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Nonsolvability for differential operators with multiple complex characteristics

By

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Introduction

This paper examines the local solvability in the Gevrey classes G^σ, $1 < \sigma \leq \infty$, ($G^\infty := C^\infty$) of linear analytic partial differential operators $P(x, D)$ of the following type

$$P = A^r + \textit{lower order terms}\,, \qquad r \in \mathbf{N},\ r \geq 2$$

with A being a classical analytic pseudodifferential operator of complex-valued principal type modelled microlocally near a point by the Mizohata operator $D_n + iy_n^h D_1$, $h \in \mathbf{N}$.

The main result of our paper states that if h is odd then $P(x, D)$ is not locally σ-solvable for every $1 < \sigma \leq \infty$, without any restrictions on the lower order terms or r. In particular we recover the results in the C^∞ category of F. Cardoso, F. Treves [2], P. Popivanov [20] for $r = 2$ and R. Goldman [10] ($r \geq 3$, nonzero subprincipal symbol) and F. Cardoso [3] (for every $1 < \sigma < \infty$, $r = 2$). We recall that in the case simple characteristics $P(x, D)$ is locally σ-(non) solvable when h is (odd) even, $1 < \sigma \leq \infty$ ($\sigma = \infty$ stands for the C^∞ case) cf. [9], [16], [20].

Microlocal σ-(non) hypoellipticity for p.d.o.-s of the type above was shown by L. Cattabriga, L. Rodino and L. Zanghirati [6] and one uses their estimates on the infinite order p.d.o.-s which are parametrices in G^σ, $1 < \sigma < r/(r-1)$. We mention also the papers of A. Corli [7], [8] on σ-nonsolvability for operators with multiple real characteristics.

Further we study the possibility h even. We exhibit a class of operators in $\mathbf{R}^2$ which are σ-solvable when $1 < \sigma < \sigma_0$ but not σ-solvable if $\sigma > \sigma_1$, $\sigma_1 > \sigma_0 > r/(r-1)$ are expressed explicitly by r, h and a positive even integer, related to the lower order term.

The main results are stated in section 1. The proof of σ-nonsolvability in the case h being odd is contained in sections 2 and 3, while the last one deals with the case h being even.

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1. Statement of the main results

We recall that if $\Omega \subset \mathbf{R}^n$ is an open set and $1 < \sigma \leq \infty$ then the class of Gevrey functions of order σ, $G^\sigma(\Omega)$, consists of all $f(x) \in C^\infty(\Omega)$ s.t.$\forall K \subset\subset \Omega$, $\exists C = C_{K,f} > 0$ s.t.

$$\sup_{x \in K} |f(x)| \leq C^{|\alpha|+1}(\alpha!)^\sigma, \qquad \alpha \in \mathbf{Z}^n_+ \tag{1}$$

and one defines inductive topology of $G^\sigma(\Omega)$ (see [14]). The set $G^1(\Omega)$ coincides with the space of all real analytic functions in Ω. If $\sigma > 1$ we put $G_0^\sigma(\Omega) = G^\sigma(\Omega) \cap C_0^\infty(\Omega)$ and write $G^{(\sigma)'}(\Omega)$ $(G_0^{(\sigma)'}(\Omega))$ for the space of σ-ultradistributions (with compact support), namely the dual of $G_0^\sigma(\Omega)$ $(G^\sigma(\Omega))$ in the corresponding topology.

An analytic partial differential operator $P(x, D)$, $x \in \Omega$, is called locally σ-solvable at $x^0 \in \Omega$, $1 < \sigma \leq \infty$ iff there exists a neighbourhood V of x_0 having the next property: $\forall f \in G_0^{(\sigma)}(V)$, $\exists u \in G^{(\sigma)'}(\Omega)$ s.t. $P(x, D)u = f$ in V. Here $G_0^\infty(\Omega)$ $(G_0^{(\infty)'}(\Omega))$ stands for $C_0^\infty(\Omega)$ $(\mathscr{D}'(\Omega))$.

We consider an analytic differential operator $P(x, D)$ of order $m \geq 2$ with principal symbol $p_m(x, \xi)$ satisfying the next requirement: there exist $\rho^0 = (x^0, \xi^0) \in T^*(\Omega)\backslash 0$, a neighbourhood ω of x^0, an open cone $\Gamma \in \mathbf{R}^n\backslash 0$ containing ξ^0 and a positive integer $r \leq m$ s.t. denoting $\Gamma_\omega = \omega \times \Gamma$ we can write

$$p_m(x, \xi) = q_{m-r}(x, \xi)(a_1(x, \xi))^r, \qquad (x, \xi) \in \Gamma_\omega \tag{2}$$

where q_{m-r} is an analytic elliptic symbol in Γ_ω, $\operatorname{ord}_\xi q_{m-r} = m - r$, $a_1(x, \xi)$ is a first order complex valued principal symbol i.e. $d_\xi a_1(x, \xi) \neq 0$ on $\Sigma = \{(x, \xi) \in \Gamma_\omega: p_m(x, \xi) = 0\}$.

Without loss of generality (multiplying a_1 by i and shrinking Γ_ω if necessary) we assume $d_\xi(\operatorname{Re} a_1(x, \xi)) \neq 0$ in Γ_ω. We demand further that ρ^0 and $\operatorname{Im} a_1(x, \xi)$ satisfy

> $\operatorname{Im} a_1(x, \xi)$ has a zero of odd order along the bicharacteristic of $\operatorname{Re} a_1(x, \xi)$ through ρ^0 and it changes its sign from $-$ to $+$.

Under the hypotheses above we claim.

Theorem 1.1. *The operator $P(x, D)$ is not locally σ-solvable at x^0 for every $1 < \sigma \leq \infty$.*

Next we study operators in $\mathbf{R}^2$ for which the order h of the vanishing of $\operatorname{Im} a_1$ on the characteristic set of $\operatorname{Re} a_1$ is even. In order to simplify the statements we consider an operator P defined by its adjoint P^* as follows

$$P^*(x, D) = (D_2 + idx_2^h D_1)^m + \sum_{j=0}^{m-1} B_j(x_2, D_1)(D_2 + idx_2^h D_1)^{m-1-j}, \tag{3}$$

where $d \in \mathbf{R}\backslash 0$, h is even and $B_j(x_2, D_1)$ is a differential operator in D_1 of order j with coefficients real analytic functions of x_2, $0 \leq j \leq m-1$, $D_s = i^{-1}\partial_{x_s}$, $s = 1, 2$.

We suppose that the subprincipal symbol of P is nonzero on the characteristic set i.e. there exists $c > 0$ s.t.

$$|B_{m-1}(t, \xi)| \geq c|\xi|^{m-1}, \qquad |t| \ll 1, \quad |\xi| \gg 1. \tag{4}$$

In view of (4) the algebraic equation with respect to λ

$$\lambda^m + \sum_{j=0}^{m-1} B_j(t, \xi)\lambda^{m-1-j} = 0$$

has for $|t| \ll 1$, $|\xi| \gg 1$, m different roots $l_k(t, \xi)$, $1 \leq k \leq m$ which are real analytic functions and they are analytic symbols of order $(m-1)/m$ (see [12])

$$l_k(t, \xi) \asymp \sum_{s=0}^{\infty} v_{k,s}^{\pm}(t)|\xi|^{(m-1-s)/m}, \qquad \pm\xi \gg 1, \ 1 \leq k \leq m. \tag{5}$$

We assume that for some $1 \leq j \leq m$ there exists $b > 0$ and positive integer μ s.t. for $|t| \ll 1$ and $v = v_{j,0}^{+}$ or $v = v_{j,0}^{-}$

$$\omega(t) = \operatorname{Im} \int_0^t v(t')dt' \geq bt^{2\mu}, \quad |v_{j,s}^{\pm}(t)| = O(|t|^{2\mu-1}) \quad s = 1, \dots, m-2 \ \textit{if} \ m \geq 3 \tag{6}$$

We demand also $m \geq 2$ and

$$h + 1 > 2\mu m/(m-1) \tag{7}$$

Here is the second principal result of the paper.

Theorem 1.2. *Let h be even and let the hypotheses* (6) *and* (7) *hold. Then the operator $P(x, D)$ given via its adjoint by* (3) *is not σ-solvable at* 0 *for $\gamma \leq \sigma \leq \infty$, where* $\gamma = \dfrac{m(h+1-2\mu)}{(h+1)(m-1) - 2\mu m}$.

Remark 1.1. According to [12] $P(x, D)$ and $P^*(x, D)$ are solvable for $1 \leq \sigma < \dfrac{mh}{mh - h - 1}$ and $P^*(x, D)$ is not σ-hypoelliptic if $\gamma \leq \sigma \leq \infty$. We point out that the previous assertion remains true also for other classes of operators, whose adjoints are not C^∞ hypoelliptic but are analytically hypoelliptic (see [6], [12], [21] for such operators).

2. Asymptotic solution to the reduced system

We take and fix arbitrary $1 < \sigma < r/(r-1)$. Evidently σ-nonsolvability implies σ'-nonsolvability for every $\sigma < \sigma' \leq \infty$.

The set of points for which $P(x, D)$ is σ-nonsolvable is closed so it is enough to show that there exists a sequence $\{x^j\}_{j=1}^{\infty}$ s.t. $\lim_{j \to \infty} x^j = x^0$ and P is not σ-solvable at x^j, $j \geq 1$. It is well known that the assumption on $a_1(x, \xi)$ imply the existence of a sequence $\{\rho^j\}_{j=1}^{\infty} \subset T^*\Omega \backslash 0$, $\rho^j = (x^j, \xi^j) \xrightarrow{j \to \infty} \rho^0$ and conic neighbourhoods $\Gamma_{\omega_j} \ni \rho^j$, $j = 1, 2, \dots$ s.t. $\operatorname{Im} a_1(x, \xi)$ vanishes of fixed odd order

h_j on Σ along each null-bicharacteristic of $\operatorname{Re} a_1(x, \xi)$, $(x, \xi) \in \Gamma_{\omega_j}$ and the change of sign is always from $-$ to $+$. So we assume without loss of generality that this property (h fixed for ρ^0 and Γ_ω) holds and moreover as in [6] after the eventual renumerating of the variables and shrinking of Γ_ω we can suppose

$$\partial_{\xi_n} \operatorname{Re} a_1(x, \xi) \neq 0\,, \qquad (x, \xi) \in \Gamma_\omega. \tag{8}$$

One observes that P^* has the same property h odd and fixed on Γ_ω but with the change of the sign from $+$ to $-$.

We recall the necessary condition for local σ-solvability namely if the operator $P(x, D)$ is σ-solvable at x^0 then for every small enough neighbourhood ω of x^0 and every $T > 0$, $\varepsilon \in (0, T)$ there exists a positive constant $C = C(\omega, T, \varepsilon)$ s.t.

$$(\|u\|_{C(\omega)})^2 \leq C \|u\|_{T-\varepsilon,\sigma} \|P^* u\|_{T-\varepsilon,\sigma}\,, \qquad \forall u \in G_0^\sigma(\omega : T)\,, \tag{9}$$

where $\|\cdot\|_{C(\omega)}$ is the supremum norm in $C(\omega)$; $G_0^\sigma(\omega : T) = G_0^\sigma(\omega) \cap G^\sigma(\omega : T)$, $G^\sigma(\omega : T)$ is the Banach space of all functions $u(x) \in G^\sigma(\omega)$ s.t.

$$\|u\|_{T,\sigma} = \sum_{\alpha \in \mathbf{Z}_+^n} (\alpha!)^{-\sigma} T^{|\alpha|} \|\partial_x^\alpha u\|_{L^2(\omega)} < \infty\,.$$

We have to use cut-off functions from G^θ, $\theta = 1 + \varepsilon$, $0 < \varepsilon \ll 1$ in the constructions of p.d.o.-s and F.I.O.-s. Indeed if $\sum_{j=0}^{\infty} A_{m-j}(x, \xi)$ is a classical analytic symbol defined in a conic neighbourhood $\omega \times \Gamma \ni (x_0, \xi^0)$ we realize it as p.d.o. with an amplitude $A(x, \xi)$ given by

$$A(x, \xi) = \varphi(x) \sum_{j=0}^{\infty} A_{m-j}(x, \xi) \kappa(\xi) \chi(\xi/(M(j + 1)))$$

with $\varphi(x) \in G_0^\theta(\mathbf{R}^n)$, $\varphi = 1$ near x^0, $\kappa(\xi) \in G^\theta(\mathbf{R}^n)$ being homogeneous of zero order for $|\xi| \geq 1$ and supported in Γ, $\kappa(\xi) = 1$ for $\xi \in \Gamma_0 \subset\subset \Gamma$, Γ_0 is a smaller conic neighbourhood of ξ^0; $\chi(\xi) \in G^\theta(\mathbf{R}^n)$, $\chi(\xi) = 1(=0)$ for $|\xi| \geq 1(|\xi| \leq 1/2)$ and the constant $M > 0$ is large enough. Then we call $A(x, \xi)$ a θ-amplitude which is suggested by the next estimates:

$$|\partial_\xi^\alpha \partial_x^\beta A(x, \xi)| \leq C^{|\alpha|+|\beta|+1} (\alpha! \beta!)^\theta \langle \xi \rangle^{m-|\alpha|}\,, \qquad \langle \xi \rangle = \sqrt{1 + |\xi|^2}\,,$$

where $C > 0$ is independent from α, $\beta \in \mathbf{Z}_+^n$, $(x, \xi) \in \mathbf{R}^{2n}$. The formal calculus of p.d.o. and F.I.O. with such amplitudes is valid as in the C^∞ case via the stationary phase method but with a loss of Gevrey regularity of the realization, namely after the composition of two operators with θ-amplitudes the amplitude of the resulting operator is $2\theta - 1(= 1 + 2\varepsilon)$-amplitude cf. [11]. As we will perform finite number of compositions, if ε is fixed small enough then the Gevrey index of the amplitudes of the resulting p.d.o.-s and F.I.O.-s will be always less than the fixed in the beginning of the proof $\sigma > 1$. In view of this observation we shall use the same letter θ for the Gevrey index of the realizations.

The conditions on the principal symbol p_m yield the existence of an analytic homogeneous canonical map $\phi\colon \Gamma_\omega \to T^*\mathbf{R}^n \backslash 0$ with $\phi(\rho^0) = (0; \tilde{1}) = \tilde{\rho}^0$, $\tilde{1} =$

$(1, 0, \dots, 0)$ and an analytic elliptic F.I.O. E_ϕ, defined mcl near $\tilde{\rho}^0$, associated to ϕ, and a θ-p.d.o. $Q(y, D)$, analytic and elliptic mcl near $\tilde{\rho}^0$ s.t. the adjoint operator P^* is written as

$$P^*(x, D) = E_\phi \circ Q(y, D) \circ \tilde{P}(y, D) \circ E_\phi^* + R(x, D)\,, \tag{10}$$

where if one sets $M_h = D_n - iy_n^h D_1$ then $\tilde{P}$ is represented in the following form

$$\tilde{P}(y, D) = (M_h)^r + \sum_{j=0}^{r-1} P_j(y, D') \circ (M_h)^j \tag{11}$$

with P_j having θ-amplitude of order $r - 1 - j$, supported in a conic neighborhood of $\tilde{\rho}^0$, analytic mcl near $\tilde{\rho}^0$, $0 \le j \le r - 1$, E_ϕ^* stands for the adjoint of E_ϕ while $R(x, D)$ is with a θ-amplitude of order m and is θ-regular mcl in Γ_ω i.e. $\exists C > 0$ s.t.

$$|\partial_\xi^\alpha \partial_x^\beta R(x, \xi)| \le C^{|\alpha|+|\beta|+1} (\alpha!\beta!)^\theta \langle \xi \rangle^{m-|\beta|} \exp(-c|\xi|^{1/\theta})\,, \tag{12}$$

$$(x, \xi) \in \Gamma_\omega\,, \quad \alpha, \beta \in \mathbf{Z}_+^n\,, \quad \langle \xi \rangle := (1 + |\xi|^2)^{1/2}\,.$$

The F.I.O. E_ϕ is defined in the following way

$$E_\phi w(x) = \int_{\mathbf{R}_{y'}^{n-1}} \int_{\mathbf{R}_{\eta'}^{n-1}} \exp(i\phi(x, \eta') - iy'\eta') e(x, y', \eta') w(y', x_n) dy' d\eta'\,, \tag{13}$$

where $e(x, y', \eta')$ is a θ-amplitude, supported in $\omega \times \tilde{\omega} \times \tilde{\Gamma}$ and is analytic symbol in a smaller conic neighbourhood $\omega_0 \times \tilde{\omega}_0 \times \tilde{\Gamma}_0$.

We recall from [6] the reduction of $\tilde{P}(y, D)$ mcl near ρ^0 to a system $M_h I + A(y, D')$, acting on vector functions $U = (u_1, \dots, u_r)^t$, where I is the $r \times r$ unit matrix and $A(y, D')$ is a $r \times r$ matrix-valued θ-p.d.o. of order $p := (r - 1)/r$, namely $A(y, D') = \{A_{ij}(y, D')\}_{i,j=1}^r$, where $A_{rj}(y, D') = P_{j-1}(y, D')|D_1|^{p(r-j)}$, $1 \le j \le r$, $A_{jj+1}(y, D') = -|D_1|^p$, $1 \le j \le r - 1$ and $A_{ij}(y, D') = 0$ in all other cases. Then u solves $\tilde{P}(y, D)u = f$ iff $u = |D_1|^{-p(r-1)} u_1$, u_1 is the first component of a solution $U = (u_1, \dots, u_r)^t$ of the system $(M_h I + A(y, D'))U = (f, 0, \dots, 0)^t$. Here $|D_1|^s$ denotes a θ-p.d.o. with symbol equal to η_1^s mcl near $\tilde{\rho}^0$ and supported in $\tilde{\omega} \times \tilde{\Gamma}$. According to Lemma 2.3. in [6] there exists a series of $r \times r$ infinite order p.d.o.-s $\sum_{\nu=0}^\infty E_\nu(y, D')$, defined in $\tilde{\omega} \times \tilde{\Gamma}$, satisfying formally

$$(M_h I + A(y, D')) \circ \sum_{\nu=0}^\infty E_\nu(y, D') = \sum_{\nu=0}^\infty E_\nu(y, D') \circ M_h I\,. \tag{14}$$

More precisely we have for $\nu \ge 0$

$$(M_h + A(y, \eta'))E_\nu(y, \eta') = F_\nu(y, \eta')\,, \qquad (y, \eta') \in \tilde{\omega} \times \tilde{\Gamma}\,, \tag{15}$$

$$E_0(y', 0, \eta') = I\,, \qquad E_\nu(y', 0, \eta') = 0\,, \qquad \nu \ge 1\,, \tag{16}$$

$$F_\nu(y, \eta') = -\sum_{k=1}^{\nu} \sum_{|\alpha|=k} (\alpha!)^{-1} D_\eta^\alpha A(y, \eta') \partial_y^\alpha E_{\nu-k}(y, \eta') \tag{17}$$

and by definition $F_0 = 0$. Moreover according to [6] we can find three positive numbers D, c and C s.t.

(18) $$\|\partial^\alpha_{\eta'}\partial^\beta_y E_\nu(y,\eta')\| \le C^{|\alpha|+|\beta|+|\nu|+1}\alpha!\beta!\nu!\langle\xi\rangle^{-\nu-|\alpha|}\exp(c|\eta'|^p),$$
$$(y,\eta')\in\tilde{\Gamma}_{\tilde{\omega}}, \quad |\eta'|\ge D,\ \beta\in\mathbf{Z}^n_+,\ \alpha\in\mathbf{Z}^{n-1}_+,\ \nu=0,1,\dots$$

Now we are ready to start the construction of an asymptotic solution to $P^*(x,D)u=0$ in G^σ. The first step is to define such solution for $\tilde{P}(y,D)$.

Let $\chi(\eta')\in G^\theta(\mathbf{R}^{n-1})$, $\operatorname{supp}\chi\subset\Gamma\cap\{|\eta'|\ge 2C_1\}$, $\chi(t\eta')=\chi(\eta')$ for $t\ge 1$, $|\eta'|\ge 3C_1$ and $\chi(\eta')=1$, $\eta'\in\Gamma_0\cap\{|\eta'|\ge 3C_1\}$, where $C_1=\max\{C,D\}$, C, D as in (18), $\Gamma_0\subset\subset\Gamma\subset\mathbf{R}^{n-1}_{\eta'}\backslash 0$ are conic neighbourhoods of $\eta'=(1,0,\dots,0)$. We choose and fix functions $\psi(y)\in G^\theta_0(\omega)$, $\psi(y)=1$ on $\omega_0\subset\subset\omega$, ω_0 is a small neighbourhood of $(0,\dots,0)$ and $\kappa(s)\in G^\theta_0(\mathbf{R})$, $\operatorname{supp}\kappa\in[1,4]$, $\kappa(s)=1$ if $2<s<3$. Setting $\kappa_\lambda(\eta')=\lambda^{-n+1}\kappa(|\eta'|/\lambda)$ for $\lambda\ge 1$ we define consecutively

(19) $$\tilde{E}(y,\eta')=\psi(y)\sum_{\nu=0}^{\infty}E_\nu(y,\eta')\chi(\eta'/(\nu+1)),$$

(20) $$\omega^\lambda(y)=\int_{\mathbf{R}^{n-1}_{\eta'}}\exp(i\Phi(y,\eta'))E(y,\eta')\kappa_\lambda(\eta')d\eta'$$

with $E(y,\eta')$ being the r-vector $\tilde{E}(y,\eta')(1,\dots,1)^t$, $\Phi=iy_n^{h+1}\eta_1/(h+1)-y'\eta'$. Note that (16) implies for some $c>0$

$$E(y',0,\eta')=(1,\dots,1)^t+O(\exp(-c|\eta'|^{1/\theta})),\qquad \eta'\in\mathbf{R}^{n-1}.$$

In view of (18) and (19) $E(y,\eta')$ verifies for $C>0$ large enough the next estimates (recall that $p=r/(r-1)$)

(21) $$\|\partial^\alpha_{\eta'}\partial^\beta_y E(y,\eta')\|\le C^{|\alpha|+|\beta|+1}(\alpha!\beta!)^\theta\langle\eta'\rangle^{-|\alpha|}\exp(C|\eta'|^p),$$
$$(y,\eta')\in\mathbf{R}^{2n-1},\ \beta\in\mathbf{Z}^n_+,\ \alpha\in\mathbf{Z}^{n-1}_+,$$

(22) $$\Big\|\partial^\alpha_{\eta'}\partial^\beta_y\Big(\tilde{E}(y,\eta')-\psi(y)\sum_{\nu=0}^{N}E_\nu(y,\eta')\chi(\eta')\Big)\Big\|$$
$$\le C^{|\alpha|+|\beta|+N+1}(\alpha!\beta!N!)^\theta$$
$$\times\langle\eta'\rangle^{-N-|\alpha|}\exp(C|\eta'|^p),\qquad (y,\eta')\in\mathbf{R}^{2n-1},\ \beta\in\mathbf{Z}^n_+,\ \alpha\in\mathbf{Z}^{n-1}_+,\ N\in\mathbf{Z}_+.$$

Put $f^\lambda(y)=(M_h+A(y,D'))w^\lambda$.

Proposition 2.1. *There exist $T_0>0$, $\delta>0$, $c>0$, $C(T)>0$ for $T\in(0,T_0]$ s.t.*

(23) $$\|w^\lambda\|_{T,\sigma}\le C(T)\exp(c\lambda^p),\qquad T\in(0,T_0],\ \lambda\ge 1$$

(24) $$\|w^\lambda\|_{C(\omega)}\ge c,\qquad \lambda\to\infty$$

(25) $$\|f^\lambda\|_{T,\sigma}\le C(T)\exp(-\delta\lambda^{1/\theta}),\qquad T\in(0,T_0],\ \lambda\ge 1.$$

Proof. The estimate (24) follows from the fact that (16) implies $w^\lambda(0)=\int\kappa_\lambda(\eta')d\eta'(1,\dots,1)^t+O(\exp(-c_0|\lambda|^{1/\theta}))$, $c_0>0$, $\lambda\ge 1$, while the inequality (23) is obtained by straight forward calculations taking into account (21) with $\alpha=0$.

Now we deal with the proof of (25). We have

$$f^{\lambda}(y)=\int_{\mathbf{R}^{n-1}}\exp(i\Phi(y,\eta'))F(y,\eta')\kappa_{\lambda}(\eta')d\eta' \tag{26}$$

where $F(y,\eta')=M_hE(y,\eta')+B(y,\eta')$, B is the amplitude of the composition $A(y,D')\circ E(y,D')$. In view of (22) and the formal calculus in [1], [11] we deduce that the remainders

$$R_B^M(y,\eta')=B(y,\eta')-\sum_{|\alpha|\le M}(\alpha!)^{-1}D_{\eta'}^{\alpha}A(y,\eta')\partial_{y'}^{\alpha}E(y,\eta')$$

and

$$R_E^M(y,\eta')=E(y,\eta')-\psi(y)\sum_{\nu=0}^{M}E_{\nu}(y,\eta')\chi(\eta')$$

satisfy the next estimates for some $C>0$ ($R^M=R_E^M$ or R_B^M):

$$|\partial_{\eta'}^{\alpha}\partial_y^{\beta}R^M(y,\eta')|\le C^{|\alpha|+|\beta|+M+1}(\alpha!\beta!M!)^{\theta}\langle\eta'\rangle^{-M-|\alpha|}\exp(C|\eta'|^{\rho}), \tag{27}$$

$$(y,\eta')\in\mathbf{R}^{2n-1},\ \beta\in\mathbf{Z}_+^n,\ \alpha\in\mathbf{Z}_+^{n-1},\ M=1,2,\dots$$

We recall that if Φ is a real analytic first order homogeneous phase function then $\exists C>0$ s.t. for $\forall\gamma\in\mathbf{Z}_+^n$, $0\le j\le|\gamma|$ we have

$$\partial_y^{\gamma}(\exp(i\Phi(y,\eta')))=\exp(i\Phi(y,\eta'))\sum_{j=0}^{|\gamma|}S_j^{\gamma}(\Phi)(y,\eta'),$$

$$\operatorname{ord}_{\eta'}S_j^{\gamma}(\Phi)=j,\qquad |S_j^{\gamma}(\Phi)(y,\eta')|\le C^{|\gamma|}|\eta'|^{j}(|\gamma|-j)! \tag{28}$$

and so we can write

$$\partial_y^{\alpha}f^{\lambda}(y)=\sum_{\beta\le\alpha}\frac{\alpha!}{\beta!(\alpha-\beta)!}\sum_{j=0}^{|\beta|}F_{\alpha,\beta}^{\lambda,j}(y), \tag{29}$$

where

$$F_{\alpha,\beta}^{\lambda,j}(y)=\int_{\mathbf{R}^{n-1}}\exp(i\Phi)S_j^{\beta}(y,\eta')\partial_y^{\alpha-\beta}F(y,\eta')\kappa_{\lambda}(\eta')d\eta'.$$

We will show that one can find $C,\ \delta>0$ s.t. if $\alpha\ge\beta\in\mathbf{Z}_+^n$, $0\le j\le|\beta|$ and $\lambda\ge1$

$$\sup_{y\in\omega}|F_{\alpha,\beta}^{\lambda,j}(y)|\le C^{|\alpha|+1}(\alpha!)^{\theta}\exp(-\delta\lambda^{1/\theta}) \tag{30}$$

which yields the same estimate for $\partial_y^{\alpha}f^{\lambda}(y)$, with C eventually larger, and hence we get (25) (taking $T\ll 1/C$).

One may write for every positive integer N

$$F_{\alpha,\beta}^{\lambda,j}(y)=I_{\alpha,\beta,N}^{\lambda,j}(y)+J_{\alpha,\beta,N}^{\lambda,j}(y)+F_{\alpha,\beta,N}^{\lambda,j}(y) \tag{31}$$

where

$$I^{\lambda,j}_{\alpha,\beta,N} = \int_{\mathbf{R}^{n-1}} \exp(i\Phi) S^\beta_j \partial_y^{\alpha-\beta}(M_h R_E^{N+j} + R_B^{N+j})(y,\eta')\kappa_\lambda(\eta')d\eta'$$

$$J^{\lambda,j}_{\alpha,\beta,N} = \int_{\mathbf{R}^{n-1}} \exp(i\Phi) S^\beta_j \partial_y^{\alpha-\beta}\left(\sum_{|\gamma|=0}^{N+j} 1/\gamma! D^\gamma_{\eta'} A(y,\eta')\partial^\gamma_{y'} R_E^{N+j-|\gamma|}(y,\eta')\right)\kappa_\lambda(\eta')d\eta'$$

$$L^{\lambda,j}_{\alpha,\beta,N} = \int_{\mathbf{R}^{n-1}} \exp(i\Phi) S^\beta_j \partial_y^{\alpha-\beta}\Bigg(M_h\left(\psi(y)\sum_{v=0}^{N+j} E_v(y,\eta')\right)$$

$$+ \sum_{|\gamma|=0}^{N+j} 1/\gamma! D^\gamma_{\eta'} A(y,\eta')\partial^\gamma_{y'}(\psi(y)\sum_{\mu=0}^{N+j-|\gamma|} E_\mu(y,\eta'))\Bigg)\chi(\eta')\kappa_\lambda(\eta')d\eta' .$$

The estimates (27) and (28) imply for some $C > 0$

$$\max\{|I^{\lambda,j}_{\alpha,\beta,N}(y)|, |J^{\lambda,j}_{\alpha,\beta,N}(y)|\} \le C^{|\alpha|+N+1}(\alpha!N!)^\theta \lambda^{-N}\exp(C\lambda^p), \tag{32}$$

$$y \in \tilde{\omega},\ \eta' \in \mathbf{R}^{n-1},\ \beta \le \alpha \in \mathbf{Z}^n_+,\ 0 \le j \le |\beta|,\ N \in \mathbf{Z}_+ .$$

The transport equations (15), (17) yield that $L^{\lambda,j}_{\alpha,\beta,N}(y)$ is written as

$$L^{\lambda,j}_{\alpha,\beta,N} = \int_{\mathbf{R}^{n-1}} \exp(i\Phi) S^\beta_j(y,\eta')\partial_y^{\alpha-\beta}(K^{\lambda,j}_{N,1} + K^{\lambda,j}_{N,2} + K^{\lambda,j}_{N,3})\chi(\eta')\kappa_\lambda(\eta')d\eta' \tag{33}$$

with

$$K^{\lambda,j}_{N,1}(y,\eta') = D_n\psi(y)\sum_{v=0}^{N+j} E_v(y,\eta'),$$

$$K^{\lambda,j}_{N,2}(y,\eta') = \sum_{\gamma\ne 0}^{N+j}\sum_{\mu=0}^{N+j-|\gamma|}\sum_{0<|\tau|\le|\gamma|} 1/(\tau!(\gamma-\tau)!) D^\gamma_{\eta'} A(y,\eta')(\partial^\tau_{y'}\psi(y))\partial^{\gamma-\tau}_{y'} E_\mu(y,\eta'),$$

$$K^{\lambda,j}_{N,3}(y,\eta') = -iy_n^h D_1\psi(y)\sum_{v=0}^{N+j} E_v(y,\eta') .$$

Recall that $\psi(y) = 1$ near $y = 0$ which leads to $\mathrm{supp}(D_n\psi) \subseteq \{y; |y_n| \ge c > 0\}$. Thus we have $|\exp(i\Phi)| = \exp(-y_n^{h+1}\eta_1/(h+1)) \le C\exp(-d\lambda)$ for $(y,\eta') \in \mathrm{supp}\, D_n\psi(y) \times \mathrm{supp}\,\kappa_\lambda(\eta')$, C, $d > 0$ independent of (y,η') and $\lambda \ge 1$. Using the well known inequality $\lambda^k \exp(-c\lambda) \le c^{-k}k!$, $k \in \mathbf{Z}_+$ we get that the first integral in (33) satisfies the estimate (32). As to the second and the third terms in the right hand side of (33) we observe that their amplitudes are supported in $\{y; |y'| \ge C > 0\}$ (because there are only derivatives of nonzero order of ψ with respect to y') and we integrate by parts with the differential operator $L^* = \langle \overline{\partial_{\eta'}\Phi}, \partial_{\eta'}\rangle/(i|\partial_{\eta'}\Phi|^2)$ using the identity $L^*(\exp(i\Phi)) = \exp(i\Phi)$ and the fact that $\mathrm{Re}\,\partial_{\eta'}\Phi = y' \ne 0$ on the support of the corresponding amplitudes. If L is the adjoint operator of L^* we can write the sum of the second and the third integral in (33) as follows:

$$\int_{\mathbf{R}^{n-1}} \exp(i\Phi) S_j^{\beta}(y,\eta') \Bigg(\sum_{\gamma=1}^{N+j} \sum_{\mu=0}^{N+j-|\gamma|} \sum_{0<|\tau|\le|\gamma|} 1/(\tau!(\gamma-\tau)!) \partial_y^{\alpha-\beta}(\partial_{y'}^{\tau}\psi(y))$$
$$\times L^{N+j-|\gamma|-\mu}(D_{\eta'}^{\gamma} A(y,\eta')\partial_{y'}^{\gamma-\tau} E_\mu(y,\eta'))$$
$$- i y_n^h D_1\psi(y) \sum_{\nu=0}^{N+j} L^{N+j-\nu}(E_\nu(y,\eta'))\Bigg)\chi(\eta')\kappa_\lambda(\eta')d\eta' .$$

Using (20), (28), the property $|(L)^s(\kappa_\lambda(\eta'))| \le C^{s+1}(s!)^\theta \lambda^{-n-s}$, $s \in \mathbf{Z}_+^n$ and the rule for differentiation of the product of several functions we estimate the amplitude in the integral above by $C^{|\alpha|+N+1}(\alpha!N!)^\theta\lambda^{-n-N}\exp(c\lambda^p)$, with $C>0$ independent from α, N and $\lambda \ge 1$. So we can write

$$|L_{\alpha,\beta,N}^{\lambda,j}(y)| \le C^{|\alpha|+N+1}(\alpha!N!)^\theta \lambda^{-n-N}\exp(c\lambda^p), \tag{34}$$
$$y\in\omega,\ \beta\le\alpha\in\mathbf{Z}_+^{n-1},\ 0\le j\le|\beta|,\ N=1,2,\dots$$

The estimates (32), (33) and (34) imply after summation over $N=0,1,\dots$ and using (31) the desired estimate (30) since $p=(r-1)/r<1/\sigma<1/\theta$.

3. Passing from the system to the equation

We set $v^\lambda(y)=w_1^\lambda(y)$—the first component of $w^\lambda(y)$. We have

$$\tilde{P}(y,D)v^\lambda = f_1^\lambda + \sum_{j=2}^{r} B_j(y,D')f_j^\lambda + S(y,D')v^\lambda, \tag{35}$$

where $B_j(y,D')$, $2\le j\le r$ and $S(y,D')$ are p.d.o.-s with θ-amplitudes and S is θ-smoothing in $\tilde\omega\times\tilde\Gamma$ i.e. $S(y,\eta')$ verifies (12) when $(y,\eta')\in\tilde\omega\times\tilde\Gamma$.

Lemma 3.1. *There exist $T_0>0$, $\delta>0$, $C(T)>0$ for $T\in(0,T_0]$ s.t.*

$$\|S(y,D')v^\lambda\|_{T,\sigma} \le C(T)\exp(-\delta\lambda^{1/\theta}), \qquad \lambda\ge1. \tag{36}$$

Proof. Using G^θ cut-off functions we decompose S in the following form: $S(y,\eta')=S_1(y,\eta')+S_2(y,\eta')$ with $S_1(y,\eta')=0$ for $(y,\eta')\in\tilde\omega\times\tilde\Gamma$, while $S_2(y,\eta')$ satisfies (12) when $(y,\eta')\in\mathbf{R}^{2n-1}$. So we can write for $\alpha\in\mathbf{Z}_+^n$

$$D_y^\alpha(S(y,D)v^\lambda) = \sum_{\beta\le\alpha}\frac{\alpha!}{\beta!(\alpha-\beta)!}S_{\alpha,\beta}^\lambda(y), \tag{37}$$

where $S_{\alpha,\beta}^\lambda(y)=S_{\alpha,\beta}^{\lambda,1}(y)+S_{a,\beta}^{\lambda,2}(y)$ and

$$S_{a,\beta}^{\lambda,l}(y) = \int_{\mathbf{R}_{\zeta'}^{n-1}}\int_{\mathbf{R}_{x'}^{n-1}}\int_{\mathbf{R}_{\eta'}^{n-1}} \exp(i(y'\eta'+\Phi(z',y_n,\zeta'))) \tag{38}$$
$$\times \eta'^{\beta'}(-i)^{\alpha_n}\sum_{j=0}^{\beta_n} S_j^{\beta_n}(\Phi)(z',y_n,\eta')$$
$$\times \partial_{y_n}^{\alpha_n-\beta_n}(D_{y'}^{\alpha'-\beta'}S_l(y,\eta')E_1(z',y_n,\zeta'))\kappa_\lambda(\zeta')d\zeta'dz'd\eta' .$$

Since $\Phi(z',0,\zeta')=iz'\zeta'$ and for some $c>0$ we have $|\eta'-\zeta'|\ge c(|\eta'|+|\zeta'|)$, if $\eta' \notin \Gamma$, $\zeta' \in \mathrm{supp}_{\zeta'}(E_1(z', y_n, \zeta') \times \kappa_\lambda(\zeta'))$, the next estimate holds for ζ' and η' as above, $c > 0$ eventually smaller, $|y_n| \ll 1$

$$(39) \qquad |\eta' - \partial_{z'}\Phi(z', y_n, \zeta')| \geq c(|\eta'| + |\zeta'|) \geq c\lambda .$$

So we integrate N times by parts with the operator $L^* = \langle \eta' - \overline{\partial_{z'}\Phi}, \partial_{z'} \rangle / (i|\eta' - \partial_{z'}\Phi|^2)$ in the integral representing $S^{\lambda,1}_{\alpha,\beta}(y)$ and get in view of (39)

$$(40) \qquad |S^{\lambda,1}_{\alpha,\beta}(y) \leq C^{|\alpha|+N+1}(\alpha! N!)^\theta \lambda^{-N} \exp(c\lambda^p)$$

$$y \in \omega,\ \beta \leq \alpha \in \mathbf{Z}^{n-1}_+,\ \lambda \geq 1,\ N = 1, 2, \ldots$$

Concerning $S^{\lambda,2}_{\alpha,\beta}(y)$ we point out that for $|\eta'| \leq \lambda/2$ we can integrate with the operator L above since $|\zeta'| \geq \lambda$ on $\mathrm{supp}(\kappa_\lambda(\zeta'))$ leads to $|\eta' - \partial_{z'}\Phi(z', y_n, \zeta')| \geq c\lambda$ for $|y_n| \ll 1$. In the case $|\eta'| \geq \lambda/2$ we use the fact that $S_2(y, \eta')$ satisfies (12). Thus the estimate (40) is true for $S^{\lambda,2}_{\alpha,\beta}(y)$ which yields (36).

Next we state a technical assertion namely if B is θ-p.d.o. and E is θ-F.I.O. as above, with compactly supported kernels, then for every $\sigma \geq \theta$ there exist $T_0 > 0$, $\delta \in (0, 1)$, $C(T) > 0$ for $T \in (0, T_0]$ s.t. for every $u \in G^\sigma_0(\omega : T)$, $T \in (0, T_0]$

$$(41) \qquad \max\{\|Bu\|_{\delta T,\sigma}, \|Eu\|_{T,\sigma}\} \leq C(T)\|u\|_{T,\sigma} .$$

We omit the proof since it is only a technical complication of the arguments in dealing with the norms $\|\cdot\|_{T,\sigma}$ cf. [1], [3], [7], [8].

Taking into account (35), (36) (with S replaced by R), (41) and (25) we deduce $\exists T_0 > 0$, $C(T) > 0$, $0 < T \leq T_0$ s.t.

$$(42) \qquad \|\tilde{P}(y, D)v^\lambda\|_{T,\sigma} \leq C(T)\exp(-\delta\lambda^{1/\theta}), \qquad \lambda \geq 1 ,$$

$$(43) \qquad \|v^\lambda\|_{T,\sigma} \leq C(T)\exp(c\lambda^p), \qquad \lambda \geq 1 ,$$

$$(44) \qquad \liminf_{\lambda \to \infty} \|v^\lambda\|_{C(\omega)} > 0 .$$

The final step is to put

$$(45) \quad u^\lambda(x) = (Jv^\lambda)(x)$$

$$= \int_{\mathbf{R}^{n-1}_{\eta'}} \int_{\mathbf{R}^{n-1}_{y'}} \exp(i(x'\eta' - \phi(y', x_n, \eta')))e(x, y', \eta')v^\lambda(y', x_n)dy'd\eta' ,$$

where J is θ-F.I.O., right inverse of E^*_ϕ mcl near ρ^0 i.e. $E^* \circ J = Id + R(y, D')$, $R(y, D')$ is θ-p.d.o., G^θ-smoothing mcl near ρ^0. Thus we obtain

$$P^*(x, D)u^\lambda = E_\phi \circ Q(y, D) \circ \tilde{P}(y, D)v^\lambda + E \circ Q(y, D) \circ R(y, D')v^\lambda$$

and (42), (41) and (36) imply $\|P(x, D)u^\lambda\|_{T,\sigma} \leq C(T)\exp(-\delta\lambda^{1/\theta})$ for $C(T) > 0$, $0 < T \leq T_0$, T_0, $\delta > 0$ independent of $\lambda \geq 1$, while (43) and (44) hold for u^λ. We contradict the necessary estimate for σ-solvability by letting $\lambda \to \infty$.

4. Proof of Theorem 1.2

In this case the Gevrey index $\sigma > r/(r-1)$ and so we are not able to apply infinite order p.d.o. but rather use the asymptotic solution constructed in [12]. We can choose an asymptotic solution to $P^*u = 0$ in the following form

$$(46) \qquad Z(x, \xi) = \exp(i\Psi(x, \xi))A(x, \xi), \qquad x \in \mathbf{R}^2, \ |x| \ll 1, \ \xi \in \mathbf{R}$$

with $\Psi(x, \xi) = -idx_2^{h+1}\xi/(h+1) + x_1\xi + \psi(x_2, \xi)$, ψ solving

$$(47) \qquad \partial_{x_2}\psi = \lambda_j(x_2, \xi), \qquad \psi(0, \xi) = 0,$$

where j is a fixed index satisfying (6). Without loss of generality we take $d = 1$, $\xi > 0$ i.e. (6) is valid for $v = v_{j,0}^+$. In view of (5) and (6) we can write $\psi = \psi_0 + \psi_1$ and $\exists C > 0$ s.t. for $\xi \gg 1$, $|x_2| \ll 1$

$$(48) \qquad \operatorname{Im} \psi_0(x_2) = \omega(x_2)\xi^{(m-1)/m}, \qquad |\psi_1(x_2, \xi)| \le C|x_2|^{2\mu}\xi^{(m-2)/m}.$$

The amplitude $A(x_2, \xi)$ is a $\theta = 1 + \varepsilon$ $(0 < \varepsilon \ll 1)$ realization of

$$\sum_{s=0}^{\infty} a_s(x_2)\xi^{-sp}, \qquad p = (m-1)/m,$$

where a_s, $s = 0, 1 \ldots$ are real analytic functions, uniquely determined as the solutions of the transport equations in the construction of a formal asymptotic solution to $P^*u = 0$, with initial data $a_0(0) = 1$, $a_s(0) = 0$ if $s \ge 1$ (see [12] for more general operators or in this case one can use the asymptotic methods for linear o.d.e. after the Fourier transformation $x_1 \to \xi$).

In view of (6) $(-\operatorname{Im}\omega(t) < -bt^{2\mu})$ we get from (48)

$$(49) \qquad |\exp(i\Psi(x, \xi))| \le \exp(x_2^{h+1}\xi/(h+1) - bx_2^{2\mu}\xi^p + C|x_2|^{2\mu}\xi^{(m-2)/m})$$

and consequently

$$|\exp(i\Psi(x, \xi))| \le \exp(C|x_2|^{2\mu}\xi^{(m-2)/m})$$

providing $x_2\xi^\tau \le c_0$, $\xi \ge 1$, $c_0 = (b(h+1))^{1/(h+1-2\mu)}$, $\tau = (m(h+1-2\mu))^{-1}$.

Let $\chi(t) \in G^\theta(\mathbf{R})$, $\operatorname{supp}\chi \subseteq (-\infty, c_1]$, $\chi(t) = 1$ on $(-\infty, c_2]$ with the constants chosen to satisfy $0 < c_2 < c_1 < c_0$ (see also [6], [12], [21]). Note that (6) and (7) lead to

$$(50) \qquad |\exp(i\Psi(x, \xi))| \le \exp(-\delta\xi^{1/\gamma}), \qquad c_2 \le x_2\xi^\tau \le c_1, \qquad \xi \ge 1$$

with $\delta > 0$ depending on c_1, c_2 and C only.

If $\kappa_\lambda(\xi)$ is the function in the previous section for $\eta' = \xi \in \mathbf{R}$ we denote

$$(51) \qquad w^\lambda(x) = \int_{\mathbf{R}} \exp(i\Psi(x, \xi))\chi(x_2\xi^\tau)\kappa_\lambda(\xi)d\xi, \qquad \lambda \ge 1.$$

Then

$$P^*(x, D)w^\lambda = \int_{\mathbf{R}} \exp(i\Psi(x, \xi))\chi(x_2\xi^\tau)R(x_2, \xi)\kappa_\lambda(\xi)d\xi + \int_{\mathbf{R}} \exp(i\Psi(x, \xi))S(x_2, \xi)\kappa_\lambda(\xi)d\xi \tag{52}$$

and for certain positive constants C, δ we have cf. [12]

$$|\partial_{x_2}^N R(x_2, \xi)| \le C^{N+1}(N!)^\theta \exp(-\delta\lambda^{(m-1)/m}), \qquad N = 0, 1, \dots \tag{53}$$

while the estimates

$$|\partial_{x_2}^l(\chi(x_2\xi^\tau))| \le C^{l+1}(l!)^\theta \xi^{l\tau}, \qquad l \ge 1$$

and the fact that in S participate derivatives with respect to x_2 of $\chi(x_2\xi^\tau)$ of order not less than 1 yield via direct calculations (or see [12]) that

$$\operatorname{supp}(S) \subseteq \{(x_2, \xi) : c_2 \le x_2\xi^\tau \le c_1, \xi \ge 1\} \tag{54}$$

$$|\partial_{x_2}^N S(x_2, \xi)| \le C^{l+1}(l!)^\theta \xi^{(m_0+\tau N)}, \qquad N = 0, 1, \dots \tag{55}$$

with $m_0 = \min\{m\tau, m-1\}$.

Applying the arguments in the previous two sections or relying on the estimates in [12] we find that $\exists T_0 > 0$, $\delta > 0$ and $C(T) > 0$ for $T \in (0, T_0]$ s.t. the next estimate holds

$$\|P^*(x, D)w^\lambda\|_{T,\sigma} \le C(T)\exp(-\delta\lambda^{1/\gamma}), \qquad \lambda \ge 1, \tag{56}$$

where the L^2-norm in (1) is taken over a small fixed open set ω containing (0, 0).

The only problem is that $w^\lambda(x)$ is not compactly supported and so we choose $g(x) \in G_0^\theta(\omega)$, $g(x)$ equals 1 near $x = 0$ and put $u^\lambda(x) = g(x)w^\lambda(x)$. Then

$$P^*(x, D)u^\lambda = g(x)P^*(x, D)w^\lambda + f_1^\lambda(x) + f_2^\lambda(x) \tag{57}$$

where f_1^λ, (f_2^λ) is represented as an oscillatory integral of the type (51), (52) with θ-amplitude supported in $\{|x_1| \ge \delta'\}(\{|x_2| \ge \delta'\})$ for certain $\delta' > 0$ small enough. Integration by parts as in the proof of Proposition 2.1 with the operator $L = (ix_1)^{-1}\partial_\xi$, $L(\exp(i\Psi)) = \exp(i\Psi)$ in the integral defining f_1^λ shows that with certain $T_0 > 0$, $\delta > 0$ and $C(T) > 0$, $0 < T \le T_0$ we obtain

$$\|f_1^\lambda\|_{T,\sigma} \le C(T)\exp(-\delta\lambda^{1/\gamma}), \qquad \lambda \ge 1, \tag{58}$$

while $\|f_2^\lambda\|_{T,\sigma}$ satisfies (58) taking into account the following two inequalities valid for certain positive constants C and c depending on δ', h, m, n, b and the constant C in (49):

$$|\exp(i\Psi(x, \xi))| \le \exp(-c\xi), \qquad |x_2| \ge \delta', \qquad \xi \ge 1$$

$$(n!)^\theta\lambda^{n\tau}\exp(-\lambda) \le C^n(n!)^{\theta+\tau}\exp(-\lambda/2) \le C^n(n!)^\gamma\exp(-\lambda/2).$$

Note that we use the equivalence of $|\xi|$ and λ on $\operatorname{supp}\kappa_\lambda(\xi)$ and the observation that (7) implies $\theta + \tau < \gamma$ providing $\varepsilon > 0$ is small enough.

The last inequality appears when we estimate the derivatives (in the definition of the norm $\|f_2^\lambda\|_{T,\sigma}$) of the function $\chi(x_2\xi^\tau)$ and we use the fact that $\sigma \geq \gamma > \theta + \tau$.

So we get that $\|P^*u^\lambda\|_{T,\sigma}$ satisfies (56) which yields the σ-nonsolvability of $P(x, D)$ at 0 since clearly $\|u^\lambda\|_{T,\sigma} \leq C(T)\exp(cO(1))$ while

$$\lim_{\lambda\to\infty} u^\lambda(0,0) = \int_\lambda^{4\lambda} \kappa_\lambda(\xi)d\xi = \int_1^4 \kappa(\xi)d\xi \geq 1$$

implies

$$\lim_{\lambda\to\infty} \inf \|u^\lambda\|_{C(\omega)} > 0 .$$

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