = 1$ for $\mathscr{F} = \mathscr{E} \otimes [K]$. So $\chi(\mathscr{F}) = 1$. We have also $h^2(\mathscr{F}) = h^0(\mathscr{E} \otimes [2K]) \leqq h^0(\mathscr{F})$. Hence $h^0(\mathscr{F}) > 0$ and we have $D \in |H + \pi^*K|$. Then $H^2D = s_2(\mathscr{E}) + s_1(\mathscr{E})K = 0$, contradicting the ampleness of H. Thus this case is ruled out.

(2.8.3) In case $c_2(\mathscr{E}) = 3$, we have $H^3 = s_2(\mathscr{E}) = 1$ and $\chi(\mathscr{E}) = 2$. Since $h^2(\mathscr{E}) = h^0(\mathscr{E} \otimes [3K])$, we infer $h^0(\mathscr{E}) \geqq 2$, so $\dim|H| \geqq 1$. In fact we have

$\dim |H| = 1$. Indeed, otherwise, $\Delta(P, H) \leqq 1$ and hence $b_2(P) \leqq 2$ by [F2; Part III]). This contradicts $b_2(M) = 9$.

For any $D \in |H|$, we have $H^2 D = 1$, so D is irreducible and reduced. Moreover, for any other member D' of $|H|$, the scheme theoretical intersection Z of D and D' is an irreducible reduced curve of arithmetic genus two with $HZ = 1$. By definition of the Segre class, $C = \pi(Z)$ represents $s_1(\mathscr{E})$, so $C \in |A|$. For every fiber X of π over $x \in C$, either $D \cap X$ or $D' \cap X$ is a simple point because otherwise $X \subset Z$. Ths implies $Z \simeq C$. This section Z of π over C yields an exact sequence $0 \to \mathscr{F} \to \mathscr{E}_C \to \mathscr{Q} \to 0$ of vector bundles over C and $\mathscr{Q}$ is identified with the restriction of H to $Z \simeq C$. So $\deg(\mathscr{Q}) = 1$ and $\deg(\mathscr{F}) = 3$.

Let δ and δ' be the section in $H^0(M, \mathscr{E}) \simeq H^0(P, H)$ corresponding to D and D'. Then the restrictions of them in $H^0(C, \mathscr{E}_C)$ come from $H^0(C, \mathscr{F})$ by construction. Thus they define members Y and Y' of $|\mathscr{F}|$. Clearly $x \in \operatorname{Supp}(Y)$ if and only if $\pi^{-1}(x) \subset D$. In particular $Y \cap Y' = \varnothing$. The structure of D is related to Y as follows:

If Y consists of three different points (this is the case if D is a general member of $|H|$), then $\pi_D \colon D \to M$ is the blowing-up at Y since $c_2(\mathscr{E}) = 3$. We have an exact sequence $0 \to \mathcal{O}[E_Y] \to \mathscr{E}_D \to \mathcal{O}[\pi_D^* A - E_Y] \to 0$ of vector bundles on D, where E_Y is the exceptional divisor over Y consisting of three (-1)-curves.

If $Y = p_1 + 2p_2$ for some $p_1 \neq p_2$, let M_2 be the blowing-up of M at p_1 and p_2. The strict transform of C meets the (-1)-curve E_2 over p_2 at a point p_3. Let M_3 be the blowing-up of M_2 at p_3. Then the strict transform E_2' of E_2 is a (-2)-curve. Contracting E_2' to an ordinary double point we obtain D.

If $Y = 3p_1$, let M_1 be the blowing-up of M at p_1 and let E_1 be the exceptional curve over p_1. Let p_2 be the meeting point of E_1 and the strict transform of C. Let M_2 be the blowing-up of M_1 at p_2 and let E_2 be the exceptional curve. Let p_3 be the meeting point of E_2 and the strict transform of C. Let M_3 be the blowing-up of M_2. Then the strict transforms of E_1 and E_2 on M_3 are (-2)-curves meeting transversally at a point. Contracting them to a rational double point of type A_2, we obtain D.

Having these observations in mind, we will now show that such a vector bundle does really exist. We take three points p_1, p_2, p_3 on M in a generic position. Since $h^0(M, A) = 4$, there is a unique member C of $|A|$ containing $Y = \bigcup p_i$. Let D be the blowing-up of M at Y and let E_Y be the exceptional divisor. Then $h^1(D, 2E_Y - A_D) = h^1(D, [-K]_D - E_Y) = 1$ since Y is in a generic position. Moreover, for any general element e of $H^1(D, 2E_Y - A_D)$, the restriction of e to E_i, the (-1)-curve over p_i, is not zero for each i, because $0 = h^1(D, [-K]_D - E_1 - E_2) = h^1(D, [-K]_D - E_2 - E_3) = h^1(D, [-K]_D - E_3 - E_1)$. Let $0 \to \mathcal{O}[E_Y] \to \mathscr{E}' \to \mathcal{O}[A_D - E_Y] \to 0$ be the extension of vector bundles on D induced by e. The restriction of it to E_i is a non-trivial extension $0 \to \mathcal{O}(-1) \to \mathscr{E}_i' \to \mathcal{O}(1) \to 0$, so the restriction $\mathscr{E}_i'$ of $\mathscr{E}'$ to E_i is trivial. Hence $\mathscr{E}'$ is the pull-back of a vector bundle $\mathscr{E}$ on M. Obviously $c_1(\mathscr{E}) = A$, $c_2(\mathscr{E}) = 3$ and $h^0(E) = \chi(\mathscr{E}) = 2$. Thus it suffices to show the ampleness of $\mathscr{E}$.

Note that $\dim |-K| = 1$ and every member F of $|-K|$ is irreducible and reduced since $(-K)F=1$ and $-K$ is ample. So there are only finitely many singular members of $|-K|$ and each of them has only finite singular points. Let S be the union of all such singular points of members of $|-K|$. Since Y is in a generic position, C is a general member of $|A|$. So we may assume that $S \cap C = \emptyset$ and C is smooth.

Let J be the Jacobian variety of C and we fix a point o on C. For any divisor X in M, we define $v(X) = [X_C - (XC)o] \in J$. This depends only on the linear equivalence class of X, and gives a mapping $v\colon \mathrm{Pic}(M) \to J$. The image N of v is a countable subset of the abelian variety J.

For any triple $m = (m_1, m_2, m_3)$ of positive integers, let $\mu_m\colon C \times C \times C \to J$ be the morphism defined by $\mu_m(x_1, x_2, x_3) = [m_1x_1 + m_2x_2 + m_3x_3 - (m_1 + m_2 + m_3)o]$. Note that $\mu = \mu_{(1,1,1)}$ factors through the symmetric product $C \times C \times C / S_3$ of C, which is a $\mathbf{P}^1$-bundle over J.

We claim that μ_m is smooth at $a = (a_1, a_2, a_3)$ if a_i's are three different points on C. To see this, set $b = \mu_m(a)$ and let $d\mu_m\colon T_{C\times C\times C,a} \simeq T_{C,a_1} \oplus T_{C,a_2} \oplus T_{C,a_3} \to T_{J,b}$ be the differential. Write $d\mu_m = \delta_1 \oplus \delta_2 \oplus \delta_3$ for $\delta_i \in \mathrm{Hom}(T_{C,a_i}, T_{J,b})$. Let μ' be the map defined by $\mu'(x) = \mu(x) - \mu(a) + b \in J$ and let $d\mu' = \delta_1' \oplus \delta_2' \oplus \delta_3'$ be its differential at a. Then $\delta_i = \mathrm{m}_i\delta_i'$. Since a_i's are different, $C \times C \times C$ is étale at a over the symmetric product. So μ and μ' are smooth at a. Hence the images of δ_i' generate $T_{J,b}$, and so do the images of $\delta_i = m_i\delta_i'$. This implies that $d\mu_m$ is surjective, as desired.

From this claim we infer that every fiber of μ_m is a curve for any m. So $J_3 = \bigcup_m \mu(\mu_m^{-1}(N))$ is a union of countably many curves.

For any pair $k = (k_1, k_2)$ of positive integers, let $\mu_k\colon C \times C \to J$ be the morphism defined by $\mu_k(x_1, x_2) = [k_1x_1 + k_2x_2 - (k_1 + k_2)o]$. Then, similarly as before, we infer that $\mu_k^{-1}(b)$ is finite for every $b \in J$ except $b = [\omega - 2o]$, where ω is the canonical bundle of C. Therefore $N' = \{(x_1, x_2) \in C \times C | x_1 + x_2 \notin |\omega|$ and $\mu_k(x_1, x_2) \in N$ for some $k\}$ is a countable set. Let $f\colon C \times C \times C \to C \times C$ be the projection and let $J_2 = \mu(f^{-1}(N'))$. Then J_2 is a union of countably many curves.

Now, since p_i's are in a generic position, we may assume $\mu(p_1, p_2, p_3) \notin J_2 \cup J_3$. We may assume also $\mathrm{Bs}\,|Y| = \emptyset$ on C. We will now prove that $H = H(\mathscr{E})$ is ample on $P = \mathbf{P}(\mathscr{E})$.

We will derive a contradiction assuming $HX = 0$ for some curve X in P. Since $h^0(P, H) = h^0(M, \mathscr{E}) = 2$, there is a member D' of $|H|$ such that $D' \cap X \neq \emptyset$. Then $X \subset D'$ since $HX = 0$. Let $Y' = p_1' + p_2' + p_3'$ be the member of $|Y|$ corresponding to D'. Then D' is obtained from M by several belowing-ups over Y, possibly followed by a contraction yielding a rational double point. In any case $D' \cap D = Z$ is the strict transform of C on D'. Since $HX = 0$, we have $Z \cap X = \emptyset$. So $\pi(Z) \cap \pi(X) \subset \mathrm{Supp}(Y')$.

If $\pi(X) \cap C$ consists of three points, then so does Y' and $\pi(X)_C = m_1p_1' + m_2p_2' + m_3p_3'$ in $\mathrm{Pic}(C)$. This implies $\mu_m(p_1', p_2', p_3') \in N$ for $m = (m_1, m_2, m_3)$ and $\mu(p_1', p_2', p_3') \in J_3$. On the other hand, $\mu(p_1', p_2', p_3') = \mu(p_1, p_2, p_3)$ since $Y' = Y$ in $\mathrm{Pic}(C)$. This contradicts $\mu(p_1, p_2, p_3) \notin J_3$.

If $\pi(X)\cap C$ consists of two points, set $\pi(X)_C = k_1p'_1 + k_2p'_2$ and $Y' = p'_1 + p'_2 + p'_3$. We have $p'_1 + p'_2 \notin |\omega|$ because otherwise $p'_3 \in \mathrm{Bs}\,|Y'| = \mathrm{Bs}\,|Y|$. Hence $(p'_1, p'_2) \in N'$ and $\mu(p_1, p_2, p_3) = \mu(p'_1, p'_2, p'_3) \in J_2$, contradicting the choice of Y.

If $\pi(X)$ meets C at only one point p', we claim that Y' is of the form $2p' + q'$ for some $q' \in C$ (possibly $q' = p'$). Indeed, otherwise, D' is isomorphic to the blowing-up M_1 of M at p' over a neighborhood of p'. So the strict transforms X_1 and C_1 of $\pi(X)$ and C on M_1 do not meet. On the other hand, the member F of $|-K|$ passing p' is smooth at p' since $S\cap C = \varnothing$. The strict transform F_1 of F on M_1 is a member of $|-K - E'|$ with $F_1^2 = 0$, where E' is the exceptional curve and K denotes the pull-back of K by abuse of notation. Then $C_1 = -K + F_1$ in $\mathrm{Pic}\,(M_1)$ and $0 = X_1C_1 \geqq -K\cdot\pi(X) > 0$. Thus we infer $Y' = 2p' + q'$. This implies $(p', q') \in N'$ as before and hence $\mu(p_1, p_2, p_3) = \mu(p'_1, p'_2, p'_3) \in J_2$, a contradiction.

Now we have proved $HX > 0$ for any curve X in P. We have $H^2D' = HZ = 1$ for any member D' of $|H|$. For any irreducible surface W of P, there is a member D' of $|H|$ such that $D'\cap W \neq \varnothing$. So $H^2W = HD'W > 0$. Of course $H^3 = 1$. Conseuently H is ample by Nakai's criterion.

(2.9) Suppose that (M, A) is of the type (2.3; 5_0). For any fiber F of $f\colon M \to \mathbf{P}^1_\beta$, we have $AF = 2$. So $r = 2$ and $\mathscr{E}_F \simeq \mathcal{O}(1)\oplus\mathcal{O}(1)$. Hence $\mathscr{E} \simeq f^*\mathscr{G}\otimes [H_\alpha]$ for some vector bundle $\mathscr{G}$ on $\mathbf{P}^1_\beta$. Restricting to a fiber of $M \to \mathbf{P}^1_\alpha$, we infer that $\mathscr{G}$ is ample. So $\mathscr{G} \simeq \mathcal{O}(2)\oplus\mathcal{O}(1)$. Thus we conclude $\mathscr{E} \simeq [H_\alpha + 2H_\beta]\oplus[H_\alpha + H_\beta]$. Hence $c_2(\mathscr{E}) = 3$ and $s_2(\mathscr{E}) = 9$.

(2.10) Suppose that (M, A) is of the type (2.3; 5_1). Set $L = H_\alpha$. There is a morphism $f\colon M \to \mathbf{P}^1_\beta$ such that $f^*H_\beta = L - E$ and every fiber F of f is $\mathbf{P}^1$. Since $AF = 2$, we infer $\mathscr{E}_F \simeq \mathcal{O}(1)\oplus\mathcal{O}(1)$ for any F. So $\mathscr{E} \simeq f^*\mathscr{G}\otimes L$ for some vector bundle $\mathscr{G}$ on $\mathbf{P}^1_\beta$. Since E is a section of f and $L_E = 0$, $\mathscr{G}$ can be identified with $\mathscr{E}_E$. In view of $AE = 2$, we infer $\mathscr{G} \simeq H_\beta\oplus H_\beta$. So $\mathscr{E} \simeq [2L - E]\oplus[2L - E]$. Hence $c_2(\mathscr{E}) = 3$ and $s_2(\mathscr{E}) = 9$.

Remark. The polarized manifold (P, H) is the Segre product of $(\Sigma_1, 2L - E)$ and $(\mathbf{P}^1_\alpha, H_\alpha)$. On the other hand, (P, H) is a scroll over $\mathbf{P}^1_\alpha\times\mathbf{P}^1_\beta$ via the morphism $i_\alpha\times f$, where i_α is the identity of $\mathbf{P}^1_\alpha$. This gives a vector bundle of the type (2.9). Thus, the polarized manifold (P, H) in case (2.9) is isomorphic to that here. This can be viewed also as a hyperquadric fibration over $\mathbf{P}^1_\beta$. Compare [F6; (3.29)].

(2.11) From now on, we study the case (2.3; 1). We have $2 = (K + A)A = A^2 = c_2(\mathscr{E}) + s_2(\mathscr{E})$. Therefore $d(P, H) = s_2(\mathscr{E}) = c_2(\mathscr{E}) = 1$. Hence $\chi(M, \mathscr{E}) = r\chi(M, \mathcal{O})$ by Riemann–Roch theorem.

We have also $h^2(M, \mathscr{E}) = 0$. Indeed, otherwise, there would be a non-trivial homomorphism $\mathscr{E} \to \mathcal{O}_M[K]$ by Serre duality. This is impossible since $\mathscr{E}$ is ample and $K \sim 0$. Thus $h^0(M, \mathscr{E}) \geqq \chi(M, \mathscr{E}) = r\chi(M, \mathcal{O})$.

(2.12) We will derive a contradiction assuming $\chi(M, \mathcal{O}) > 0$. By classification theory M is a K3-surface or an Enriques surface.

If M is K3, we have $h^0(P, H) = h^0(M, \mathscr{E}) \geqq 2r$. So $\Delta(P, H) = (r+1) + 1 - h^0(P, H) \leqq 2 - r$. By the theory of Δ-genus we infer $r = 2$ and $(P, H) \simeq (\mathbf{P}^3, \mathcal{O}(1))$. This is absurd.

If M is Enriques, let $\tilde{M}$ be the universal covering of M. Let the pull-backs to $\tilde{M}$ be denoted by $\tilde{}$. We have $h^2(\tilde{M}, \tilde{\mathscr{E}}) = 0$ similarly as in (2.11). So $h^0(\tilde{P}, \tilde{H}) = h^0(\tilde{M}, \tilde{\mathscr{E}}) \geqq \chi(\tilde{M}, \tilde{\mathscr{E}}) = 2\chi(M, \mathscr{E}) = 2r$. Since $d(\tilde{P}, \tilde{H}) = 2$, this implies $\Delta(\tilde{P}, \tilde{H}) = r + 3 - h^0(\tilde{P}, \tilde{H}) \leqq 3 - r \leqq 1$.

When $\Delta(\tilde{P}, \tilde{H}) = 0$, then $(\tilde{P}, \tilde{H})$ is a hyperquadric. So $b_2(\tilde{P}) = 1$. This is absurd since $\tilde{P}$ is a scroll over $\tilde{M}$.

When $\Delta(\tilde{P}, \tilde{H}) = 1$, we infer from $d(\tilde{P}, \tilde{H}) = 2$ that $\tilde{P}$ is a double covering of $\mathbf{P}^{r+1}$ with $\tilde{H}$ being the pull-back of $\mathcal{O}(1)$ (see [F2]). Then, by [F3; (3.11)], $b_2(\tilde{P}) = 1$, yielding a contradiction.

(2.13) Thus we have $\chi(M, \mathcal{O}) \leqq 0$. By the classification theory M is a complex torus or a hyperelliptic surface. In the latter case, the Albanese variety is an elliptic curve and the Albanese mapping makes M a fiber bundle over it with all fibers being isomorphic to an elliptic curve. Moreover $b_2(M) = 2$.

(2.14) Lemma. *Let $f: S \to C$ be an analytic fiber bundle over a curve C with all fibers being isomorphic to an elliptic curve. Let $\mathscr{F}$ be an ample locally free sheaf on S of rank $r \geqq 2$ such that $c_1(\mathscr{F})X = 1$ for every fiber X of f. Then*

1) *$\mathscr{L} = f_*\mathscr{F}$ is an invertible sheaf on C with $\delta = \deg(\mathscr{L}) > 0$,*
2) *$\mathscr{C} = \mathrm{Coker}(f^*\mathscr{L} \to \mathscr{F})$ is locally free,*
3) *$\mathscr{G} = f_*\mathscr{C}$ is an invertible sheaf of degree δ,*
4) *f has a section Z such that $\det(\mathscr{F}) = \mathcal{O}_S[Z + f^*B]$ for some line bundle B on C of degree $r\delta$, and*
5) *$c_1(\mathscr{F})^2 = 2r\delta$.*

Proof. We have $h^0(X, \mathscr{F}_X) = c_1(\mathscr{F}_X) = 1$ for any fiber X. Moreover, any section of $\mathscr{F}_X$ comes from a trivial subbundle of $\mathscr{F}_X$. So $\mathscr{L}$ is invertible and $\mathscr{C}$ is locally free. We have also $h^1(X, \mathscr{F}_X) = 0$ and $R^1f_*\mathscr{F} = 0$. So, using the exact sequence $0 \to f^*\mathscr{L} \to \mathscr{F} \to \mathscr{C} \to 0$, we infer $f_*\mathscr{C} \simeq (R^1f_*)(f^*\mathscr{L})$. Let N be a line bundle on C such that f^*N is the relative canonical bundle of f. Then $\deg(N) = 0$ since f is an analytic fiber bundle. By the Grothendieck duality we have $((R^1f_*)(f^*\mathscr{L}))^\vee \simeq f_*((f^*\mathscr{L})^\vee \otimes f^*N) \simeq \mathscr{L}^\vee \otimes N$. Putting things together we obtain $f_*\mathscr{C} \simeq \mathscr{L} \otimes N^\vee$, which implies 3).

In order to show 4), we use the induction on r. If $r = 2$, $\mathscr{C}$ is invertible and $\deg(\mathscr{C}_X) = 1$. So the support Z of $\mathrm{Coker}(f^*\mathscr{G} \to \mathscr{C})$ is a section of f and $\mathscr{C} = f^*\mathscr{G} \otimes [Z]$. Hence $\det(\mathscr{F}) = f^*(\mathscr{L} \otimes \mathscr{G}) \otimes [Z]$, as desired. If $r > 2$, we apply the induction hypothesis to $\mathscr{C}$. So $\det(\mathscr{C}) = \mathcal{O}[Z + f^*U]$ with $\deg(U) = (r-1)\delta$ by 3). Since $\det(\mathscr{F}) = \det(\mathscr{C}) \otimes f^*\mathscr{L}$, we get 4) for $\mathscr{F}$.

By the adjunction formula we have $(Z + N)_Z = 0$. So $Z^2 = 0$, which implies 5). Finally we get $\delta > 0$ since $\det(\mathscr{F})$ is ample. Thus we complete the proof of the lemma.

(2.15) Lemma. *Let $f: S \to C$ be the Albanese fibration of a hyperelliptic surface S and let A be an ample line bundle on S with $A^2 = 2$. Then $|A|$ contains a member of the form $Z + X$, where Z is a section of f and X is a fiber of f.*

Proof. It is known (cf., e.g., [BPV]) that there exists a curve Y in S such that the restriction f_Y of f to Y is étale and is of degree $\mu \leqq 3$. Among such curves we choose and fix a curve Y where μ attains the minimum. Note that $Y^2 = 0$.

Since $h^0(S, A) = \chi(S, A) = 1$, the member D of $|A|$ is unique. Let X be a fiber of f. Then X and Y form a basis of $H^2(S; \mathbf{Q})$ since $b_2(S) = 2$. Set $\mu A \sim yX + xY$. Then $y = AY$, $x = AX$ and $xy = \mu \leqq 3$.

We will derive a contradiction assuming $x > 1$. This implies $x = \mu$ and $y = 1$. We claim that D is irreducible and reduced. Indeed, otherwise, $D = D_1 + D_2$ for some prime divisors D_1, D_2 with $AD_1 = AD_2 = 1$. We may assume $YD_1 = 1$ and $YD_2 = 0$ since $YD = AY = 1$. Then D_1 is not a fiber of f because $\mu > 1$. So $XD_1 > 0$. Hence $XD_2 < AX = \mu$. On the other hand $XD_2 > 0$ since $Y + tX$ is ample for $t \gg 0$. From $Y^2 = YD_2 = 0$ we infer ${D_2}^2 \leqq 0$ by index theorem. But S contains no rational curve and hence ${D_2}^2 \geqq 0$. So ${D_2}^2 = 0$. Hence $D_2 \to C$ is étale and of degree $< \mu$, contradicting the choice of Y. Thus we prove that D is prime.

Let S' be the fiber product of S and Y over C. Then $G = \mathrm{Gal}(Y/C)$ is a cyclic group of order μ. For each $\tau \in G$, let $\sigma_\tau: Y \to S'$ be the morphism induced by the inclusion $Y \to S$ and $f_Y \circ \tau: Y \to C$. Let $Y_\tau = \mathrm{Im}(\sigma_\tau)$. They are μ disjoint sections of $f': S' \to Y$. We have $D'Y_\tau = 1$ for the pull-back D' of D since $AY = 1$.

If S' is a complex torus, then Y_τ's are subtori and $Q = S'/Y_\tau$ is an elliptic curve. Moreover D' is a section of $S' \to Q$ since $D'Y_\tau = 1$. This is absurd since $(D')^2 > 0$ and $g(D') > 1$. Hence S' is hyperelliptic and f' is the Albanese fibration of it.

For any fiber X' of f', we have $D' \sim X' + \sum_{\tau \in G} Y_\tau$ since $b_2(S') = 2$. Hence $D' - X' - \sum_\tau Y_\tau$ comes from $\mathrm{Pic}^0(S') \simeq \mathrm{Pic}^0(Y)$. Hence $D' = \sum_\tau Y_\tau$ in $\mathrm{Pic}(X')$. Let X be the image of X' in S. Then $X' \simeq X$ and the above relation implies $D_X = Y_X$ in $\mathrm{Pic}(X)$. Thus $(A - Y)_X = 0$ in $\mathrm{Pic}(X)$ for every fiber X of f. Hence $\mathscr{L} = f_*(\mathcal{O}[A - Y])$ is invertible on C, $\mathcal{O}[A - Y] = f^*\mathscr{L}$ and $1 = Y(A - Y) = \mu \cdot \deg(\mathscr{L})$. This contradicts $\mu > 1$.

Now we conclude $x = 1$. So $y = \mu$. Moreover $h^0(X, A_X) = 1$ for every fiber X and hence $\mathscr{L} = f_*(\mathcal{O}[A])$ is invertible on C. The support Z of $\mathrm{Coker}(f^*\mathscr{L} \to \mathcal{O}[A])$ is a section of f and $Z + X'' \in |A|$ for $X'' \in |f^*L|$. We have $AZ = AX'' = 1$ and hence X'' is a fiber of f. Thus we complete the proof.

(2.16) Suppose that M is hyperelliptic in (2.13). Let $f: M \to C$ be the Albanese fibration. Then, by (2.15), $AX = 1$ for any fiber X of f. So $2 = A^2 = 2r\delta$ by (2.14; 5). This contradicts $r \geqq 2$. Thus we conclude that M is a complex torus.

(2.17) Since $h^0(M, A) = \chi(M, A) = 1$, there is a unique member C of $|A|$. We claim that C is a smooth curve of genus two.

Indeed, otherwise, $C = X + Y$ for some elliptic curves X, Y with $X^2 = Y^2 = 0$ and $XY = 1$. Then X is a subtorus and $Q = M/X$ is an elliptic curve (in fact $M \simeq X \times Y$). So, applying (2.14) to $f: M \to Q$, we obtain a contradiction.

Thus M is the Jacobian variety of C.

(2.18) Before classifying vector bundles of the above type, we exhibit here examples (cf. [*S*]).

Let o be a point on a smooth curve C of genus two. Let $N_n = C \times \cdots \times C$ be the product of n-copies of C and let $P_n = N_n/S_n$ be the symmetric product. Let M be the Jacobian variety of C and let $\pi_n: P_n \to M$ be the morphism induced by the Albanese mapping of (C, o). Thus, $\pi_n(x_1, \ldots, x_n)$ is the class of $x_1 + \cdots + x_n - no$ in $\mathrm{Pic}(C)$. It is well-known that π_n is a $\mathbf{P}^{n-2}$-bundle for $n > 2$ and π_2 is the blowing-up at the point $p = [\omega - 2o] \in M$.

Let $f_i: N_n \to C$ be the i-th projection and set $\Delta_n = \sum_{i=1}^n f_i^* o$. Let D_n be the divisor on P_n whose pull-back to N_n is Δ_n. Then $D_n^n = 1$ and D_n is ample since so is Δ_n on N_n and $\Delta_n^n = n!$. Moreover $D_n \simeq P_{n-1}$ and the restriction of π_n to D_n is identified with π_{n-1}.

Let H_n be the line bundle $\mathcal{O}[D_n]$ on P_n. We claim that the restriction of H_n to $D_n \simeq P_{n-1}$ is identified with H_{n-1}. Indeed, $N_{n-1} \simeq f_i^{-1}(o)$ and the pull-back of H_n via $f_i^{-1}(o) \to D_n$ is the restriction of $[\Delta_n]$, so it is $[\Delta_{n-1}]$. Going down via $N_{n-1} \to D_n$, we prove the claim.

For any $y \in M$, $D_n \cap \pi_n^{-1}(y)$ is a hyperplane in $\pi_n^{-1}(y) \simeq \mathbf{P}^{n-2}$ unless $n = 3$ and $y = p$. Hence (P_n, H_n) is a scroll over M and $\mathcal{E}_{n-1} = (\pi_n)_* \mathcal{O}[H_n]$ is a vector bundle of rank $n - 1$ for $n > 2$. The section in $H^0(P_n, H_n) \simeq H^0(M, \mathcal{E}_{n-1})$ defining D_n yields an exact sequence $0 \to \mathcal{O}_M \to \mathcal{E}_{n-1} \to \mathcal{E}_{n-2} \to 0$ by the above claim. As for $\mathcal{E}_2$, we have $0 \to \mathcal{O}[F] \to \mu^*\mathcal{E}_2 \to \mathcal{O}[D_2] \to 0$, where $\mu: P_2 \to M$ is the blowing-up at p, F is the exceptional curve over p and $D_2 \simeq P_1 \simeq C$.

Now we see $c_1(\mathcal{E}_r) = s_1(\mathcal{E}_r) = [\mu(D_2)] = [\pi_1(P_1)]$ and $c_2(\mathcal{E}_r) = 1 = s_2(\mathcal{E}_r)$. Thus $\mathcal{E}_r$ is a vector bundle of the type in question. This will be called the *Jacobian bundle* of rank r of (C, o) and will be denoted by $\mathcal{E}_r(C, o)$. $(P_n, H_n) = (\mathbf{P}(\mathcal{E}_{n-1}), \mathcal{O}(1))$ will be called the *Jacobian scroll* of (C, o).

(2.19) We continue to study vector bundles of the type (2.17).

Lemma. *Let $\mu: M' \to M$ be a birational morphism and let $\mathcal{Q}$ be a quotient bundle of $\mu^*\mathcal{E}$. Then $H^2(M', \mathcal{Q} \otimes N') = 0$ for any $N' \in \mathrm{Pic}^0(M')$.*

Proof. We have $h^2(\mathcal{Q} \otimes N') \leqq h^2(\mu^*\mathcal{E} \otimes N') = h^2(M, \mathcal{E} \otimes N)$, where $N \in \mathrm{Pic}^0(M) \simeq \mathrm{Pic}(M')$ and $\mu^* N = N'$. If this is not zero, we have a non-zero homomorphism $\mathcal{E} \to N^\vee$ by Serre duality. This is impossible since $\mathcal{E}$ is ample.

(2.20) Lemma. *Let p be a point on M and let $\mu: M' \to M$ be the blowing-up at p. Then there is no vector bundle $\mathcal{Q}$ on M' satisfying the following properties*:

1) $q = \mathrm{rank}(\mathcal{Q}) > 1$.
2) *$\mathcal{Q}$ is a quotient bundle of $\mu^*\mathcal{E}$.*
3) *$c_1(\mathcal{Q}) = \mu^* A - E_p$, where E_p is the exceptional curve over p.*

4) $c_2(\mathscr{Q}) = 0$.

5) $h^0(\mathscr{Q}) = h^1(\mathscr{Q}) > 0$.

Proof. We will derive a contradiction from these conditions. We use the induction on q. Suppose that $q = 2$. We have a non-zero homomorphism $\mathcal{O} \to \mathscr{Q}$ by 5). Let $\mathscr{Q}^\vee \to \mathcal{O}$ be the dual and let $\mathscr{I}$ be the image of it. Let $M'' \to M'$ be the blowing-up of $\mathscr{I}$ and let F be the divisor defined by the pull-back of $\mathscr{I}$. Then we have an exact sequence $0 \to \mathcal{O}[F] \to \mathscr{Q}'' \to \mathcal{O}[A'' - E_p'' - F] \to 0$ of vector bundles on M'', where the symbol $''$ denote the pull-back to M''. Then $0 = c_2(\mathscr{Q}'') = F(A - E_p - F)$, where we omit the symbol $''$ for the sake of convenience. We consider the restriction of the above sequence over a general member Γ of $|tA|$ for $t \gg 0$. Then $A(A - E_p - F) > 0$ since $A - E_p - F$ is a quotient of the ample vector bundle $\mathscr{E}_\Gamma$. Moreover, $A - E_p - F$ is nef on M'' since it is a quotient of $\mathscr{E}''$. Therefore $F(A - E_p - F) = 0$ implies $0 \leqq F(A - E_p) = F^2 \leqq 0$ by the index theorem. Thus $(A - E_p)F = F^2 = 0$. Since $(A - E_p)^2 = 1$, this implies $F \sim 0$ by the index theorem. Hence $\mathscr{I} = \mathcal{O}$ and $M'' = M'$. We have $0 \to \mathcal{O} \to \mathscr{Q} \to \mathcal{O}[A - E_p] \to 0$ on M'. Recall that $A - E_p$ is nef since it is a quotient of $\mu^*\mathscr{E}$. Hence $H^1(E_p - A) = 0$ by the vanishing theorem. Therefore the above sequence splits and $h^2(\mathscr{Q}) \geqq h^2(\mathcal{O}) = 1$. This contradicts (2.19).

Now we consider the case $q \geqq 3$. Similarly as before, there is a birational morphism $M'' \to M'$, an effective divisor F on M'' and an exact sequence $0 \to \mathcal{O}[F] \to \mathscr{Q}'' \to \mathscr{V} \to 0$ of vector bundles on M''. Since $\mathscr{V}$ is a quotient of $\mathscr{E}''$, we have $c_2(\mathscr{V}) \geqq 0$ and $c_1(\mathscr{V}) = A - E_p - F$ is nef (see Appendix). So $0 = c_2(\mathscr{Q}'') = c_2(\mathscr{V}) + c_1(\mathscr{V})F \geqq 0$ and hence $0 = c_2(\mathscr{V}) = c_1(\mathscr{V})F = F(A - E_p - F)$. Similarly as before we obtain $A(A - E_p - F) > 0$, hence $F^2 \leqq 0$, so $0 = (A - E_p)F = F^2$ and $F = 0$. Therefore $M'' = M'$. It is then clear that $\mathscr{V}$ satisfies 2), 3) and 4). So $\chi(\mathscr{V}) = 0$ by the Riemann–Roch theroem. Hence $h^0(\mathscr{V}) = h^1(\mathscr{V})$ by (2.19). Moreover $h^1(\mathscr{V}) \geqq h^2(\mathcal{O}) = 1$ since $h^2(\mathscr{Q}'') = 0$ by (2.19). Thus $\mathscr{V}$ satisfies 5) too, contradicting the induction hypothesis.

(2.21) Lemma. *Let $\mathscr{E}$ be an ample vector bundle on M such that $r =$ rank $(\mathscr{E}) > 1$, det $(\mathscr{E}) = A$, $c_2(\mathscr{E}) = 1$ and $h^0(\mathscr{E}) > 0$. Then*

1) *$h^0(\mathscr{E}) = h^1(\mathscr{E}) = 1$, and*

2) *the natural mapping $H^1(\mathscr{E}^\vee) \otimes H^0(\mathscr{E}) \to H^1(\mathcal{O}_M)$ is injective, where $\mathscr{E}^\vee$ is the dual bundle of $\mathscr{E}$.*

Proof. We use the induction on r.

i) When $r = 2$, similarly as in (2.20), there are a birational morphism μ: $M'' \to M$, an effective divisor F on M'' and an exact sequence

$$(\#) \quad 0 \to \mathcal{O}[F] \to \mathscr{E}'' \to \mathcal{O}[A'' - F] \to 0$$

of vector bundles on M'', where the symbol $''$ denote the pull-back to M''. We obtain $A''(A'' - F) > 0$ by restricting over a general member of $|tA|$ for $t \gg 0$.

ii) Assume that $A''F > 0$. Then $A''F = 1$ since $(A'')^2 = 2$. So $Z = \mu_*F$ is prime since $AZ = A''F = 1$. Moreover $(A - 2Z)A = 0$ implies $0 \geqq (A - 2Z)^2 =$

$4Z^2 - 2$ by the index theorem. Hence $Z^2 = 0$ and Z is a subtorus of M. This yields a contradiction as in (2.17). Thus we conclude $A''F = 0$.

iii) We have $F^2 = -1$ since $1 = c_2(\mathscr{E}) = F(A'' - F)$. Hence we may assume that μ is the blowing-up at a point p on M and F is the exceptional curve over p.

Using the exact sequence $0 \to \mathcal{O}_{M''} \to \mathcal{O}_{M''}[F] \to \mathcal{O}_F(-1) \to \mathcal{O}$, we get $h^i(M'', F) = h^i(M'', \mathcal{O}_{M''}) = h^i(M, \mathcal{O}_M)$ for any i. So, by the exact sequence (#), we have $h^1(M'', A'' - F) \geqq h^2(M'', F) = 1$ since $h^2(\mathscr{E}'') = 0$ by (2.19). On the other hand $\chi(M'', A'' - F) = 0$ by the Riemann–Roch theorem. Hence $h^0(M'', A'' - F) > 0$. This implies that p is a point on the unique member C of $|A|$. The strict transform Y of C is the unique member of $|A'' - F|$.

iv) Let $e \in H^1(M'', 2F - A'') = H^1(M'', F - Y)$ be the extension class of the exact sequence (#). Then $e_F \in H^1(F, \mathcal{O}(-2))$ is not zero since $\mathscr{E}''_F$ is trivial. On the other hand $\operatorname{Ker}(H^1(M'', F - Y) \to H^1(F, \mathcal{O}(-2)) = H^1(M'', -Y) = 0$ by Ramanujam's vanishing theorem. Hence $H^1(M'', F - Y)$ is one-dimensional and e is a generator of it.

v) Let $0 \to \mathcal{O}[F - Y] \to \mathcal{O}[F] \to \mathcal{O}_Y[F] \to 0$ be the natural exact sequence. Using the long exact sequence we infer that $H^1(M'', F - Y) \to H^1(M'', F)$ is injective since $h^0(M'', F - Y) = 0$, $h^0(M'', F) = 1$ and $h^0(Y, F) = 1$. Hence $H^0(M'', Y) \otimes H^1(M'', F - Y) \to H^1(M'', F)$ is injective. So the mapping $H^0(M'', Y) \to H^1(M'', F)$ induced by (#) is injective. This implies $h^0(M'', \mathscr{E}'') = h^0(M'', F) = 1$ and $h^1(M'', \mathscr{E}'') = 1$.

vi) Since $H^1(M'', \mathcal{O}) \simeq H^1(M'', F)$, the mapping $\varphi\colon H^1(M'', F) \to H^1(M'', \mathscr{E}'')$ induced by (#) is essentially the Serre dual of $H^1(M, \mathscr{E}^{\vee}) \to H^1(M, \mathcal{O})$ induced by a generator of $H^0(\mathscr{E}) \simeq \mathbf{C}$. By the preceding observations we see that φ is surjective. Combining them we prove 2).

vii) Now we consider the case $r > 2$. We claim that there is an exact sequence

$$(\#\#) \quad 0 \to \mathcal{O} \to \mathscr{E} \to \mathscr{Q} \to 0$$

of vector bundles on M.

Indeed, since $h^0(\mathscr{E}) > 0$, there are a birational morphism $\mu\colon M'' \to M$, an effective divisor F on M'' and an exact sequence $0 \to \mathcal{O}[F] \to \mathscr{E}'' \to \mathscr{Q} \to 0$ of vector bundles on M''. It suffices to derive a contradiction assuming $F \neq 0$. We have $A''(A'' - F) > 0$ as in i), and $A''F = 0$ exactly as in ii). This implies $F^2 < 0$ by the index theorem. So $1 = c_2(\mathscr{E}'') = c_2(\mathscr{Q}) + F(A'' - F) = c_2(\mathscr{Q}) - F^2$. On the other hand $c_2(\mathscr{Q}) \geqq 0$ (see Appendix). Hence $F^2 = -1$ and $c_2(\mathscr{Q}) = 0$. So we may assume that μ is the blowing-up at a point and F is the exceptional curve. Since $H^2(\mathscr{E}'') = 0$, we have $h^1(\mathscr{Q}) \geqq h^2(M'', F) = 1$. Now we get a contradiction to (2.20).

viii) Since $c_1(\mathscr{Q}) = A$ and $c_2(\mathscr{Q}) = c_2(\mathscr{E}) = 1$, we have $\chi(\mathscr{Q}) = 0$ by Riemann–Roch theorem. We have also $h^1(\mathscr{Q}) \geqq h^2(\mathcal{O}) - h^2(\mathscr{E}) = 1$. Hence, by the induction hypothesis, $h^0(\mathscr{Q}) = h^1(\mathscr{Q}) = 1$ and $H^1(\mathscr{Q}^{\vee}) \otimes H^0(\mathscr{Q}) \to H^1(\mathcal{O}_M)$ is injective.

ix) Let $e \in H^1(\mathscr{Q}^{\vee})$ be the extension class of the exact sequence (##). The mapping $H^0(\mathscr{Q}) \to H^1(\mathcal{O}_M)$ in the long exact sequence is injective since e is a generator of $H^1(\mathscr{Q}^{\vee})$. Hence $h^0(\mathscr{E}) = h^1(\mathscr{E}) = 1$, proving 1). Moreover, the mapping $H^1(\mathcal{O}_M) \to H^1(\mathscr{E})$ in the long exact sequence is surjective and is the Serre dual of $H^1(\mathscr{E}^\vee) \to H^1(\mathcal{O}_M)$ induced by $\varphi \in H^0(\mathscr{E})$ giving (# #). From this we obtain 2).

Thus we complete the proof of (2.21).

(2.22) Lemma. *Let $\mathscr{E}$ be an ample vector bundle as in* (2.21). *Then there is a point o on C such that $\mathscr{E} \simeq \mathscr{E}_r(C, o)$.*

Proof. We use the induction on r. When $r = 2$, for some point p on C, we have an exact sequence $0 \to \mathcal{O}[F] \to \mu^*\mathscr{E} \to \mathcal{O}[\mu^*A - F] \to 0$ as in (2.21), where $\mu\colon M'' \to M$ is the blowing-up at p and F is the exceptional curve. Let o be the point on C such that $p + o$ is a canonical divisor on C. Then, by (2.18), we have an exact sequence $0 \to \mathcal{O}[F] \to \mu^*\mathscr{E}_2(C, o) \to \mathcal{O}[\mu^*A - F] \to 0$. Since $h^1(2F - \mu^*A) = 1$, these sequences are isomorphic and hence $\mathscr{E} \simeq \mathscr{E}_2(C, o)$. When $r > 2$, let $0 \to \mathcal{O} \to \mathscr{E} \to \mathscr{Q} \to 0$ be the sequence (# #) in (2.21; vii). By the induction hypothesis $\mathscr{Q} \simeq \mathscr{E}_{r-1}(C, o)$ for some point o on C. So (# #) is isomorphic to $0 \to \mathcal{O} \to \mathscr{E}_r(C, o) \to \mathscr{E}_{r-1}(C, o) \to 0$ in (2.18) since $h^1(\mathscr{Q}^\vee) = 1$. Thus we prove the lemma.

(2.23) Now we come back to the situation (2.17). We claim $h^0(\mathscr{E} \otimes N) > 0$ for some $N \in M^*$, the Picard scheme $\mathrm{Pic}^0(M)$ of M.

We will derive a contradiction assuming the contrary. Let L be the Poincaré bundle on $M \times M^*$ and let f (resp. g) be the projection onto M (resp. M^*). Set $\mathscr{F} = f^*\mathscr{E} \otimes L$. Since $\chi(\mathscr{E} \otimes N) = 0$ by Riemann–Roch theorem, we have $h^q(\mathscr{E} \otimes N) = 0$ for any $q \geqq 0$ and $N \in M^*$ by assumption and (2.19). So $R^q g_* \mathscr{F} = 0$ and hence $H^q(M \times M^*, \mathscr{F}) = 0$ for any q.

Let o be the origin of M regarded as the Picard scheme of M^*. Set $M_x^* = f^{-1}(x)$ for $x \in M$ and let L_x (resp. $\mathscr{F}_x$) be the restriction of L (resp. $\mathscr{F}$) to M_x^*. Then $h^q(M_x^*, L_x) = 0$ for any q if $x \neq 0$. So $h^q(M_x^*, \mathscr{F}_x) = 0$ since $\mathscr{F}_x$ is a direct sum of L_x's. Hence $R^q f_* \mathscr{F} = 0$ on $M - \{o\}$. We have in fact $\mathrm{Supp}\,(R^2 f_* \mathscr{F}) = \{o\}$ since $h^2(M_o^*, \mathscr{F}_o) = rh^2(M_o^*, L_o) = r$. Now, using the Leray spectral seuqnce of $\mathscr{F}$ with respect to f, we infer $h^2(M \times M'', \mathscr{F}) = h^0(M, R^2 f_* \mathscr{F}) > 0$, contradicting the preceding observation. Thus we prove the claim.

Combining this claim and (2.22), we infer $\mathscr{E} \simeq \mathscr{E}_r(C, o) \otimes N$ for some point o on C and a numerically trivial line bundle N on M. So $(P, H) \simeq (P_{r+1}(C, o), H_{r+1}(C, o) \otimes \pi^*N)$, where $(P_{r+1}(C, o), H_{r+1}(C, o))$ is the Jacobian scroll of (C, o) as in (2.18).

(2.24) Now we study the case $n = \dim M \geqq 3$ and $g(M, A) = 2$. Let $D_1, \ldots, D_{n-2}$ be general members of $|tA|$ for $t \gg 0$ and let $S = \bigcap_j D_j$. Then S is a smooth surface and $\mathscr{E}_S$ is ample. Hence $A^{n-2}c_2(\mathscr{E}) = t^{2-n}c_2(\mathscr{E}_S) > 0$. Similarly $A^{n-2}s_2(\mathscr{E}) > 0$. So $A^n = (c_2(\mathscr{E}) + s_2(\mathscr{E}))A^{n-2} \geqq 2$. In addition, by (1.3), we have $AC \geqq r \geqq 2$ for any rational curve C in M. Therefore, using [F6; (1.10)], we infer that M is a double covering of $\mathbf{P}_\alpha^n$ with branch locus being a smooth hypersurface of degree six and A is the pull-back of H_α. The intersection T of $(n - 2)$ general members of

$|A|$ is a K3-surface. The restriction $\mathscr{E}_T$ is ample and $g(T, \det(\mathscr{E}_T)) = 2$. This contradicts (2.12).

Thus, the case $n \geqq 3$ is ruled out.

(2.25) Summarizing we obtain the following

Theorem. *Let $\mathscr{E}$ be an ample vector bundle of rank $r \geqq 2$ on a manifold M of dimension $n \geqq 2$. Suppose that $g(M, A) = 2$ for $A = \det(\mathscr{E})$. Then $n = 2$ and one of the following conditions is satisfied. The associated scroll of $(M, \mathscr{E})$ is denoted by (P, H) below.*

1) *M is the Jacobian variety of a smooth curve C of genus two and $\mathscr{E} \simeq \mathscr{E}_r(C, o) \otimes N$ for some numerically trivial line bundle N on M, where $\mathscr{E}_r(C, o)$ is the Jacobian bundle for some point o on C (cf. (2.18)). $A^2 = 2$ and $H^{r+1} = 1$.*

2) *$M \simeq \mathbf{P}(\mathscr{F})$ for some stable vector bundle $\mathscr{F}$ of rank two on an elliptic curve C with $c_1(\mathscr{F}) = 1$. There is an exact sequence $0 \to \mathcal{O}_M[2H(\mathscr{F}) + \rho^*G] \to \mathscr{E} \to \mathcal{O}_M[H(\mathscr{F}) + \rho^*T] \to 0$, where G and T are line bundles on C and ρ is the morphism $M \to C$. $A^2 = 3$ and we have either*

2-i) $\deg(T) = 1$, $\deg(G) = -2$ *and* $H^3 = 1$, *or*

2-ii) $\deg(T) = 0$, $\deg(G) = -1$ *and* $H^3 = 2$ *(cf. (2.7)).*

$2^{\#}$) *M, $\mathscr{F}$, C and ρ are as in 2) and $\mathscr{E} \simeq \rho^*\mathscr{G} \otimes H(\mathscr{F})$ for some stable vector bundle $\mathscr{G}$ of rank three on C with $c_1(\mathscr{G}) = -1$. $A^2 = 3$ and $H^4 = 2$ (cf. (2.6)).*

3) *$M \simeq \mathbf{P}(\mathscr{F})$ and $\mathscr{E} \simeq \rho^*\mathscr{G} \otimes H(\mathscr{F})$ for some semistable vector bundles $\mathscr{F}$ and $\mathscr{G}$ of rank two on an elliptic curve C, where ρ is the morphism $M \to C$. Moreover $(c_1(\mathscr{F}), c_1(\mathscr{G})) = (1, 0)$ or $(0, 1)$. P is the fiber product of $\mathbf{P}(\mathscr{F})$ and $\mathbf{P}(\mathscr{G})$ over C. $A^2 = 4$ and $H^3 = 3$ (cf. (2.4)).*

4) *$-K$ is ample, $K^2 = 1$ and $A = -2K$. M is the blowing-up of $\mathbf{P}^2$ at eight points. Moreover we have either*

4-a) *$\mathscr{E} \simeq [-K] \oplus [-K]$ and $H^3 = 3$, or*

4-b) *$c_2(\mathscr{E}) = 3$, $r = 2$ and $H^3 = 1$ (cf. (2.8)).*

5_0) *$M \simeq \mathbf{P}^1_\alpha \times \mathbf{P}^1_\beta$ and $\mathscr{E} \simeq [H_\alpha + 2H_\beta] \oplus [H_\alpha + H_\beta]$. $A^2 = 12$ and $H^3 = 9$ (cf. (2.9)).*

5_1) *M is the blowing-up of $\mathbf{P}^2_\alpha$ at a point and $\mathscr{E} \simeq [2H_\alpha - E] \oplus [2H_\alpha - E]$, where H_α is the pull-back of $\mathcal{O}(1)$ of $\mathbf{P}^2_\alpha$ and E is the exceptional curve. $A^2 = 12$ and $H^3 = 9$ (cf. (2.10)).*

Remark. The existence of a vector bundle of the above type 2) is uncertain. The others do really exist.

§3. $\mathcal{O}(1)$-sectional genus

(3.1) For an ample vector bundle $\mathscr{E}$ of rank $r \geqq 2$ on a manifold M, the $\mathcal{O}(1)$-sectional genus is defined to be $g(P, H)$, where $P = \mathbf{P}(\mathscr{E})$ and H is the tautological line bundle on it. As a part of classification theory of polarized manifolds of small sectional genera (cf. [F5] and [F6]), we get the following results. Proofs are easy and omitted.

(3.2) Theorem *$g(P, H) = 0$ if and only if either*

1) *$M \simeq \mathbf{P}^1$, or*
2) *$M \simeq \mathbf{P}^n_\alpha$ and $\mathscr{E} \simeq H_\alpha \oplus H_\alpha$.*

(3.3) Theorem. *$g(P, H) = 1$ if and only if*

1) *M is an elliptic curve,*
2) *$M \simeq \mathbf{P}^2_\alpha$ and $\mathscr{E} \simeq 2H_\alpha \oplus H_\alpha$,*
3) *$M \simeq \mathbf{P}^2_\alpha$ and $\mathscr{E} \simeq H_\alpha \oplus H_\alpha \oplus H_\alpha$,*
4) *$M \simeq \mathbf{P}^2_\alpha$ and $\mathscr{E}$ is the tangent bundle, or*
5) *$M \simeq \mathbf{P}^1_\sigma \times \mathbf{P}^1_\tau$ and $\mathscr{E} \simeq [H_\sigma + H_\tau] \oplus [H_\sigma + H_\tau]$.*

(3.4) Theorem. *$g(P, H) = 2$ if and only if*

1) *M is a smooth curve of genus two,*
2) *$(M, \mathscr{E})$ is of one of the types in (2.25), or*
3) *M is a hyperquadric in $\mathbf{P}^4$ and $\mathscr{E} \simeq \mathscr{O}(1) \oplus \mathscr{O}(1)$.*

Appendix. Chern classes of semipositive vector bundles

Definition. A vector bundle $\mathscr{E}$ on a variety V is said to be semipositive if the tautological line bundle $H(\mathscr{E})$ on $\mathbf{P}_V(\mathscr{E})$ is nef, i.e., $H(\mathscr{E})C \geqq 0$ for any curve C in $\mathbf{P}_V(\mathscr{E})$.

The following facts are obvious by definition.

(1) *$f^*\mathscr{E}$ is semipositive for any morphism $f\colon W \to V$.*

(2) *Any quotient bundle of $\mathscr{E}$ is semipositive.*

(3) *$\mathscr{E} \otimes A$ is ample for any ample line bundle A.*

Besides these, many (I should say *most*) results on ample vector bundles have semipositive versions. For example we have:

Theorem. *$c_n(\mathscr{E}) \geqq 0$ for $n = \dim V$.*

This fact is well-known among experts, but I do not know a good reference. So we give here a proof since we use this in the text.

Proof of the theorem. By base change we may assume that V is smooth, projective and hence is a submanifold of $\mathbf{P}^N_\alpha$ with homogeneous coordinate $(\alpha_0 : \cdots : \alpha_N)$. We may further assume that the hyperplane section $D_i = V \cap \{\alpha_i = 0\}$ is smooth for each i and that $D = D_0 + \cdots + D_N$ has no singularity other than normal crossings. Suppose that $c_n(\mathscr{E}) < 0$. Then $\sum_{j=0}^n c_{n-j}(\mathscr{E}) H^j_\alpha / m^j < 0$ for some large integer m. Let $f\colon \mathbf{P}^N_\beta \to \mathbf{P}^N_\alpha$ be the morphism defined by $f(\beta_0 : \cdots : \beta_N) = (\beta_0^m : \cdots : \beta_N^m)$. Then $M = f^{-1}(V)$ is smooth and $f^*H_\alpha = mH_\beta$. Hence $c_n(\mathscr{E}_M \otimes H_\beta) = \sum_{j=0}^n c_{n-j}(\mathscr{E}_M) H^j_\beta < 0$ by the choice of m. This contradicts [BG] since $\mathscr{E}_M \otimes H_\beta$ is ample on M by (1) and (3). Thus we conclude $c_n(\mathscr{E}) \geqq 0$.

Corollary. *$c_1(\mathscr{E}) = \det(\mathscr{E})$ is nef.*

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