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Cauchy problem for abstract evolution equations of parabolic type

By

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§0. Introduction

In this paper, we treat the Cauchy problem of the abstract evolution equation

$$(0.1)\qquad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}u + A(t)u = f(t) & 0 < t \leqq T_0 \\ u(0) = u_0 \end{cases}$$

in a Banach space $\mathbf{E}$, where $u(t)$ and $f(t)$ are $\mathbf{E}$-valued functions on $[0, T_0]$. $A(t)$ is a function on $[0, T_0]$ to the set of unbounded operators acting on $\mathbf{E}$ and we assume that the definition domain of $A(t)$ is dense in $\mathbf{E}$ and independent of t.

On this type of equation H. Tanabe [4] and P. E. Sobolevski [3] treated under the condition that $A(t)A(s)^{-1}$ is uniformly Hörder continuous in t in the uniform operator topology.

The object of this paper is to weaken such an assumption on the continuity of $A(t)A(s)^{-1}$. Our weaker assumptions are presented in §I (particularly condition (A_4)). We prove the existence and uniqueness theorem about (0.1) by constructing the fundamental solution. We use Sobolevski's method. (see also A. Friedman [1]).

Our new condition contains the case that

$$\| A(t)A(s)^{-1} - A(\tau)A(s)^{-1} \| \leqq C|\log|t-\tau||^{\alpha},$$

in a neighborhood of $t = \tau$, where $\alpha < -2$ and C is a positive constant. Remark that $\omega(r) = C|\log r|^{\alpha}$ is not uniformly Hörder continuous near $r = 0$. Regarding the case that the definition domain of $A(t)$ depends on t, see T. Kato and H. Tanabe [2] and A. Yagi [5].

The paper is organized as follows. In §I we present the basic assumptions and, definitions and the statement of the Main Theorem. In §II we construct fundamental solution and prove its strong continuity. In §III we prove local strong differentiability of fundamental solution near the diagonal set. In §IV, by establishing local uniqueness of fundamental solution near the diagonal set, we prove global existence and uniqueness of fundamental solution. We then prove

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our Main Theorem in § V. In § VI we give an application of our Main Theorem to an initial-boundary value problem of a parabolic equation. Finally, in § VII we consider an example about the necessity of regularity assumption.

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§ I. Assumptions and The Main Theorem

We consider the initial value problem of an evolution equation

$$\tag{1.1} \begin{cases} \dfrac{d}{dt}u(t) + A(t)u(t) = f(t) \\ u(0) = u_0, \end{cases}$$

where $0 < t \leqq T_0 < \infty$ or $0 < t < \infty$.

Here $u(t)$ is the unknown function with values in a Banach space $\mathbf{E}$, $A(t)$ is an unbounded linear operator acting on $\mathbf{E}$, $f(t)$ is a given function with values in $\mathbf{E}$, $u_0 \in \mathbf{E}$ and T_0 is a given positive number. We denote by $\mathbf{B(E)}$ the Banach space of all bounded linear operators in $\mathbf{E}$ endowed with the topology defined by operator norm. In what follows we restrict ourselves to the case of $0 < t \leqq T_0 < \infty$. We need the following assumptions (A_1), (A_2), (A_3) and (A_4) on $A(t)$. If the constants C_1, C_2 and C_3 appearing in (1.2), (1.3) and (1.4) can be chosen independently of T_0 as $T_0 \to \infty$, then T_0 is considered as finite but arbitrarily large.

(A_1): For each $t \in [0, T_0]$ the domain of $A(t)$ is dense in $\mathbf{E}$ and is independent of t, and $A(t)$ is a closed operator.

(A_2): For each $t \in [0, T_0]$ $(A(t) + \lambda I)^{-1}$ exists for any λ with $\mathrm{Re}\lambda \geqq 0$ and

$$\tag{1.2} \|(A(t) + \lambda I)^{-1}\| \leqq \frac{C_1}{|\lambda| + 1},$$

where C_1 is a positive constant independent both of t and λ.

Under the assumption (A_2) each $-A(t)(t \in [0, T_0])$ generates an analytic semigroup $\exp(-\tau A(t))(\tau \geqq 0)$ and there exists a positive constant C_2 independent both of t and s suth that

$$\tag{1.3} \|A(t)^n \exp(-\tau A(t))\| \leqq \frac{C_2}{\tau^n},$$

where $n = 0, 1, 2,\ \tau > 0,\ t \in [0, T_0]$.

(A_3): There exists a positive constant C_3 independent both of t and s such that

$$\tag{1.4} \|A(t)A(s)^{-1}\| \leqq C_3 \quad \text{for } t, s \in [0, T_0].$$

(A_4): For any $t, s, \tau \in [0, T_0]$

$$\tag{1.5} \|(A(t) - A(\tau))A(s)^{-1}\| \leqq \omega(|t - \tau|),$$

where $\omega(r)$ is a positive and monotone increasing function defined in the interval $(0, \infty)$ and satisfying the following conditions (1.6) and (1.7).

That is, putting $\tilde{\omega}(r) = \omega(r)/r$,

$$\int_0^{T_0} \tilde{\omega}(r)|\log r|\mathrm{d}r < \infty \tag{1.6}$$

$$\omega(r)|\log r| \longrightarrow 0 \ \textit{as} \ r \downarrow 0. \tag{1.7}$$

From (1.6) it is easy to see that

$$\int_0^{T_0} \tilde{\omega}(r)\mathrm{d}r < \infty \tag{1.8}$$

Recall that in Sobolevski [1] $\omega(r) = r^\alpha$ $(0 < \alpha \leqq 1)$.

Here we define a fundamental solution.

Definition. An operator-valued function $U(t, \tau)$, with values in $\mathbf{B(E)}$, defined and strongly continuous in t, τ for $0 \leqq \tau \leqq t \leqq T_0$ is called a fundamental solution of the homogeneous equation

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) + A(t)u(t) = 0 \qquad \text{for } 0 < t \leqq T_0 \tag{1.9}$$

if

(1): the derivative $\dfrac{\partial}{\partial t}U(t, \tau)$ exists in the strong topology and belongs to $\mathbf{B(E)}$ for $0 \leqq \tau < t \leqq T_0$, and it is also strongly continuous in t, τ for $0 \leqq \tau < t \leqq T_0$;

(2): the range of $U(t, \tau)$ is included in D_A for $0 \leqq \tau < t \leqq T_0$;

and if

(3): for any $x \in \mathbf{E}$

$$\frac{\partial}{\partial t}U(t, \tau)x + A(t)U(t, \tau)x = 0 \qquad \text{for } 0 \leqq \tau < t \leqq T_0 \tag{1.10}$$

$$\lim_{t \downarrow \tau} U(t, \tau)x = x. \tag{1.11}$$

Proposition. *Assume that the conditions* (A_1), (A_2), (A_3) *and* (A_4) *hold. Then there exists a fundamental solution* $U(t, \tau)$ *of the homogeneous equation* (1.9).

We now state the assumption (F) for the inhomogeneous term $f(t)$;

(F): $f(t)$ is a continuous function with values in $\mathbf{E}$ and satisfies

$$\int_\tau^t \frac{\|f(t) - f(s)\|}{t - s}\mathrm{d}s \longrightarrow 0 \ \textit{as} \ t - \tau \downarrow 0, \tag{1.12}$$

provided that $0 \leqq \tau < t \leqq T_0$.

We define a strong solution of the initial value problem (1.1).

Definition. A function $u(t)$ with values in $\mathbf{E}$ is called a strong solution of the initial value problem (1.1), if

(1): $u(t)$ is continuous in t for $0 \leqq t \leqq T_0$;

(2): $u(t)$ is continuously differentiable in t for $0 < t \leqq T_0$;

and if

(3): $u(t)$ satisfies (1.1).

The main theorem of the prosent paper is the following.

Theorem. *Assume that the condition* (A_1), (A_2), (A_3), *and* (A_4) *for* $A(t)$ *and* (F) *for* $f(t)$ *are satisfied. Then for any given* $u_0 \in \mathbf{E}$, *there existits a unique strong sulution* $u(t)$ *of the initial value problem* (1.1). *Moreover, if* $u_0 \in D_A$ *then* $u(t)$ *is continuously differentiable up to* $t = 0$. *This solution can be expressed by*

$$u(t) = U(t, 0)u_0 + \int_0^t U(t, s)f(s)\mathrm{d}s \qquad \text{for } 0 \leqq t \leqq T_0, \tag{1.13}$$

where $U(t, \tau)$ is a fundamental solution of (1.9).

§ II. Construction and continuity of a fundamental solution

In this section, we define at first a fundamental solution $U(t, \tau)$ as a solution of an integral equation. We prove also the strong continuity of $U(t, \tau)$ with respect to t, τ. The differentiability with respect to t will be investigated in the next section.

II-1. Formal construction of a fundamental solution. By definition, $U(t, \tau)$ is a solution of the problem

$$\begin{cases} \dfrac{\partial}{\partial t} U(t, \tau) + A(t)U(t, \tau) = 0 & \text{for } \tau < t \\ U(\tau, \tau) = I. \end{cases} \tag{2.1}$$

So, if we put

$$W(t, \tau) = U(t, \tau) - \exp(-(t - \tau)A(\tau)),$$

$W(t, \tau)$ satisfies

$$\begin{cases} \dfrac{\partial}{\partial t} W(t, \tau) + A(t)W(t, \tau) = (A(\tau) - A(t))\exp(-(t - \tau)A(\tau)) \\ W(\tau, \tau) = 0. \end{cases}$$

So let us put

$$\phi_1(t, \tau) = (A(\tau) - A(t))\exp(-(t - \tau)A(\tau)) \tag{2.2}$$

Then by virtue of Duhamel's Principle we get

$$w(t, \tau) = \int_\tau^t U(t, s)\,\phi_1(s, \tau)\mathrm{d}s.$$

Therefore $U(t, \tau)$ is a solution of the integral equation of Volterra type

$$U(t, \tau) = \exp(-(t-\tau)A(\tau)) + \int_\tau^t U(t, s)\,\phi_1(s, \tau)\mathrm{d}s. \tag{2.3}$$

In (2.3) and in the sequel, the integrals of operator-valued functions are defined in the sense of Riemann with respect to the strong topology. A solution of (2.3) can be obtained by successive approximation, that is, if we put

$$\phi_{k+1}(t, \tau) = \int_\tau^t \phi_k(t, s)\,\phi_1(s, \tau)\mathrm{d}s \tag{2.4}$$

for $k = 1, 2, 3, \cdots,$

$$\Phi(t, \tau) = \sum_{k=1}^{\infty} \phi_k(t, \tau), \tag{2.5}$$

and

$$U(t, \tau) = \exp(-(t-\tau)A(\tau)) + \int_\tau^t \exp(-(t-s)A(s))\,\Phi(s, \tau)\mathrm{d}s, \tag{2.6}$$

then $U(t, \tau)$ is a solution of (2.3). And if we use Fubini's theorem we see that $\Phi(t, \tau)$ satisfies

$$\Phi(t, \tau) = \phi_1(t, \tau) + \int_\tau^t \phi_1(t, s)\,\Phi(s, \tau)\mathrm{d}s. \tag{2.7}$$

II-2. Convergence of the series $\Phi(t, \tau)$. Under the assumption (1.8), there exists a positive constant δ_0 such that

$$C_2 \int_0^{\delta_0} \tilde{\omega}(r)\mathrm{d}r < 1/2$$

and let us write

$$C_4 = 2C_2 \int_0^{\delta_0} \tilde{\omega}(r)\mathrm{d}r \qquad (<1). \tag{2.8}$$

We prove the convergence of the series $\Phi(t, \tau)$ for $0 < t - \tau \leqq \delta_0$. For this, we estimate the integral

$$\int_0^r \tilde{\omega}(r-s)\tilde{\omega}(s)\mathrm{d}s.$$

Lemma 2.1.

(2.9)
$$\int_0^r \tilde{\omega}(r-s)\tilde{\omega}(s)\,ds \leqq 2\tilde{\omega}(r)\int_0^r \tilde{\omega}(s)\,ds.$$

Proof.

$$\int_0^r \tilde{\omega}(r-s)\tilde{\omega}(s)\,ds = \int_0^{r/2} \tilde{\omega}(r-s)\tilde{\omega}(s)\,ds + \int_{r/2}^r \tilde{\omega}(r-s)\tilde{\omega}(s)\,ds$$
$$= 2\int_0^{r/2} \tilde{\omega}(r-s)\tilde{\omega}(s)\,ds$$

since $\omega(r)$ is monotone increasing,

$$\leqq 2\omega(r)\int_0^{r/2} \frac{\tilde{\omega}(s)}{r-s}\,ds$$
$$= 2\tilde{\omega}(r)\int_0^{r/2}\left(\frac{1}{r-s}+\frac{1}{s}\right)\omega(s)\,ds$$
$$\leqq 2\tilde{\omega}(r)\int_0^r \tilde{\omega}(s)\,ds.$$

Q.E.D.

Now we prove the convergence of $\Phi(t, \tau)$. By virtue of (1.3), (1.5) and (2.2) we get

(2.10)
$$\|\phi_1(t, \tau)\| = \|(A(t)-A(\tau))A(\tau)^{-1}\, A(\tau)\exp(-(t-\tau)A(\tau))\| \leqq C_2\tilde{\omega}(t-\tau)$$

Using lemma 2.1, it is easy to see that

(2.11)
$$\|\phi_k(t.\ \tau)\| \leqq C_2 C_4^{k-1}\tilde{\omega}(t-\tau)$$

for $k = 1, 2, 3, \cdots$.
Since $C_4 < 1$, the series $\Phi(t, \tau)$ converges and satisfies

(2.12)
$$\|\Phi(t, \tau)\| \leqq \frac{C_2}{1-C_4}\tilde{\omega}(t-\tau)$$

II-3. Strong continuity. We prove that $U(t, \tau)$, which is given by the formura (2.6), belongs to **B(E)** and strongly continuous in t, τ for $0 \leqq t-\tau \leqq \delta_0$. We need some lemmas.

Lemma 2.2. *The following inequalities hold for all* t, s, ξ, $\eta \in [0, T_0]$ and $\tau \geqq 0$

(2.13)
$$\|(A(t)-A(s))\exp(-\tau A(\xi))\| \leqq C\frac{\omega(|t-s|)}{\tau}$$

(2,14)
$$\|(\exp(-\tau A(t)) - \exp(-\tau A(s)))A(\eta)^{-1}\| \leqq C\tau\omega(|t-s|)$$

(2.15)
$$\|\exp(-\tau A(t)) - \exp(-\tau A(s))\| \leqq C\omega(|t-s|)$$

(2.16)
$$\|A(\xi)(\exp(-\tau A(t)) - \exp(-\tau A(s)))A(\eta)^{-1}\| \leqq C\omega(|t-s|)$$

(2.17) $$\| A(\xi)(\exp(-\tau A(t)) - \exp(-\tau A(s)))\| \leqq C\frac{\omega(|t-s|)}{\tau}$$

where C denotes a positive constant independent all of t, s, ξ, η, τ.

Proof. Writing

$$(A(t) - A(s))\exp(-\tau A(\xi)) = (A(t) - A(s))A(\xi)^{-1}A(\xi)\exp(-\tau A(\xi))$$

and use (1.3) and (1.5), (2.13) holds.
Let us define an operator valued function

$$\phi(\zeta) = \exp(-(\tau - \zeta)A(t))\exp(-\zeta A(s))A(s)^{-1}.$$

Then $\phi(\zeta)$ is continuously differentiable in the sense of operator norm. So integrating $\phi'(\zeta)$ from 0 to τ, we have

(2.18) $$\begin{aligned}&(\exp(-\tau A(s)) - \exp(-\tau A(t)))A(s)^{-1}\\ &= \int_0^\tau \exp(-(\tau-\zeta)A(t))(A(t) - A(s))A(s)^{-1}\exp(-\zeta A(s))d\zeta.\end{aligned}$$

Using (1.3), (1.5) and (2.18), we get (2.14).
To prove (2.15), (2.16) and (2.17) we divide $\exp(-\tau A(t)) - \exp(-\tau A(s))$ into several parts. Writing

$$I_1 = \left(\exp\left(-\frac{\tau}{2}A(t)\right) - \exp\left(-\frac{\tau}{2}A(s)\right)\right)A(s)^{-1}A(s)\exp\left(-\frac{\tau}{2}A(s)\right),$$

$$I_2 = \exp\left(-\frac{\tau}{2}A(t)\right)\left(\exp\left(-\frac{\tau}{2}A(t)\right) - \exp\left(-\frac{\tau}{2}A(s)\right)\right),$$

we get

$$\begin{aligned}&\exp(-\tau A(t)) - \exp(-\tau A(s))\\ &= I_1 + I_2.\end{aligned}$$

Writing

$$J_1 = \exp(-\tau A(t))(A(s) - A(t))A(s)^{-1}$$

$$J_2 = A(t)\exp\left(-\frac{\tau}{2}A(t)\right)\left(\exp\left(-\frac{\tau}{2}A(t)\right) - \exp\left(-\frac{\tau}{2}A(s)\right)\right)A(s)^{-1}$$

$$J_3 = \exp\left(-\frac{\tau}{2}A(t)\right)(A(t) - A(s))A(s)^{-1}\exp\left(-\frac{\tau}{2}A(s)\right),$$

we get

$$I_2 = J_1 + J_2 + J_3.$$

And writing

$$K_1 = (A(s) - A(t))\exp\left(-\frac{\tau}{2}A(t)\right)\exp\left(-\frac{\tau}{2}A(s)\right)$$

$$K_2 = \exp\left(-\frac{\tau}{2}A(t)\right)(A(t) - A(s))\exp\left(-\frac{\tau}{2}A(s)\right)$$

$$K_3 = \left(\exp\left(-\frac{\tau}{2}A(t)\right) - \exp\left(-\frac{\tau}{2}A(s)\right)\right)A(s)\exp\left(-\frac{\tau}{2}A(s)\right),$$

we get

$$A(s)I_1 = K_1 + K_2 + K_3.$$

Using (1.3), (1.5) and (2.14), we get the bounds $\|I_1\|, \|J_1\|, \|J_2\|, \|J_3\| \leqq C\omega(|t-s|)$, so we have (2.15).

In view of (1.4) it suffices to consider the case $\xi = s$ for the proof of (2.17). Using (1.3), (2.13) and (2.15), we get the bounds $\|A(s)I_2\|, \|K_1\|, \|K_2\|, \|K_3\| \leqq C\dfrac{\omega(|t-s|)}{\tau}$, so we get (2.17). The proof of (2.16) is similar to that of (2.17). Q.E.D.

Lemma 2.3. *The following inequalities hold for all* $\tau, \xi, \eta \in [0, T_0]$ *and* $t, s \geqq 0$.

$$\|(\exp(-tA(\tau)) - \exp(-sA(\tau)))A(\eta)^{-1}\| \leqq C|t-s| \tag{2.19}$$

$$\|A(\xi)(\exp(-tA(\tau)) - \exp(-sA(\tau)))A(\eta)^{-1}\| \leqq \frac{C|t-s|}{\min(t, s)} \tag{2.20}$$

where C *denotes a constant independent all of* t, s, τ, ξ, η.

Proof. Since

$$\begin{aligned} A(\tau)^n(\exp(-tA(\tau)) - \exp(-sA(\tau)))A(\tau)^{-1} \\ = \int_s^t A(\tau)^n \exp(-\zeta A(\tau))d\zeta \end{aligned} \tag{2.21}$$

where $n = 0, 1$ and $t, s > 0$ when $n = 1$,
using (1.3) and (1.4), we get (2.18) and (2.19). Q.E.D.

Lemma 2.4. *The following inequality hold for all* t, s, ξ.

$$\|A(\xi)(A(t)^{-1} - A(s)^{-1})\| \leqq C\omega(|t-s|) \tag{2.22}$$

$$\|A(t)^{-1} - A(s)^{-1}\| \leqq C\omega(|t-s|) \tag{2.13}$$

where C *denotes a positive constsant independent all of* t, s, ξ.

Proof. Using (1.4) and (1.5) we see that

$$\begin{aligned} \|A(\xi)(A(t)^{-1} - A(s)^{-1})\| &= \|A(\xi)A(t)^{-1}(A(t) - A(s))A(s)^{-1}\| \\ &\leqq C\omega(|t-s|). \end{aligned}$$

So we get (2.22). And (2.23) is clear from (2.22) and (1.4). Q.E.D.

Lemma 2.5. *The operator-valued function* $A(t)\exp(-\tau A(s))$ *is uniformly continuous in the sense of operator norm with respect to* t, τ, s *where* $0 \leqq t \leqq T_0$, $\varepsilon \leqq \tau \leqq T_0$, $0 \leqq s \leqq T_0$ *for any positive number* ε.

Proof. If $0 \leqq t + \Delta t \leqq T_0$, $\varepsilon \leqq \tau + \Delta\tau \leqq T_0$, $0 \leqq s + \Delta s \leqq T_0$ then

$$\begin{aligned}
&A(t + \Delta t)\exp(-(\tau + \Delta\tau)A(s + \Delta s)) - A(t)\exp(-\tau A(s)) \\
&\quad = (A(t + \Delta t) - A(t))A(s + \Delta s)^{-1}A(s + \Delta s)\exp(-(\tau + \Delta\tau)A(s + \Delta s)) \\
&\qquad + \left(A(t)\left(\exp\left(-\left(\frac{\tau}{2} + \Delta\tau\right)A(s + \Delta s)\right)\right.\right. \\
&\qquad \left.\left. - \exp\left(-\frac{\tau}{2}A(s + \Delta s)\right)\right)A(s + \Delta s)^{-1}A(s + \Delta s)\exp\left(-\frac{\tau}{2}A(s + \Delta s)\right)\right) \\
&\qquad + A(t)(\exp(-\tau A(s + \Delta s)) - \exp(-\tau A(s)))
\end{aligned}$$

Applying (1.3), (1.5), (2.17) and (2.20), the lemma follows. Q.E.D.

Lemma 2.6. *The operator-valued functions* $(A(\tau) - A(t))\exp(-(t - \tau)A(t))$, $(A(\tau) - A(t))\exp(-(t - \tau)A(t))$, $\exp(-(t - \tau)A(t))$, $\exp(-(t - \tau)A(\tau))$ *are uniformly continuous in the sense of operator norm with respect to* t, τ *provided* $t - \tau \geqq \varepsilon$, $t, \tau \in [0, T_0]$, *for any positive number* ε.

Proof. The continuity of $(A(\tau) - A(t))\exp(-(t - \tau)A(\tau))$ and $(A(\tau) - A(t))\exp(-(t - \tau)A(t))$ is obvious from lemma 2.5. Writing $\exp(-(t - \tau)A(\tau)) = A(\tau)\exp(-(t - \tau)A(\tau))A(\tau)^{-1}$ and $\exp(-(t - \tau)A(t)) = A(t)\exp(-(t - \tau)A(t))A(t)^{-1}$, the continuity of these two functions is obvious from (1.3), (2.23) and lemma 2.5.

Lemma 2.7. *For any* $x \in \mathbf{E}$, the function $A(t)\exp(-\tau A(s))A(\xi)^{-1}x$ *is continuous with respect to* $t, \tau, s, \xi \in [0, T_0]$.

Proof. Writing

$$\begin{aligned}
I_1 &= (A(t+\Delta t) - A(t))A(s+\Delta s)^{-1}\exp(-(\tau+\Delta\tau)A(s+\Delta s))A(s+\Delta s)A(\xi+\Delta\xi)^{-1} \\
I_2 &= A(t)\exp(-(\tau+\Delta\tau)A(s+\Delta s))A(\xi)^{-1}(A(\xi) - A(\xi+\Delta\xi))A(\xi+\Delta\xi)^{-1} \\
I_3 &= A(t)(\exp(-(\tau+\Delta\tau)A(s+\Delta s)) - \exp(-(\tau+\Delta\tau)A(s)))A(\xi)^{-1} \\
I_4 &= A(t)A(s)^{-1}(\exp(-(\tau+\Delta\tau)A(s)) - \exp(-\tau A(s)))A(s)A(\xi)^{-1},
\end{aligned}$$

we get

$$\begin{aligned}
&A(t + \Delta t)\exp(-(\tau + \Delta\tau)A(s + \Delta s))A(\xi + \Delta\xi)^{-1} - A(t)\exp(-\tau A(s))A(\xi)^{-1} \\
&\quad = I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Using (1.3), (1.4), (1.5) and (2.16) we see that $\|I_1\| \leqq C\omega(|\Delta t|)$, $\|I_2\| \leqq C\omega(|\Delta\xi|)$ and $\|I_3\| \leqq C\omega(|\Delta s|)$. As for I_4, from the strong continuity of the semigroup exp

$(-\tau A(s))$ and (1.4), for any $x \in \mathbf{E}$

$$\|I_4 x\| \longrightarrow 0 \qquad as\ \Delta\tau \longrightarrow 0$$

Thus we have proved this lemma. Q.E.D.

Corollary. *The operator-valued functions* $\exp(-(t-\tau)A(\tau))$, $\exp(-(t-\tau)A(t))$, $(A(\tau)-A(t))\exp(-(t-\tau)A(\tau))A(\tau)^{-1}$ *are strongly continuous in* t, τ, *where* $0 \leqq \tau \leqq t \leqq T_0$.

Lemma 2.8. If $0 \leqq t-\tau \leqq \delta_0$, $U(t, \tau)$, which is given by the formula (2.6), defines a $\mathbf{B(E)}$-valued function. And $U(t, \tau)$ is strongly continuous with respect to t, τ. Morever if $0 < t-\tau \leqq \delta_0$, $U(t, \tau)$ is continuous in the sense of operator norm.

Proof. From the convergence of the series $\Phi(t, \tau)$ in the sense of operator norm and lemma 2.6, we conclude that $U(t, \tau)$ is continuous in the sense of operator norm provided $0 < t-\tau \leqq \delta_0$. From the corollary to lemma 2.7 and

$$\int_\tau^t \|\exp(-(t-s)A(s))\Phi(s, \tau)\|\,ds < \infty,$$

we conclude that $U(t, \tau)$ is strongly continuous in t, τ for $0 \leqq t-\tau \leqq \delta_0$.Q.E.D.

§III. Local strong differentiability of a fundamental solution

In this section we prove that, for any $x \in \mathbf{E}$, the strong derivative $\frac{\partial}{\partial t}U(t, \tau)x$ exists and is continuous with respect to t, τ for $0 < t-\tau \leqq \delta_0$, that $U(t, \tau)x \in D_A$, and that $\frac{\partial}{\partial t}U(t, \tau)x + A(t)U(t, \tau)x = 0$ holds. Remark that this is local differentiability near the diagonal set. For the proof of global differentiability, we need a lemma on the uniqueness of solution of homogeneous equation, which will be proved in the next section.

Let us consider an operator-valued function $U_\rho(t, \tau)$ for $\varepsilon \leqq t-\tau \leqq \delta_0$,

$$U_\rho(t, \tau) = \exp(-(t-\tau)A(\tau)) + \int_\tau^{t-\rho} \exp(-(t-s)A(s))\Phi(s, \tau)ds, \tag{3.1}$$

where ε is any fixed positive number and $0 < \rho < \varepsilon/3$. By virtue of the continuity of the continuity of $\Phi(s, \tau)$ and $\exp(-(t-s)A(s))$ and the differentiability of analytic semigroup, we see that, for any $x \in \mathbf{E}$, $U_\rho(t, \tau)x$ is differentiable with respect to t. And $U_\rho(t, \tau)$ satisfies

$$\begin{aligned} \frac{\partial}{\partial t}U_\rho(t, \tau)x = &- A(\tau)\exp(-(t-\tau)A(\tau))x \\ &+ \exp(-\rho A(t-\rho))\Phi(t-\rho, \tau)x \end{aligned} \tag{3.2}$$

$$-\int_{\tau}^{t-\rho} A(s)\exp(-(t-s)A(s))\Phi(s,\tau)x\,ds.$$

Using (2.2) and (2.7), we see that

$$\begin{aligned}(3.3)\qquad \frac{\partial}{\partial t}U_\rho(t,\tau)x &= -A(t)\exp(-(t-\tau)A(\tau))x \\ &+ (\exp(-\rho A(t-\rho)) - I)\Phi(t,\tau)x \\ &+ (I - \exp(-\rho A(t-\rho)))\int_{t-\rho}^{t} \phi_1(t,s)\Phi(s,\tau)x\,ds \\ &+ \exp(-\rho A(t-\rho))(\phi_1(t-\rho,\tau) - \phi_1(t,\tau))x \\ &+ \exp(-\rho A(t-\rho))\int_{\tau}^{t-\rho} (\phi_1(t-\rho,s) - \phi_1(t,s))\Phi(s,\tau)x\,ds \\ &- A(t)\int_{\tau}^{t-\rho} \exp(-(t-s)A(s))\Phi(s,\tau)x\,ds \\ &= I_1 + I_2 + I_3 + I_4 + I_5 - I_6.\end{aligned}$$

Note that $I_1 - I_6 = -A(t)U_\rho(t,\tau)$ so we have

$$\frac{\partial}{\partial t}U_\rho(t,\tau)x + A(t)U_\rho(t,\tau)x = I_2 + I_3 + I_4 + I_5.$$

To prove uniform convergence of $\frac{\partial}{\partial t}U_\rho(t,\tau)x$ as $\rho\downarrow 0$ with respect to t, τ for $\varepsilon \leqq t-\tau \leqq \delta_0$, we need the following lemma 3.1 and lemma 3.2.

Lemma 3.1. $\|I_2\|$, $\|I_3\|/\|x\|$, $\|I_4\|/\|x\|$, $\|I_5\|/\|x\|$ tends to 0 as $\rho\downarrow 0$ uniformly with respect to t, τ for $\varepsilon \leqq t-\tau \leqq \delta_0$.

Lemma 3.2. I_6 tends to a limit as $\rho\downarrow 0$ uniformly with respect to t, τ for $\varepsilon \leqq t-\tau \leqq \delta_0$.

Admitting lemma 3.1 and lemma 3.2 for the moment, let us finish the proof of local differentiability. Since $A(t)$ is a closed operator, it follows from lemma 3.2 that

$$(3.4)\qquad \begin{cases} \displaystyle\int_{\tau}^{t} \exp(-(t-s)A(s))\Phi(s,\tau)x\,ds \in D_A, \\ \displaystyle\lim_{\rho\downarrow 0} A(t)\int_{\tau}^{t-\rho} \exp(-(t-s)A(s))\Phi(s,\tau)x\,ds \\ \qquad \displaystyle = A(t)\int_{\tau}^{t} \exp(-(t-s)A(s))\Phi(s,\tau)x\,ds. \end{cases}$$

In view of lemma 2.6, it is easy to see that $U_\rho(t, \tau)x$ and $\frac{\partial}{\partial t}U_\rho(t, \tau)x$ are uniformly continuous with respect to t, τ for $\varepsilon \leqq t - \tau \leqq \delta_0$. And it is clear that $U_\rho(t, \tau)x$ tends to $U(t, \tau)x$ as $\rho \downarrow 0$. Moreover, from (2.6), (3.3), (3.4), lemma 3.1 and lemma 3.2, we see that $U(t, \tau)x \in D_A$ and $\frac{\partial}{\partial t}U_\rho(t, \tau)x$ tends to $-A(t)U(t, \tau)x$ uniformly with respect to t, τ for $\varepsilon \leqq t - \tau \leqq \delta_0$. Thus we conclude that $U(t, \tau)x \in D_A$ and $\frac{\partial}{\partial t}U(t, \tau)x$ exists and is continuous with respect to t, τ for $0 < t - \tau \leqq \delta_0$ and that the equation

$$\frac{\partial}{\partial t}U(t, \tau)x + A(t)U(t, \tau)x = 0$$

holds.

Before beginning the proof of lemma 3.1 and lemma 3.2, we divide $\phi_1(s, \tau) - \phi_1(t, \tau)$ and $\Phi(s, \tau) - \Phi(t, \tau)$ into several parts and get the estimate of each part, which will be used in the proof of lemma 3.1 and lemma 3.2, and we state these estimates as lemma 3.3.

In what follows the letter C denotes various positve constants independent of t, τ, ρ, but possibly depending on δ_0 and ε. Using (2.2), we get

$$\begin{aligned} \phi_1(s, \tau) - \phi_1(t, \tau) &= (A(t) - A(s))\exp(-(t-\tau)A(\tau)) \\ &\quad + (A(\tau) - A(s))(\exp(-(s-\tau)A(\tau)) - \exp(-(t-\tau)A(\tau))) \\ &= J_1(t, s, \tau) + J_2(t, s, \tau) \end{aligned} \tag{3.5}$$

and using (2.7) and (3.5), we get

$$\begin{aligned} \Phi(s, \tau) - \Phi(t, \tau) &= (\phi_1(s, \tau) - \phi_1(t, \tau)) \\ &\quad + \int_\tau^s (\phi_1(s, r) - \phi_1(t, r))\Phi(r, \tau)dr \\ &\quad - \int_s^t \phi_1(t, r)\Phi(r, \tau)dr \\ &= J_1(t, s, \tau) + J_2(t, s, \tau) \\ &\quad + \int_\tau^s J_1(t, s, r)\Phi(r, \tau)dr + \int_\tau^s J_2(t, s, r)\Phi(r, \tau)dr \\ &\quad - \int_s^t \phi_1(t, r)\Phi(r, \tau)dr \\ &= J_1(t, s, \tau) + J_2(t, s, \tau) \\ &\quad + K_1(t, s, \tau) + K_2(t, s, \tau) \\ &\quad - K_3(t, s, \tau) \end{aligned} \tag{3.6}$$

Lemma 3.3. *If $0 \leqq \tau < s < t \leqq T_0$ we have the following inequalities.*

$$\| J_1(t, s, \tau) \| \leqq C \frac{\omega(t-s)}{t-\tau} \tag{3.7}$$

$$\| J_2(t, s, \tau) \| \leqq C \frac{(t-s)\tilde{\omega}(s-\tau)}{t-\tau} \tag{3.8}$$

$$\| K_1(t, s, \tau) \| \leqq C\omega(t-s) \int_\tau^s \frac{\tilde{\omega}(r-\tau)}{t-r} \mathrm{d}r \tag{3.9}$$

$$\| K_2(t, s, \tau) \| \leqq C(t-s) \int_\tau^s \frac{\tilde{\omega}(s-r)\tilde{\omega}(r-\tau)}{t-r} \mathrm{d}r \tag{3.10}$$

$$\| K_3(t, s, \tau) \| \leqq C \int_s^t \tilde{\omega}(t-r)\tilde{\omega}(r-\tau) \mathrm{d}r \tag{3.11}$$

Proof of lemma 3.3. (3.7) follows from (2.13). Writing

$$J_2(t, s, \tau) = (A(s) - A(\tau))A(\tau)^{-1} \int_t^s A(\tau)^2 \exp(-(\zeta - \tau)A(\tau)) \mathrm{d}\zeta,$$

and using (1.3) and (1.5), we have

$$\| J_2(t, s, \tau) \| \leqq C\omega(s-\tau) \int_s^t \frac{1}{(\zeta-\tau)^2} \mathrm{d}\zeta$$
$$= C \frac{t-s}{t-\tau} \tilde{\omega}(s-\tau),$$

thus we get (3.8). Using (3.7), (3.8) and (2.12), we get (3.9) and (3.10). And (3.11) follows from (2.10) and (2.12).

We proceed now to the proof of lemma 3.1 and lemma 3.2.

Proof of lemma 3.1. At first, let us estimate I_2. Since $\Phi(t, \tau)x$ is uniformly continuous with respect to t, τ for $\varepsilon \leqq t - \tau \leqq \delta_0$, for any positive number γ, there exists a finite set $\Gamma_0 = \{(t_1, \tau_1), \cdots, (t_N, \tau_N)\}$ of points of the set $\Gamma = \{(t, \tau) | \varepsilon \leqq t - \tau \leqq \delta_0, t \leqq T_0\}$ such that, for any $(t, \tau) \in \Gamma$, we can take $(t_i, \tau_i) \in \Gamma_0$ which satisfies

$$\| \Phi(t_i, \tau_i)x - \Phi(t, \tau)x \| \leqq \frac{\gamma}{2(C_2+1)} \tag{3.12}$$

where C_2 is the constant in (1.3). By lemma 2.7, $\exp(-\xi A(\eta))x$ is uniformly continuous for $\xi, \eta \in [0, T_0]$, so there exists a positive number δ such that, if $\rho < \delta$ then

$$\| (\exp(-\rho A(t-\rho)) - I) \Phi(t_i, \tau_i)x \| \leqq \frac{\gamma}{2} \tag{3.13}$$

for any $t \in [\varepsilon, T_0]$ and $i = 1, 2, \cdots, N$. Therefore, if $\rho < \delta$ then for any $(t, \tau) \in \Gamma$ we can take $(t_i, \tau_i) \in \Gamma_0$ which satisfies (3.12) and we see from (3.13) that for any t, τ

$$\begin{aligned}\|I_2\| &\leqq \|(\exp(-\rho A(t-\rho))-I)(\Phi(t,\tau)x-\Phi(t_i,\tau_i)x)\| \\ &\quad+\|(\exp(-\rho A(t-\rho))-I)\Phi(t_i,\tau_i)x\| \\ &\leqq \frac{\gamma}{2}+\frac{\gamma}{2}=\gamma,\end{aligned}$$

provided $\varepsilon \leqq t-\tau \leqq \delta_0$. So we conclude that $\|I_2\|$ tends to 0 as $\rho \downarrow 0$ uniformly with respect to t, τ for $\varepsilon \leqq t-\tau \leqq \delta_0$.

To estimate I_3, we use (1.3), (2.10), (2.12) and $\varepsilon \leqq t-\tau \leqq \delta_0$, $\rho < \varepsilon/3$. If we put $t-s=r$, we see from (1.8) that

$$\|I_3\| \leqq C\int_{t-\rho}^{t} \tilde{\omega}(t-s)\tilde{\omega}(s-\tau)\mathrm{d}s\|x\| \leqq C\frac{\omega(\delta_0)}{2\varepsilon/3}\int_0^{\rho} \tilde{\omega}(r)\mathrm{d}r\|x\|.$$

So we conclude that $\|I_3\|/\|x\|$ tends to 0 as $\rho \downarrow 0$ uniformly with respect to t, τ for $\varepsilon \leqq t-\tau \leqq \delta_0$.

To estimate I_4, we use (3.5). Then we have

$$\|I_4\| = \|\exp(-\rho A(t-\rho))(J_1(t,t-\rho,\tau)+J_2(t,t-\rho,\tau))x\|.$$

Using (3.7), (3.8), $\varepsilon \leqq t-\tau \leqq \delta_0$ and $\rho < \varepsilon/3$, we see that

$$\|I_4\| \leqq C\left(\frac{\omega(\rho)}{\varepsilon}+\frac{\rho\omega(\delta_0)}{2\varepsilon^2/3}\right)\|x\|.$$

So we conclude that $\|I_4\|/\|x\|$ tends to 0 as $\rho \downarrow 0$ uniformly with respect to t, τ for $\varepsilon \leqq t-\tau \leqq \delta_0$.

To estimate I_5, we use (3.5) and (3.6). Then we have

$$I_5 = \exp(-\rho A(t-\rho))(K_1(t,t-\rho,\tau)+K_2(t,t-\rho,\tau))x.$$

Using (1.3), (3.9) and (3.10), we have

$$\begin{aligned}\|I_5\| &\leqq C\|x\|\omega(\rho)\int_{\tau}^{t-\rho}\frac{\tilde{\omega}(r-\tau)}{t-r}\mathrm{d}r \\ &\quad+C\|x\|\rho\int_{\tau}^{t-\rho}\frac{\tilde{\omega}(t-\rho-r)\tilde{\omega}(r-\tau)}{t-r}\mathrm{d}r\end{aligned} \tag{3.14}$$

As for the first term of the right hand side of (3.14), put $t-\tau=T$ and $r-\tau=r'$, and use $\varepsilon \leqq T \leqq \delta_0$, we see from (1.7) that

$$\begin{aligned}\omega(\rho)\int_{\tau}^{t-\rho}\frac{\tilde{\omega}(r-\tau)}{t-r}\mathrm{d}r &= \omega(\rho)\int_0^{T-\rho}\frac{\tilde{\omega}(r')}{T-r'}\mathrm{d}r' \\ &= \frac{1}{T}\omega(\rho)\int_0^{T-\rho}\left(\frac{\omega(r')}{T-r'}+\tilde{\omega}(r')\right)\mathrm{d}r' \\ &\leqq \frac{1}{\varepsilon}\omega(\rho)(\omega(\delta_0)(|\log\delta_0|+|\log\rho|)+\int_0^{\delta_0}\tilde{\omega}(r')\mathrm{d}r') \\ &\longrightarrow 0 \text{ as } \rho \downarrow 0,\end{aligned}$$

where the convergence is uniform with respect to t, τ for $\varepsilon \leqq t-\tau = T \leqq \delta_0$. As for the second term of the right hand side of (3.14), put $t-\tau = T$, then we have,

$$\begin{aligned} I_{5.1} &= \rho \int_\tau^{t-\rho} \frac{\tilde{\omega}(t-\rho-r)\tilde{\omega}(r-\tau)}{t-r} \mathrm{d}r \\ &= \frac{\rho}{T} \int_\tau^{t-\rho} \frac{\tilde{\omega}(t-\rho-r)\omega(r-\tau)}{t-r} \mathrm{d}r + \frac{\rho}{T} \int_\tau^{t-\rho} \tilde{\omega}(t-\rho-r)\tilde{\omega}(r-\tau)\mathrm{d}r, \end{aligned}$$

put $t-\rho-r = r'$ and apply (2.9) to the second term and use $\varepsilon \leqq T \leqq \delta_0$, $0<\rho<\varepsilon/3$, then we have

$$\begin{aligned} I_{5.1} &\leqq \frac{\omega(\delta_0)}{\varepsilon} \int_0^{T-\rho} \frac{\tilde{\omega}(r')\rho}{\rho+r'} \mathrm{d}r' + \frac{2\rho\omega(\delta_0)}{\varepsilon(T-\rho)} \int_0^{T-\rho} \tilde{\omega}(r')\mathrm{d}r' \\ &\leqq C \int_0^{\delta_0} \frac{\tilde{\omega}(r')\rho}{\rho+r'} \mathrm{d}r' + C\rho \int_0^{\delta_0} \tilde{\omega}(r')\mathrm{d}r' \end{aligned}$$

So, by dominated convergence theorem, the first term tends to 0 as $\rho \downarrow 0$ uniformly with respect to t, τ for $\varepsilon \leqq t-\tau \leqq \delta_0$. And the second term obviously tends to 0 as $\rho \downarrow 0$ uniformly with respect to t, τ for $\varepsilon \leqq t-\tau \leqq \delta_0$. This completes the proof of lemma 3.1. Q.E.D.

Before beginning the proof of lemma 3.2, we prove a lemma, which will be used in the proof of lemma 3.2 and in §V.

Lemma 3.4. *There exists a constant C independent of* t, τ, *such that*

$$\left\| A(t) \int_\tau^t \exp(-(t-s)A(s))\mathrm{d}s \right\| \leqq C \tag{3.15}$$

provided $0 < t-\tau \leqq \delta_0$.

Proof of lemma 3.4. Let $N_\rho(t, \tau)$ and $N(t, \tau)$ be the operators defined by

$$N_\rho(t, \tau)x = A(t) \int_\tau^{t-\rho} \exp(-(t-s)A(s))x\mathrm{d}s, \qquad \text{for } x \in \mathbf{E}, \tag{3.16}$$

and

$$N(t, \tau)x = A(t) \int_\tau^{t} \exp(-(t-s)A(s))x\mathrm{d}s, \qquad \text{for } x \in D_A, \tag{3.17}$$

respectively. Note that if $x \in D_A$ and $0<\rho<\rho'$ then

$$\| N_\rho(t, \tau)x - N_{\rho'}(t, \tau)x \| \leqq C'\rho', \tag{3.18}$$

where C' is a constant independent of t, τ, ρ, ρ'.

Indeed, since $x \in D_A$, if we write $x = A(\tau)^{-1}x' = A(s)^{-1}A(s)A(\tau)^{-1}x'$ (3.18) is obvious. As for $N_\rho(t, \tau)$, we prove that if $0<t-\tau \leqq \delta_0$ and $0<\rho$ then

$$\| N_\rho(t, \tau) \| \leqq C'', \tag{3.19}$$

where C'' is a constant independent of t, τ, ρ.

Admitting (3.19) for the moment let us finish the proof of lemma 3.4. For any positive number $\gamma > 0$, there exists $x' \in D_A$ such that $\|x - x'\| \leqq \frac{\gamma}{3C''}$. So if $0 < \rho < \rho' < \frac{\gamma}{3C'}$, we see from (3.18) and (3.19) that

$$\begin{aligned} \|N_\rho(t, \tau)x - N_{\rho'}(t, \tau)x\| &\leqq \|N_\rho(t, \tau)(x - x')\| \\ &\quad + \|N_\rho(t, \tau)x' - N_{\rho'}(t, \tau)x'\| \\ &\quad + \|N_{\rho'}(t, \tau)(x' - x)\| \\ &\leqq \frac{\gamma}{3} + \frac{\gamma}{3} + \frac{\gamma}{3} = \gamma \end{aligned}$$

Thus, for any $x \in \mathbf{E}$, the improper integral

$$\lim_{\rho \downarrow 0} N_\rho(t, \tau)x = \lim_{\rho \downarrow 0} \int_\tau^{t-\rho} A(t)\exp(-(t-s)A(s))x\mathrm{d}s,$$

exists. Since $A(t)$ is a closed operator we conclude that $N(t, \tau)$ is defined for any $x \in \mathbf{E}$ and, from the inequality (3.19), we have (3.15).

We proceed now to the proof of (3.19). Since $N_\rho(t, \tau)$ is a bounded operator for $\rho > 0$ and $N_\rho(t, \tau)x = N(t, \tau)x - N(t, t-\rho)x$ for $x \in D_A$ it suffices to prove

$$\|N(t, \tau)x\| \leqq C\|x\| \qquad \text{for } x \in D_A. \tag{3.20}$$

In view of Gronwall's inequality, (3.20) is derived from

$$\|N(t, \tau)x\| \leqq C\|x\| + C\int_\tau^t \tilde{\omega}(t-\xi)\|N(\xi, \tau)x\|\mathrm{d}\xi \qquad \text{for } x \in D_A. \tag{3.21}$$

So we prove (3.21). Using a variant of (2.18) we have

$$\begin{aligned} N(t, \tau)x &= (I - \exp(-(t-\tau)A(t)))\left(I - \int_\tau^t \phi_1(t, s)\mathrm{d}s\right)x \\ &\quad + \int_\tau^t A(t)\exp(-(t-\xi)A(t))\mathrm{d}\xi\left(\int_\xi^t \phi_1(t,s)\mathrm{d}s + \int_\tau^\xi (\phi_1(t,s) - \phi_1(\xi,s))\mathrm{d}s\right)x \\ &\quad + \int_\tau^t A(t)\exp(-(t-\xi)A(t))(A(t) - A(\xi))A(\xi)^{-1}N(\xi, \tau)x\mathrm{d}\xi. \end{aligned} \tag{3.22}$$

Using (1.3), (1.5), (1.8), (2.10), (3.5), we have

$$\begin{aligned} \|N(t, \tau)x\| &\leqq C\|x\| + C\|x\|\int_\tau^t \frac{1}{t-\xi}\mathrm{d}\xi\int_\xi^t \tilde{\omega}(t-s)\mathrm{d}s \\ &\quad + C\|x\|\int_\tau^t \frac{1}{t-\xi}\mathrm{d}\xi\|J_1(t, \xi, s) + J_2(t, \xi, s)\|\mathrm{d}s \\ &\quad + C\int_\tau^t \tilde{\omega}(t-\xi)\|N(\xi, \tau)x\|\mathrm{d}\xi. \end{aligned}$$

So, using (3.5), (3.7) and (3.8), we have

$$\|N(t,\tau)x\| \leqq C\|x\|$$
$$(3.23)\qquad + C\|x\|\left(\int_\tau^t \frac{1}{t-\xi}\mathrm{d}\xi\int_\xi^t \tilde{\omega}(t-s)\mathrm{d}s + \int_\tau^t \tilde{\omega}(t-\xi)\mathrm{d}\xi\int_\tau^\xi \frac{1}{t-s}\mathrm{d}s\right.$$
$$\left. + \int_\tau^t \mathrm{d}\xi\int_\tau^\xi \frac{\tilde{\omega}(\xi-s)}{t-s}\mathrm{d}s\right) + C\int_\tau^t \tilde{\omega}(t-\xi)\|N(\xi,\tau)x\|\mathrm{d}\xi.$$

It is easy to see that three double integrals in the parenthesis are equal. To estimate the second double integral we put $t-\tau=T$, $t-s=s'$ and $t-\xi=\xi'$. Then we see, from (1.6) and (1.8), that

$$\int_\tau^t \tilde{\omega}(t-\xi)\mathrm{d}\xi\int_\tau^\xi \frac{1}{t-s}\mathrm{d}s = \int_0^T \tilde{\omega}(\xi')\mathrm{d}\xi'\int_{\xi'}^T \frac{1}{s'}\mathrm{d}s'$$
$$(3.24)\qquad \leqq \int_0^{\delta_0} \tilde{\omega}(\xi')(|\log\delta_0| + |\log\xi'|)\mathrm{d}\xi'$$
$$\leqq C \text{ for } 0 < t-\tau \leqq \delta_0.$$

Using (3.24), we get (3.21) from (3.23). This completes the proof of lemma 3.4.

We begin the proof of lemma 3.2. Writing

$$I_6 = A(t)\int_\tau^{t-\rho} \exp(-(t-s)A(s))(\Phi(s,\tau)-\Phi(t,\tau))x\mathrm{d}s$$
$$+ A(t)\int_\tau^{t-\rho} \exp(-(t-s)A(s))\Phi(t,\tau)x\mathrm{d}s$$
$$= L_1(\rho) + L_2(\rho),$$

it suffices to prove that $\|L_1(\rho)-L_1(\rho')\|$ and $\|L_2(\rho)-L_2(\rho')\|$ tends to 0 as $\rho, \rho' \downarrow 0$ uniformly with respect to t, τ for $\varepsilon \leqq t-\tau \leqq \delta_0$. As for $\|L_2(\rho)-L_2(\rho')\|$, let ρ, ρ' be $0<\rho<\rho'<\varepsilon/3$ and use (3.16), then we see that

$$L_2(\rho)-L_2(\rho') = A(t)\int_{t-\rho'}^{t-\rho} \exp(-(t-s)A(s))\Phi(t,\tau)x\mathrm{d}s$$
$$= N_\rho(t,\tau)\Phi(t,\tau)x - N_{\rho'}(t,\tau)\Phi(t,\tau)x.$$

Since $\Phi(t,\tau)x$ is uniformly continuous in t, τ for $\varepsilon \leqq t-\tau \leqq \delta_0$ and $t \leqq T_0$, for any positive number γ there exists a finite set $\Gamma_0 = \{(t_1,\tau_1),(t_2,\tau_2),\cdots(t_N,\tau_N)\}$ of points of the set $\Gamma = \{(t,\tau)|\varepsilon \leqq t-\tau \leqq \delta_0, t \leqq T_0\}$ such that for any $(t,\tau)\in\Gamma$ we can take $(t_i,\tau_i)\in\Gamma_0$ which satisfies

$$(3.25)\qquad \|\Phi(t,\tau)x - \Phi(t_i,\tau_i)x\| \leqq \frac{\gamma}{5C''},$$

where C'' is the constant in (3.19). And since D_A is dense in $\mathbf{E}$, there exist $x_1, x_2, \cdots, x_N \in D_A$ such that

$$\|\Phi(t_i, \tau_i)x - x_i\| \leqq \frac{\gamma}{5C''} \qquad \text{for } i = 1, 2, \cdots, N. \tag{3.26}$$

So for any $(t, \tau) \in \Gamma$ take $(t_i, \tau_i) \in \Gamma_0$ which satisfies (3.25) and let ρ, ρ' be $0 < \rho < \rho' < \frac{\gamma}{5C'}$, then we see from (3.18), (3.19) and (3.26) that

$$\begin{aligned} \|L_2(\rho) - L_2(\rho')\| \leqq\; & \|N_\rho(t, \tau)(\Phi(t, \tau)x - \Phi(t_i, \tau_i)x)\| \\ & + \|N_\rho(t, \tau)(\Phi(t_i, \tau_i)x - x_i)\| \\ & + \|N_\rho(t, \tau)x_i - N_{\rho'}(t, \tau)x_i\| \\ & + \|N_{\rho'}(t, \tau)(x_i - \Phi(t_i, \tau_i)x)\| \\ & + \|N_{\rho'}(t, \tau)(\Phi(t_i, \tau_i)x - \Phi(t, \tau)x)\| \\ \leqq\; & \frac{\gamma}{5} + \frac{\gamma}{5} + \frac{\gamma}{5} + \frac{\gamma}{5} + \frac{\gamma}{5} = \gamma. \end{aligned}$$

Thus we conclude that $\|L_2(\rho) - L_2(\rho')\|$ tends to 0 as $\rho, \rho' \downarrow 0$ uniformly with respect to t, τ for $\varepsilon \leqq t - \tau \leqq \delta_0$. As for $\|L_1(\rho) - L_1(\rho')\|$, let ρ, ρ' be $0 < \rho < \rho' < \varepsilon/3$ and use (3.6), then we see that

$$\begin{aligned} L_1(\rho) - L_1(\rho') &= \int_{t-\rho'}^{t-\rho} A(t)\exp(-(t-s)A(s))(\Phi(s, \tau) - \Phi(t, \tau))x\mathrm{d}s \\ &= \int_{t-\rho'}^{t-\rho} A(t)\exp(-(t-s)A(s))(J_1(t, s, \tau) + J_2(t, s, \tau) \\ &\quad + K_1(t, s, \tau) + K_2(t, s, \tau) - K_3(t, s, \tau))x\mathrm{d}s, \end{aligned}$$

then using (1.3), (3.7), (3.8), (3.9), (3.10), (3.11) and $\varepsilon \leqq t - \tau \leqq \delta_0$, and writing

$$\begin{aligned} M_1(\rho, \rho') &= \int_{t-\rho'}^{t-\rho} \frac{\tilde{\omega}(t-s)}{t-\tau}\mathrm{d}s \\ M_2(\rho, \rho') &= \int_{t-\rho'}^{t-\rho} \frac{\tilde{\omega}(s-\tau)}{t-\tau}\mathrm{d}s \\ M_3(\rho, \rho') &= \int_{t-\rho'}^{t-\rho} \tilde{\omega}(t-s)\mathrm{d}s \int_\tau^s \frac{\tilde{\omega}(r-\tau)}{t-r}\mathrm{d}r \\ M_4(\rho, \rho') &= \int_{t-\rho'}^{t-\rho} \mathrm{d}s \int_\tau^s \frac{\tilde{\omega}(s-r)\tilde{\omega}(r-\tau)}{t-r}\mathrm{d}r \\ M_5(\rho, \rho') &= \int_{t-\rho'}^{t-\rho} \mathrm{d}s \int_s^t \tilde{\omega}(t-r)\tilde{\omega}(r-\tau)\mathrm{d}r, \end{aligned}$$

we see that

$$\begin{aligned} & \|L_1(\rho) - L_1(\rho')\| \\ & \quad \leqq C\|x\|(M_1(\rho, \rho') + M_2(\rho, \rho') + M_3(\rho, \rho') + M_4(\rho, \rho') \\ & \qquad + M_5(\rho, \rho')). \end{aligned} \tag{3.27}$$

It is obvious that $M_1(\rho, \rho')$, $M_2(\rho, \rho')$ tends to 0 as $\rho, \rho' \downarrow 0$ uniformly with respect to t, τ for $\varepsilon \leqq t-\tau \leqq \delta_0$. To estimate $M_3(\rho, \rho')$, put $r-\tau = r'$, $t-s = s'$, $t-\tau = T$ and use (1.6), (1.8) and $\varepsilon \leqq T \leqq \delta_0$, we have

$$M_3(\rho, \rho') \leqq \frac{1}{\varepsilon}\int_\rho^{\rho'} \tilde{\omega}(s')ds' \int_0^{\delta_0} \tilde{\omega}(r')dr'$$

$$+ \frac{\omega(\delta_0)}{\varepsilon}\int_\rho^{\rho'} \tilde{\omega}(s')(|\log T| + |\log s'|)ds' \longrightarrow 0 \text{ as } \rho, \rho' \downarrow 0$$

uniformly with respect to t, τ for $\varepsilon \leqq t-\tau \leqq \delta_0$.

To estimate $M_5(\rho, \rho')$, put $t-s = s'$, $t-r = r'$, $t-\tau = T$ *and use* $0 < \rho < \rho' < \varepsilon/3$, $\varepsilon \leqq T \leqq \delta_0$, $T-r' > r'$, we have

$$\begin{aligned} M_5(\rho, \rho') &= \frac{1}{T}\int_\rho^{\rho'} \frac{1}{s'}ds' \int_0^{s'} \left(\frac{1}{r'} + \frac{1}{T-r'}\right)\omega(r')\omega(T-r')dr' \\ &\leqq 2\frac{\omega(\delta_0)}{\varepsilon}\int_0^{\rho'} \frac{1}{s'}ds' \int_0^{s'} \tilde{\omega}(r')dr' \\ &= 2\frac{\omega(\delta_0)}{\varepsilon}\int_0^{\rho'} \tilde{\omega}(r')dr' \int_{r'}^{\rho'} \frac{1}{s'}ds' \\ &= 2\frac{\omega(\delta_0)}{\varepsilon}\int_0^{\rho'} \tilde{\omega}(r')|\log r'|dr' \longrightarrow 0 \text{ as } \rho' \downarrow 0, \end{aligned}$$

uniformly with respect to t, τ for $\varepsilon \leqq t-\tau \leqq \delta_0$.

To estimate $M_4(\rho, \rho')$, writing

$$M_{4.1}(\rho, \rho') = \int_{t-\rho'}^{t-\rho} ds \int_\tau^{\tau+\frac{\varepsilon}{3}} \frac{\tilde{\omega}(s-r)\tilde{\omega}(r-\tau)}{t-r}dr$$

$$M_{4.2}(\rho, \rho') = \int_{t-\rho'}^{t-\rho} ds \int_{\tau+\frac{\varepsilon}{3}}^{s} \frac{\tilde{\omega}(s-r)\tilde{\omega}(r-\tau)}{t-r}dr,$$

we see that

$$M_4(\rho, \rho') = M_{4.1}(\rho, \rho') + M_{4.2}(\rho, \rho').$$

As for $M_{4.1}(\rho, \rho')$, use $t-r \geqq \frac{2}{3}\varepsilon$, $s-r \geqq \frac{\varepsilon}{3}$, $\omega(s-r) \leqq \omega(\delta_0)$ and put $r-\tau = r'$, then we have

$$M_{4.1}(\rho, \rho') \leqq \frac{\omega(\delta_0)}{2\varepsilon^2/9}(\rho'-\rho)\int_0^{\varepsilon/3} \tilde{\omega}(r')dr' \longrightarrow 0 \text{ as } \rho, \rho' \downarrow 0$$

As for $M_{4.2}(\rho, \rho')$, use $\omega(r-\tau) \leqq \omega(\delta_0)$, $r-\tau \geqq \frac{\varepsilon}{3}$ and put $t-r = r'$, $s-r = s'$, then we have

$$M_{4.2}(\rho, \rho') \leqq \frac{\omega(\delta_0)}{\varepsilon/3} \int_{t-\rho'}^{t} ds \int_{\tau}^{s} \frac{\tilde{\omega}(s-r)}{t-r} dr$$
$$= \iint_D \frac{\tilde{\omega}(r')}{s'} dr' ds' \longrightarrow 0 \text{ as } \rho' \downarrow 0,$$

where $D = \{(r', s') | r' \leqq t - \tau \leqq \delta_0,\ s' \geqq 0,\ 0 \leqq r' - s' \leqq \rho'\}$.

Thus we conclude that $M_4(\rho, \rho')$ tends to 0 as $\rho, \rho' \downarrow 0$ uniformly with respect to t, τ for $\varepsilon \leqq t - \tau \leqq \delta_0$. This completes the proof of lemma 3.2.

§ IV. Uniqueness of fundamental solution

In this section we prove global existence and uniqueness of fundamental solution. We first prove local uniqueness near the diagonal set.

Definition. A function $u(t)$ with values in $\mathbf{E}$ is called a strong solution of the initial value problem (4.1) on $[\tau, \chi]$

$$\text{(4.1)} \qquad \begin{cases} \dfrac{d}{dt} u(t) + A(t) u(t) = 0 & (\tau < t \leqq \chi) \\ u(\tau) = u_0, \end{cases}$$

if

(1): $u(t)$ is continuous in t for $\tau \leqq t \leqq \chi$;

(2): $u(t)$ is continuously differentiable in t $\tau < t \leqq \chi$;

and if

(3): $u(t)$ satisfies (4.1)

Lemma 4.1. *Let the assumption* $(A_1) \sim (A_4)$ *hold. Then, for any* $u_0 \in \mathbf{E}$ and $\tau \in [0, T_0)$, *if we put* $\chi = \min\{\tau + \delta_0, T_0\}$ *there exists a unique strong solution of the initial value problem* (4.1) *on* $[\tau, \chi]$. *And it is given by* $u(t) = U(t, \tau) u_0$, *where* $U(t, \tau)$ *is the local fundamental solution near the diagonal set.*

Admitting lemma 4.1 for the moment let us finish the proof of local uniqueness of fundamental slution. Indeed, if there exists another fundamental solution for $0 \leqq t - \tau \leqq \delta_0$, say $\tilde{U}(t, \tau)$, then $\tilde{u}(t) = \tilde{U}(t, \tau) u_0$ is also a strong solution of the initial value problem (4.1) on $[\tau, \chi]$. We have $\tilde{u}(t) = u(t)$ for $\tau \leqq t \leqq \chi$, that is, $\tilde{U}(t, \tau) u_0 = U(t, \tau) u_0$ if $\tau \leqq t \leqq \chi$ and $u_0 \in \mathbf{E}$. Therefore $\tilde{U}(t, \tau) = U(t, \tau)$.

Before beginning the proof of lemma 4.1 we prove the following lemma.

Lemma 4.2. *For any* $x \in \mathbf{E}$, *the function* $W(t, \tau) = A(t) U(t, \tau) A(\tau)^{-1} x$ is continuous with respect to t, τ for $0 \leqq t - \tau \leqq \delta_0$.

Proof. For $\tau \leqq s \leqq t$, writing

$$\phi(s) = \exp(-(t-s) A(t)) U(s, \tau) A(\tau)^{-1} x$$

its derivative is

$$\phi'(s) = \exp(-(t-s)A(t))(A(t)-A(s))\,U(s,\tau)A(\tau)^{-1}x \tag{4.2}$$

Integrating (4.2) and applying $A(t)$ to the resulting equation, we have

$$\begin{aligned} W(t,\tau) &= A(t)\exp(-(t-\tau)A(t))A(\tau)^{-1}x \\ &\quad + \int_\tau^t A(t)\exp(-(t-s)A(t))(A(t)-A(s))A(s)^{-1}\,\mathrm{W}(s,\tau)x\,ds. \end{aligned} \tag{4.3}$$

By lemma 2.7 the first term on the right hand side of (4.3) is continuous with respect to t, τ for $0 \leqq t-\tau \leqq \delta_0$. And by (2.23) $A(\tau)^{-1}$ is continuous, so $W(t,\tau)$ is continuous with respect to t, τ for $0 < t-\tau \leqq \delta_0$. Thus it remains to show that the second term on the right hand side of (4.3) tends to 0 as $t-\tau \downarrow 0$. Writing this term in the form

$$\int_\tau^t \Psi(t,s)\,W(s,\tau)x\,ds, \tag{4.4}$$

where $\Psi(t,s) = A(t)\exp(-(t-s)A(t))(A(t)-A(s))A(s)^{-1}$, and using (1.3) and (1.5) we have

$$\|\Psi(t,s)\| \leqq C\tilde{\omega}(t-s), \tag{4.5}$$

where C is a constant independent of t, s. Hence,

$$\|W(t,\tau)x\| \leqq C\|x\| + C\int_\tau^t \tilde{\omega}(t-s)\|W(s,\tau)x\|\,ds.$$

So, by Gronwall's inequality, we have

$$\|W(t,\tau)x\| \leqq C\|x\|, \tag{4.6}$$

where C is a constant independent of t, τ provided $0 < t-\tau \leqq \delta_0$. From (4.5) and (4.6) it follows that the integral in (4.4) tends to 0 as $t-\tau \downarrow 0$. This completes the proof of lemma 4.2. Incidentally we have also proved the next corollary.

Corollary. *For all t, τ with $0 \leqq t-\tau \leqq \delta_0$,*

$$\|A(t)\,U(t,\tau)A(\tau)^{-1}\| \leqq C, \tag{4.7}$$

where C is a constant independent of t, τ.

We proceed now to the proof of lemma 4.1

Proof of lemma 4.1. Let $U(t,s)$ be the local fundamental solution constructed in §II. It is obvious that $U(t,\tau)u_0$ is a strong solution of (4.1). So it remains to prove the uniqueness of the strong solution. For $n = 1, 2, \cdots$, let $A_n(t)$ be the bounded operators

$$A_n(t) = A(t)\left(I + \frac{1}{n}A(t)\right)^{-1}.$$

Then we have

$$\|(A_n(t)+\lambda I)^{-1}\| \leqq \frac{C}{|\lambda|+1} \tag{4.8}$$

$$\|(A_n(t)-A_n(\tau))A_n(s)^{-1}\| \leqq C\omega(t-\tau), \tag{4.9}$$

where $\mathrm{Re}\lambda \geqq 0$ and C is a constant independent of t, s, τ, λ, n.

From (4.8), (4.9), §II and §III, it follows that the local fundamental solution $U_n(t, \tau)$ of the homogeneous equation $\frac{\mathrm{d}}{\mathrm{d}t}u(t)+A_n(t)u(t)=0$ exists and satisfies

$$\|U_n(t, \tau)\| \leqq C, \tag{4.10}$$

where $0 \leqq t-\tau \leqq \delta_0$ and C is a constant independent of t, τ, n.

At first we prove that a strong solution $u(t)$ is unique under the asuumption that $u(t)$ is continuously differentiable up to $t=\tau$.

Assume that $u(t)$ is a strong solution of (4.1) on $[\tau, \chi]$ and continuously differentiable up to $t=\tau$. Since $u_n(t)=U_n(t,\tau)u_0$ is a strong solution of

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u_n(t)+A_n(t)u_n(t)=0 & \text{for } \tau<t\leqq\chi \\ u_n(\tau)=u_0, \end{cases} \tag{4.11}$$

on $[\tau, \chi]$, the function $w_n(t)=u(t)-u_n(t)$ satisfies

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}w_n(t)+A_n(t)w_n(t)=(A_n(t)-A(t))u(t) & \text{for } \tau<t\leqq\chi \\ w_n(\tau)=0. \end{cases} \tag{4.12}$$

Since $\|A_n(t)\| \leqq Cn$, where C is a constant independent of t, it follows from (4.9) that $(A_n(t)-A(t))u(t)$ is continuous on $[\tau, \chi]$. Thus we see that (4.12) has a unique strong solution on $[\tau, \chi]$ and it is written by

$$w_n(t)=\int_\tau^t U_n(t, s)(A_n(s)-A(s))u(s)ds. \tag{4.13}$$

From (1.2) and (1.4) we have, for any $x \in D_A$,

$$\|(A_n(s)-A(s))A(s)^{-1}\| \leqq \frac{C_1C_3}{n+1}\|A(0)x\|.$$

So, we have, for any $x \in \mathbf{E}$,

$$(A_n(s)-A(s))A(s)^{-1}x \longrightarrow 0 \text{ as } n \longrightarrow \infty, \tag{4.14}$$

uniformly with respect to s for $\tau \leqq s \leqq \chi$.

From the present assumption $A(s)u(s)$ is uniformly continuous on $[\tau, \chi]$. So, from (4.14), we have

$$(A_n(s)-A(s))u(s) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

uniformly with respect to s for $\tau \leqq s \leqq \chi$. From this, (4.10) and (4.13) we have $w_n(t) \to 0$ as $n \to \infty$, that is

$$u(t) = \lim_{n \to \infty} u_n(t).$$

Since $u_n(t)$ is uniquely defined as the strong solution of (4.11), we have the uniqueness of $u(t)$.

Now let $u(t)$ be an arbitrary strong solution of (4.1) on $[\tau, \chi]$ (not necessarily continuously differentiable up to $t = \tau$). For any $\tau < s < \chi$, $u(t)$ is a continuously differentiable strong solution on $[s, \chi]$. Writing $U(t, s)u(s) = U(t, s)A(s)^{-1}(A(s)u(s))$, we see from lemma 4.2 that $U(t, s)u(s)$ is also a continuously differentiable solution on $[s, \chi]$ and satisfies intial condidion $U(s, s)u(s) = u(s)$. Thus we have $u(t) = U(t, s)u(s)$ for $\tau < s \leqq t \leqq \chi$. Taking $s \downarrow \tau$, we have $u(t) = U(t, \tau)u_0$.

This completes the proof of lemma 4.1.

Corollary. *The equality*

$$U(t, \tau) = U(t, s)U(s, \tau) \qquad \text{for } \tau \leqq s \leqq t \leqq \chi \tag{4.15}$$

holds.

Now we prove global uniqueness of solution of the Cauchy problem of the homogeneous equation.

Lemma 4.3. *Assume that $(A_1) \sim (A_4)$ hold. Then, for any $u_0 \in \mathbf{E}$ and $\tau \in [0, T_0)$, there exists a unique strong solution of the initial value problem,*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}u(t) + A(t)u(t) = 0 & \text{for } \tau < t \leqq T_0 \\ u(\tau) = u_0, \end{cases} \tag{4.16}$$

on $[\tau, T_0]$.

Proof. Let $u(t)$ and $v(t)$ be strong solutions of (4.16) on $[\tau, T_0]$. And let $t_0, \cdots, t_n$ be $\tau = t_0 < t_1 < \cdots < t_n = T_0$ and $t_j - t_{j-1} < \delta_0 (j = 1, 2, \cdots, n)$. Applying lemma 4.1 sequentially, we have $u(t) = v(t)$ for $\tau \leqq t \leqq T_0$.

Finally we prove global existence and uniqueness of fundamental solution.

Lemma 4.4. *If t, τ are points with $0 \leqq \tau < t \leqq T_0$ and if $t_0, t_1, \cdots t_m, s_0, s_1, \cdots, s_n$ are points with $\tau = t_0 < t_1 < \cdots < t_m = t$, $0 < t_j - t_{j-1} < \delta_0 (j = 1, \cdots m)$, $\tau = s_0 < s_1 < \cdots < s_n = t$ and $0 < s_k - s_{k-1} < \delta_0 (k = 1, \cdots, n)$, then*

$$U(t_m, t_{m-1}) \cdots U(t_2, t_1)U(t_1, t_0) = U(s_n, s_{n-1}) \cdots U(s_2, s_1)U(s_1, s_0). \tag{4.17}$$

Proof. Let $r_0, r_1, \cdots, r_p$ be points with $\{r_0, r_1, \cdots, r_p\} = \{t_0, t_1, \cdots, t_m\} \cup \{s_0, s_1, \cdots, s_n\}$ and $\tau = r_0 < r_1 < \cdots < r_p = t$. Then, in view of local uniqueness of fundamental solution and (4.15), it is clear that

$$U(t_m, t_{m-1})\cdots U(t_2, t_2)U(t_1, t_0) = U(r_p, r_{p-1})\cdots U(r_2, r_1)U(r_1, r_1)$$

and

$$U(s_n, s_{n-1})\cdots U(s_2, s_1)U(s_1, s_0) = U(r_p, r_{p-1})\cdots U(r_2, r_1)U(r_1, r_1)$$

holds. Thus we have (4.17).

In view of lemma 4.4, we can define $U(t, \tau)$ by

$$U(t, \tau) = U(t_m, t_{m-1})\cdots U(t_1, t_0).$$

Then by lemma 4.3 we have global uniqueness of fundamental solution.

§ V. Solution of the Cauchy problem

In this section we prove our main theorem. In view of lem,ma 4.1 it remains to prove that the function

$$w(t) = \int_\tau^t U(t, s)f(s)\mathrm{d}s \tag{5.1}$$

satisfies the equatoin

$$\frac{\mathrm{d}}{dt}w(t) + A(t)w(t) = f(t) \qquad \text{for } \tau \leqq t \leqq \chi \tag{5.2}$$

$$w(\tau) = 0 \tag{5.3}$$

and that $\frac{\mathrm{d}}{\mathrm{d}t}w(t)$ is continuous for $\tau \leqq t \leqq \chi$, where $0 \leqq \tau < T_0$ and $\chi = \min\{\tau + \delta_0, T_0\}$.

By virtue of (2.6) we have $\| U(t, \tau)\| \leqq C$, so, it follows that (5.3) holds. As for the proof of (5.2) and the continuity of $\frac{\mathrm{d}}{\mathrm{d}t}w(t)$, we formally write, for $h > 0$,

(5.4)

$$\begin{aligned}\frac{w(t+h) - w(t)}{h} &= \frac{1}{h}\int_t^{t+h} U(t+h, s)f(s)\mathrm{d}s \\ &\quad + \left(\frac{1}{h}(U(t+h, t) - I)A(t)^{-1}\right)\left(A(t)\int_\tau^t U(t, s)f(s)\mathrm{d}s\right) \\ &= I_1(h) + I_2(h)L(t, \tau)\end{aligned}$$

$$\begin{aligned}L(t, \tau) = &\int_\tau^t A(t)U(t, s)(f(s) - f(t))\mathrm{d}s \\ &+ \int_\tau^t A(t)(U(t, s) - \exp(-(t-s)A(t)))\mathrm{d}sf(t)\end{aligned} \tag{5.5}$$

$$+\int_\tau^t A(t)\exp(-(t-s)A(t))f(t)\mathrm{d}s$$
$$= L_1(t,\tau) + L_2(t,\tau)f(t) + (I - \exp(-(t-\tau)A(t)))f(t).$$

And we write

$$\tilde{L}_1(t,\tau) = \int_\tau^t \|A(t)U(t,s)(f(s) - f(t))\|\mathrm{d}s \tag{5.6}$$

$$\tilde{L}_2(t,\tau) = \int_\tau^t \|A(t)(U(t,s) - \exp(-(t-s)A(t)))\|\mathrm{d}s. \tag{5.7}$$

We need the following two lemmas.

Lemma 5.1. *For any* $x \in \mathbf{E}$

$$\frac{U(t+h)-I}{h}A(t)^{-1}x \longrightarrow -x \qquad \text{as } h \downarrow 0 \tag{5.8}$$

provided $\tau \leqq t < \chi$

Lemma 5.2.

$$\tilde{L}_1(t,\tau) \longrightarrow 0 \text{ as } t-\tau \downarrow 0 \tag{5.9}$$

and

$$\tilde{L}_2(t,\tau) \longrightarrow 0 \text{ as } t-\tau \downarrow 0 \tag{5.10}$$

where $0 \leqq \tau < t \leqq \chi$.

And we borrow the following two lemmas from calculus.

Lemma 5.3. *For any* $\varepsilon > 0$, *let* $g(t,s)$ *be a uniformly continuous function with values in* $\mathbf{E}$, *for* $0 \leqq \tau \leqq s < t \leqq \chi$ *and* $t - s \geqq \varepsilon$. *If* $g(t,s)$ *satisfies*

$$\int_\tau^t \|g(t,s)\|\mathrm{d}s \longrightarrow 0 \qquad \text{as } t-\tau \downarrow 0,$$

then $\int_\tau^t g(t,s)\mathrm{d}s$ *is a uniformly continuous function in* t, τ *for* $0 \leqq \tau \leqq t \leqq \chi$.

Lemma 5.4. *Let* $f(s)$ *be a continuous function with values in* $\mathbf{E}$ *for* $\tau \leqq s < t$. If $\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^+ f(s) = \lim_{h\downarrow 0}\frac{f(s+h)-f(s)}{h}$ *exists and is uniformly continuous for* $\tau \leqq s < t$ *then* $\frac{\mathrm{d}}{\mathrm{d}s}f(s)$ *exists and* $\frac{\mathrm{d}}{\mathrm{d}s}f(s) = \left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^+ f(s)$ *for* $\tau \leqq s < t$.

Admitting lemma 5.1 and lemma 5.2 for the moment, let us finish the proof of (5.2) and the continuity of $\frac{\mathrm{d}}{\mathrm{d}t}w(t)$. Indeed, by lemma 2.8, $I_1(h)$ tends to $f(t)$ as

$h \downarrow 0$. By lemma 5.1 and lemma 5.2 we have $w(t) \in D_A$ and $I_2(h)L(t, \tau)$ tends to $-A(t)w(t)$ as $h \downarrow 0$. So, $\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^+ w(t)$ exists and satisfies

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^+ w(t) + A(t)w(t) = f(t) \qquad \text{for } \tau \leqq t < \chi$$

Then, by lemma 5.2, lemma 5.3, lemma 5.4, we see that $\frac{\mathrm{d}}{\mathrm{d}t} w(t)$ exists and is continuous for $\tau \leqq t \leqq \chi$ and that (5.2) holds.

We proceed now to the proof of lemma 5.1.

Proof of lemma 5.1. Consider the function $\phi(s) = \exp(-(t-s)A(t))U(s, \tau)x$ for $x \in \mathrm{E}$. It is continuously differentiable in (τ, t) and

$$\phi'(s) = \exp(-(t-s)A(t))(A(t) - A(s))U(s, \tau)x.$$

Integrating the both sides, we have

(5.11)

$$U(t, \tau)x = \exp(-(t-\tau)A(t))x + \int_\tau^t \exp(-(t-s)A(t))(A(t) - A(s))U(s, \tau)x\mathrm{d}s.$$

In view of (5.11), we can write, with $A(t)^{-1}x$ instead of x,

$$\begin{aligned} \frac{U(t+h) - I}{h} A(t)^{-1}x &= \frac{\exp(-hA(t+h)) - I}{h} A(t)^{-1}x \\ &+ \frac{1}{h}\int_\tau^{t+h} \exp(-(t+h-s)A(t+h))(A(t+h) \\ &- A(s))A(s)^{-1}(A(s)U(s, \tau)A(t)^{-1})x\mathrm{d}s \\ &= B_1(h) + B_2(h) \end{aligned}$$

Using (1.3), (1.5), (4.7), we have

$$\|B_2(h)\| \leqq \frac{C}{h}\|x\| \int_t^{t+h} \omega(t+h-s)\mathrm{d}s \leqq c\|x\|\omega(h) \longrightarrow 0 \quad \text{as } h \downarrow 0.$$

And using lemma 2.7, we have

$$B_1(h) = \frac{-1}{h}\int_0^h A(t+h)\exp(-\sigma A(t+h))A(t)^{-1}x\mathrm{d}\sigma \longrightarrow -x \text{ as } h \downarrow 0.$$

This completes the proof of lemma 5.1.

We proceed to the proof of lemma 5.2.

Proof of lemma 5.2. We write

$$\tilde{L}_1(t, \tau) \leqq \int_\tau^t \|A(t)(U(t, s) - \exp(-(t - s)A(t)))\| \; \|f(s) - f(t)\| \mathrm{d}s$$

$$+ \int_\tau^t \|A(t)\exp(-(t - s)A(t))\| \; \|f(s) - f(t)\| \mathrm{d}s$$

$$\leqq C \int_\tau^t \|A(t)(U(t, s) - \exp(-(t - s)A(t)))\| \mathrm{d}s$$

$$+ C \int_\tau^t \frac{\|f(t) - f(s)\|}{t - s} \mathrm{d}s,$$

where C denotes a constant independent of t, τ. In view of (1.12), it suffices to prove (5.7). So, we write

$$A(t)(U(t, s) - \exp(-(t - s)A(t)))$$

$$= A(t)(\exp(-(t - s)A(s)) - \exp(-(t - s)A(t)))$$

$$+ A(t) \int_s^t \exp(-(t - r)A(r))\Phi(r, s)\mathrm{d}r$$

$$= A(t)M_1(t, s) + A(t)M_2(t, s).$$

Using (2.17), we have

$$\int_\tau^t \|A(t)M_1(t, s)\| \mathrm{d}s \leqq C \int_\tau^t \tilde{\omega}(t - s)\mathrm{d}s = C \int_0^{t-\tau} \tilde{\omega}(r)\mathrm{d}r \longrightarrow 0$$

$$\text{as } t - \tau \downarrow 0.$$

As for $M_2(t, s)$, we write

$$M_2(t, s) = \int_s^t \exp(-(t - r)A(r))(\Phi(r, s) - \Phi(t, s))\mathrm{d}r$$

$$+ \int_s^t \exp(-(t - r)A(r))\mathrm{d}r\, \Phi(t, s)$$

$$= M_{2.1}(t, s) + M_{2.2}(t, s).$$

Using (3.24), (2.12), we have

$$\int_\tau^t \|A(t)M_{2.2}(t, s)\| \mathrm{d}s \leqq C \int_\tau^t \tilde{\omega}(t - s)\mathrm{d}s = C \int_0^{t-\tau} \tilde{\omega}(r)\mathrm{d}r \longrightarrow 0$$

$$\text{as } t - \tau \downarrow 0.$$

Using (3.6) and lemma 3.3, we have

$$\int_\tau^t \|A(t)M_{2.1}(t, s)\| \mathrm{d}s$$

$$\leqq \int_\tau^t \mathrm{d}s \int_s^t \|A(t)\exp(-(t - r)A(r))\| (\|J_1(t, r, s)\| + \|J_2(t, r, s)\|$$

$$+ \|K_1(t, r, s)\| + \|K_2(t, r, s)\| + \|K_3(t, r, s)\|)\mathrm{d}r$$

$$\leqq \int_\tau^t \mathrm{d}s \int_s^t \frac{\tilde{\omega}(t-r)}{t-s}\mathrm{d}r + \int_\tau^t \mathrm{d}s \int_s^t \frac{\tilde{\omega}(r-s)}{t-s}\mathrm{d}r$$

$$+ \int_\tau^t \mathrm{d}s \int_s^t \tilde{\omega}(t-r)\mathrm{d}r \int_s^r \frac{\tilde{\omega}(u-s)}{t-u}\mathrm{d}u$$

$$+ \int_\tau^t \mathrm{d}s \int_s^t \mathrm{d}r \int_s^r \frac{\tilde{\omega}(r-u)\tilde{\omega}(u-s)}{t-u}\mathrm{d}u$$

$$+ \int_\tau^t \mathrm{d}s \int_s^t \frac{1}{t-r}\mathrm{d}r \int_r^t \tilde{\omega}(t-u)\tilde{\omega}(u-s)\mathrm{d}u$$

$$= N_1 + N_2 + N_3 + N_4 + N_5.$$

It is easy to see that $N_1 = N_2$. Putting $t - s = s'$ $t - r = r'$, and using (1.6), (1.8), we have

(5.13)

$$N_1 = \int_0^{t-\tau} \tilde{\omega}(r')\mathrm{d}r' \int_{r'}^{t-\tau} \frac{1}{s'}\mathrm{d}s' \leqq \int_0^{t-\tau} \tilde{\omega}(r')(|\log(t-\tau)| + |\log r'|)\mathrm{d}r'$$

$$\longrightarrow 0 \text{ as } t - \tau \downarrow 0$$

It is easy to see that $N_3 = N_4$. Putting $u - s = u'$, $t - r = r'$, $t - s = s'$, we have

$$N_3 = \int_0^{t-\tau} \frac{1}{s'}\mathrm{d}s' \int_0^{s'} \omega(r')\mathrm{d}r' \int_0^{s'-r'} \left(\tilde{\omega}(u') + \frac{\omega(u')}{s'-u'}\right)\mathrm{d}u'$$

$$\leqq \int_0^{t-\tau} \frac{1}{s'}\mathrm{d}s' \int_0^{s'} \tilde{\omega}(r')\mathrm{d}r' \int_0^{t-\tau} \tilde{\omega}(r')\mathrm{d}r'$$

$$+ \int_0^{t-\tau} \tilde{\omega}(s')|\log s'|\mathrm{d}s' \int_0^{t-\tau} \tilde{\omega}(r')\mathrm{d}r'$$

$$+ \int_0^{t-\tau} \tilde{\omega}(s')|\log s'|\mathrm{d}s' \int_0^{t-\tau} \tilde{\omega}(r')\mathrm{d}r'$$

$$\leqq \int_0^{t-\tau} \tilde{\omega}(r')\mathrm{d}r' \int_0^{t-\tau} \tilde{\omega}(s')(|\log(t-\tau)| + 3|\log s'|)\mathrm{d}s'.$$

Using (1.6), (1.8) we see that N_3 tends to 0 as $t - \tau \downarrow 0$. As for N_5, putting $t - r = r'$, $t - u = u'$, $t - s = s'$, we have

$$N_5 = \int_0^{t-\tau} \mathrm{d}s' \int_0^{s'} \tilde{\omega}(u')\tilde{\omega}(s'-u')\mathrm{d}u' \int_{u'}^{s'} \frac{1}{r'}\mathrm{d}r'$$

$$\leqq \int_0^{t-\tau} \mathrm{d}s' \int_0^{s'} \tilde{\omega}(u')\tilde{\omega}(s'-u')|\log s'|\mathrm{d}u'$$

$$+ \int_0^{t-\tau} \mathrm{d}s' \int_0^{s'} \tilde{\omega}(u')\tilde{\omega}(s'-u')|\log u'|\mathrm{d}u'$$

$$= N_{5.1} + N_{5.2}.$$

As for $N_{5.1}$, using (2.9), (1.6) and (1.8), we have

$$N_{5.1} \leqq 2\int_0^{t-\tau} \tilde{\omega}(s')|\log s'| \int_0^{t-\tau} \tilde{\omega}(u')du' \longrightarrow 0 \text{ as } t-\tau \downarrow 0.$$

As for $N_{5.1}$, we write

$$\begin{aligned} N_{5.2} &\leqq \int_0^{t-\tau} ds' \int_0^{s'/2} \tilde{\omega}(u')\tilde{\omega}(s'-u')|\log u'|du' \\ &+ \int_0^{t-\tau} ds' \int_{s'/2}^{s'} \tilde{\omega}(u')\tilde{\omega}(s'-u')|\log u'|du' \\ &\leqq 2\int_0^{t-\tau} \tilde{\omega}(s')ds' \int_0^{t-\tau} \tilde{\omega}(u')|\log u'|du' \\ &+ 2\int_0^{t-\tau} \max(|\log s'|, |\log(s'/2)|)\tilde{\omega}(s')ds' \int_0^{t-\tau} \omega(r)dr \end{aligned}$$

Using (1.6) and (1.8), we see that $N_{5.2}$ tends to 0 as $t-\tau \downarrow 0$. This completes the proof of lemma 5.2.

§ VI. Application to an initial-boundary value problem

(See A. Friedmann [1] Part 2.9) Let Ω be a bounded domain in the real n-dimensional Euclidean space $\mathbf{R}^n$. For any $0 < T < \infty$, denote by Q_T the cylinder $\{(x, t);\, x \in \Omega,\, 0 < t < T\}$ and by $\partial\Omega$ the boundary of Ω. We consider differential operators

$$(6.1) \qquad Lu = \frac{\partial u}{\partial t} + A(x, t, D)u = \frac{\partial u}{\partial t} + \sum_{|\alpha| \leqq 2m} a_\alpha(x, t) D^\alpha u$$

with complex coefficients $a_\alpha(x, t)$ defined in Q_T, where α denotes the multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$.

Definition. For $t_0 \in [0, T]$ $A(x, t^0, D)$ is said to be strongly elliptic at a point x^0 if

$$(-1)^m \operatorname{Re}\{\sum_{|\alpha| = 2m} a_\alpha(x^0, t^0)\xi^\alpha\} > 0$$

for any real vector $\xi = (\xi_1, \cdots, \xi_n) \neq 0$, where $\xi^\alpha = \xi_1^{\alpha} \cdots \xi_n^{\alpha}$. L is said to be parabolic at a point (x^0, t^0) if $A(x, t^0, D)$ is strongly elliptic at x^0. L is said to be parabolic on a set U if L is parabolic at each point of U. L is said to be uniformly parabolic in $\bar{Q}_T$ if the coefficients $a_\alpha(x, t)$ are bounded in $\bar{Q}_T$ and if

$$(-1)^m \mathrm{Re}\{\sum_{|\alpha|=2m} a_\alpha(x, t)\xi^\alpha\} \geqq C|\xi|^{2m}$$

for all $(x, t) \in \bar{Q}_T$ and for all real ξ, where C is a positive constant independent of x, t, ξ and $|\xi| = (\xi_1^2 + \cdots + \xi_n^2)^{1/2}$.

We consider the Cauchy problem of the evolution equation

$$\text{(6.1)} \qquad \begin{cases} \dfrac{du}{dt} + A(t)u = f(t) & 0 < t \leqq T \\ u(0) = u_0, \end{cases}$$

in the Hilbert space $L^2(\Omega)$, where for each t, $f(t)$ is the function $f(x, t)$ belonging to $L^2(\Omega)$, and $A(t)$ is the operator with domain $D_A = H^{2m}(\Omega) \cap H_0^m(\Omega)$ given by $A(t)v(x) = A(x, t, D)v(x)$. And u_0 is a function in $L^2(\Omega)$.

Definition. $u(t)$ is said to be a strong solution of (6.1) if $u(t) \in C^0([0, T], L^2(\Omega)) \cap C^1((0, T,], L^2(\Omega))$ and satisfies (6.1).

We assume the following assumptions:

(E_1) The coefficients $a_\alpha(x, t)$ are continuous in $\bar{Q}_T$, and

$$|a_\alpha(x, t) - a_\alpha(x, t')| \leqq C|\log|t - t'||^\gamma$$

for all $x \in \Omega$, $t \in [0, T]$ and $t' \in [0, T]$, where C, γ are constants independent of x, t, t' and $\gamma < -2$.

(E_2) The inhomogeneous term $f(t)$ is continuous in $[0, T]$, and

$$\|f(t) - f(t')\| \leqq C|\log|t - t'||^\delta$$

for all $t \in [0, T]$, $t' \in [0, T]$, where C, δ are constants independent of x, t, t' and $\delta < -2$.

Proposition. *Assume that L is uniformly parabolic in $\bar{Q}_T$, that the conditions* (E_1), (E_2) *hold, and that $\partial\Omega$ is of class C^{2m}. Then there exists a unique strong solution of* (6.1).

Proof. From the a priori inequality of the elliptic operator $A(t)$, we see that, for some positive constant k, $\tilde{A}(t) = A(t) + kI$ satisfies the conditions (A_1), (A_2) and (A_3). From the assumption (E_1) and (E_2), we see that, for $\omega(r) = |\log r|^\gamma$, $\tilde{A}(t)$ satisfies the conditions (A_4) and $f(t)$ satisfies the condition (F). Set $v(t) = \exp(-kt)u(t)$. Then (6.1) take the equivalent form

$$\text{(6.2)} \qquad \begin{cases} \dfrac{dv}{dt} + \tilde{A}(t)v = \exp(-kt)f(t) \\ v(0) = u_0. \end{cases}$$

We can thus apply our Main Theorem to deduce this proposition. Q.E.D.

§ VII. Necessity of regularity assumption

Though our conditions are not a set of necessary conditions to obtain general existence and uniqueness theorem, here we give an example of the inhomogeneous term which fails to satisfy our condition (F) and for which the existence and uniqueness theorem in the sense stated bellow fails to hold.

We consider the Chauchy problem

$$(6.3) \qquad \begin{cases} \dfrac{\partial u}{\partial t}(t, x) = \dfrac{\partial^2 u}{\partial t^2}(t, x) + f(t, x) & -\infty < x < \infty,\ 1/2 < t \leqq T \\ u(1/2, x) = 0. \end{cases}$$

We call $u(t, x)$ a strong solution of (6.3) if $u(t, x)$ belongs to $C^0([1/2, T]; L^2_x(R)) \cap C^1((1/2, T]; L^2_x(R))$.

We show an example of $f(t, x) \in C^0([1/2, T]; L^2_x(R))$ which fails to satisfy our condition (F), and for which the existence and uniqueness of the strong solution in the above sense fails to hold. Remark that $f(t, x)$ is included at least in $C^0([1/2, T]; L^2_x(R))$. Let $1 < T < 3/2$ and $f(t, x) = F_\xi^{-1}(g(t, \xi))$ for

$$g(t, \xi) = \begin{cases} \dfrac{|t-1|^{1/4}}{|\log|t-1||^\alpha(|\xi||t-1|^{1/2}+1)} & t \neq 1 \\ 0 & t = 1 \end{cases}$$

where $0 < \alpha \leqq 1$ and F_ξ^{-1} is the Fourier transform in L^2 with respect to ξ. If the existence and uniqueness of the strong solution of (6.3) hold then the solution must take the form $u(t, x) = F_\xi^{-1}(v(t, \xi))$ where

$$v(t, \xi) = \int_0^t \exp(-\xi^2(t-s))g(s, \xi)ds,$$

and $v(t, \xi)$ must take the form

$$(6.4) \qquad v(t, \xi) = g(t, \xi) + \int_0^t \xi^2 \exp(-\xi^2(t-s))g(s, \xi)ds$$

So, to deduce our assertion by contradiction it suffices to prove the following lemma.

Lemma.

$$(6.5) \qquad \int_{1/2}^1 \frac{\|g(1, \xi) - g(s, \xi)\|}{1-s} ds = \infty$$

$$(6.6) \qquad g(t, \xi) \in C^0([1/2, T]; L^2_\xi(R))$$

$$(6.7) \qquad \frac{d}{dt} v(t, \xi) \notin L^2_\xi(R) \qquad \text{at } t = 1.$$

Remark that (6.5) indicates that $f(t, x)$ fails to satisfy the condition (F).

Proof. Since

$$\begin{aligned}
&\int_{1/2}^{1} \frac{\|g(1, \xi) - g(s, \xi)\|}{1 - s} \mathrm{d}s \\
&= \int_{1/2}^{1} \mathrm{d}s \left(\int_{0}^{\infty} \frac{1}{(|\xi|(1 - s)^{1/2} + 1)^2} \mathrm{d}\xi \frac{(1 - s)^{1/2}}{|\log|1 - s||^{2\alpha}} \right)^{1/2} \\
&\geqq C \int_{1/2}^{1} \frac{1}{(1 - s)|\log(1 - s)|^{\alpha}} \mathrm{d}s = \infty,
\end{aligned}$$

(6.5) holds. As for (6.6) $g(t, \xi) \in L_{\xi}^2(R)$ is clear. If $t \neq 1$ then by Lebesgue convergence theorem we have

$$\int_{-\infty}^{\infty} |g(t + h, \xi) - g(t, \xi)|^2 \mathrm{d}\xi \longrightarrow 0 \qquad \text{as } h \longrightarrow 0.$$

And we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} |g(1 + h, \xi) - g(1, \xi)|^2 \mathrm{d}\xi \\
&\leqq C \int_{0}^{\infty} \left(\frac{1}{\xi |h|^{1/2} + 1} \right)^2 \mathrm{d}\xi \frac{|h|^{1/2}}{|\log|h||^{2\alpha}} \\
&\leqq C \frac{|h|^{1/2}}{|h|^{1/2} |\log|h||^{2\alpha}} \longrightarrow 0 \qquad \text{as } h \longrightarrow 0.
\end{aligned}$$

Thus (6.6) hols. As for (6.7), in view of (6.4), it suffices to prove

$$\int_{1/2}^{1} \xi^2 \exp(-\xi^2(1 - s)) g(s, \xi) \mathrm{d}s \notin L_{\xi}^2(R).$$

Writing

$$I = \int_{-\infty}^{\infty} \mathrm{d}\xi \left(\int_{1/2}^{1} \xi^2 \exp(-\xi^2(1 - s)) \frac{(1 - s)^{1/4}}{|\log|1 - s||^{\alpha}(|\xi||1 - s|^{1/2} + 1)} \mathrm{d}s \right)^2,$$

and putting $1 - s = a$, we have

$$I = 2 \int_{0}^{\infty} \xi^4 \mathrm{d}\xi \left(\int_{1/2}^{1} \exp(-\xi^2 a) \frac{a^{1/4}}{|\log a|^{\alpha}(\xi a^{1/2} + 1)} \mathrm{d}a \right)^2.$$

So putiing $\xi^2 a = x$ we have

$$\begin{aligned}
I &= 2 \int_{0}^{\infty} \xi^{-1} d\xi \left(\int_{\xi^2/2}^{\xi^2} \exp(-x) \frac{x^{1/4}}{|\log(x/\xi^2)|^{\alpha}(\sqrt{x} + 1)} dx \right)^2 \\
&\geqq 2 \int_{4}^{\infty} \xi^{-1} d\xi \left(\int_{\xi^2/2}^{\xi^2} \exp(-x) \frac{x^{1/4}}{|\log(1/\xi^2)|^{\alpha}(\sqrt{x} + 1)} dx \right)^2 \\
&\geqq C \int_{4}^{\infty} \frac{1}{\xi |\log \xi|^{\alpha}} d\xi \left(\int_{\xi^2/2}^{\xi^2} \exp(-x) \frac{x^{1/4}}{\sqrt{x} + 1} dx \right)^2
\end{aligned}$$

$$\geqq C \int_0^\infty \frac{1}{\xi |\log \xi|^\alpha} d\xi = \infty$$

This completes the proof of the lemma.

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Added in proof: The condition (1.7) is deduced from the condition (1.6).

At the proofreading stage, the author was informed that T. Muramatu [6], [7] treat similar themes from the Besov spaces point of view.