J. Math. Kyoto Univ. (JMKYAZ)
36-1 (1996) 123-128

A remark on the homotopy type of the classifying space of certain gauge groups

By

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1. Introduction

Let P_k be the principal SU (2) bundle over a closed simply connected 4-manifold X with $c_2(P_k) = k$, $\mathfrak{g}_k$ its gauge group and $\mathfrak{g}_k^0$ its based gauge group. For integers m and n, let (m, n) denote the GCD of m and n if $mn \neq 0$, $(m, 0) = (0, m) = m$. In [3], it is shown that $\mathfrak{g}_k \simeq \mathfrak{g}_{k'}$ if and only if $(12/d(X), k) = (12/d(X), k')$ where $d(X) = 1$ if the intersection form of X is even and $d(X) = 2$ if odd.

In this paper we study the homotopy type of $B\mathfrak{g}_k$. The purpose of this paper is to show the following result.

Theorem 1.1. *If $B\mathfrak{g}_k$ is homotopy equivalent to $B\mathfrak{g}_{k'}$, then $(k, p) = (k', p)$ for any prime p.*

Note that the result *for* $p = 2, 3$ is obtained from the result of [3].

Theorem is proved by computing the Postnikov invariant of $(B\mathfrak{g}_k)_{(p)}$ which is $B\mathfrak{g}_k$ localized at p.

By [1], we have two homotopy equivalences

$$B\mathfrak{g}_k \simeq \mathrm{Map}_k(X, BSU(2))$$

and

$$B\mathfrak{g}_k^0 \simeq \mathrm{Map}_k^*(X, BSU(2)).$$

For a fixed prime $p \geq 5$, denote $\mathbf{H}P^\infty{}_{(p)}$ by B and put (cf. [2])

$$\begin{aligned} M_{k,X} &= \mathrm{Map}_k(X, \mathbf{H}P^\infty)_{(p)} \\ &\simeq \mathrm{Map}_k(X, B) \\ M_{k,X}^* &= \mathrm{Map}_k^*(X, \mathbf{H}P^\infty)_{(p)} \\ &\simeq \mathrm{Map}_k^*(X, B). \end{aligned}$$

Consider the Postnikov invariant $\mathbf{k}^{2p-2}(M_{k,X})$. If $B\mathfrak{g}_k \simeq B\mathfrak{g}_{k'}$, then $\mathbf{k}^{2p-2}(M_{k,X}) = \mathbf{k}^{2p-2}(M_{k',X})$ for all p. So, to prove Theorem 1.1, we have only to show

Received May 8, 1995

Proposition 1.2. *$(k, p) = p$ if and only if $\mathbf{k}^{2p-2}(M_{k,X}) = 0$.*

which will be proved in §2.

The author would like to thank Professor Akira Kono for his advices and encouragements.

2. Proof of Proposition 1.2

For a space Y and a non negative integer q, let $Y < q > = Y \cup e^{q+1}{}_{\alpha} \cup \cdots\cdots$ be a space obtained from Y by killing the homotopy groups in dimension $\geq q$. Consider the fibering

$$M^*_{k,X} \longrightarrow M_{k,X} \xrightarrow{\pi} B.$$

The map π induces a map $\tilde{\pi}$: $M_{k,X} < q > \to B < q >$ such that the following commutes,

$$\begin{array}{ccccc}
M^*_{k,X} & \longrightarrow & M_{k,X} & \xrightarrow{\pi} & B. \\
 & & \tilde{f}\downarrow & & \downarrow f \\
 & & M_{k,X} < q > & \xrightarrow{\tilde{\pi}} & B < q > .
\end{array}$$

Lemma 2.1. *If $\pi_q(B) = 0$, then the homotopy fibre of $\tilde{\pi}$ is $M^*_{k,X} < q >$.*

Proof. Let F be the homotopy fibre of $\tilde{\pi}$, then we have a commutative diagram,

$$\begin{array}{ccccc}
M^*_{k,X} & \longrightarrow & M_{k,X} & \xrightarrow{\pi} & B. \\
\downarrow & & \tilde{f}\downarrow & & \downarrow f \\
F & \longrightarrow & M_{k,X} < q > & \xrightarrow{\tilde{\pi}} & B < q > .
\end{array}$$

By the homotopy exact sequences for the fibrations which correspond to the horizontal rows of above diagram and the morphism between these exact sequences induced from the vertical maps of the diagram, we have $M^*_{k,X} < q > \simeq F$.

Recall that the p-component of $\pi_{3+k}(S^3)$ is (see [4])

$$\pi^{(p)}{}_{3+k}(S^3) = \begin{cases} 0 & 0 < k < 2p-3,\ 2p-3 < k < 4p-6 \\ \mathbf{Z}/p & k = 2p-3. \end{cases}$$

Therefore we have

$$\pi_j(M^*_{k,S^4}) = \begin{cases} 0 & 0 \leq j < 2p-3,\ 2p-3 < j < 4p-6 \\ \mathbf{Z}/p & j = 2p-3 \end{cases} \tag{2.1}$$

$$\pi_j(B)=\begin{cases}0 & 0\leq j<4,\ 4<j<2p+1,\ 2p+1<j<4p-2\\ \mathbf{Z}_{(p)} & j=4\\ \mathbf{Z}/p & j=2p+1,\end{cases} \tag{2.2}$$

in paticular $\pi_{2p}(B)=0$ and we have a commutative diagram,

$$\begin{array}{ccccc} M^*_{k,X} & \longrightarrow & M_{k,X} & \xrightarrow{\pi} & B. \\ \downarrow & & \tilde{f}\downarrow & & \downarrow f \\ M^*_{k,X}<2p> & \longrightarrow & M_{k,X}<2p> & \xrightarrow{\tilde{\pi}} & B<2p>. \end{array}$$

Denote $\mathbf{H}P^{(p-1)/2}$ by A and consider the map

$$\lambda\colon A=\mathbf{H}P^{(p-1)/2}\to\mathbf{H}P^{\infty}\xrightarrow{\ l\ } B,$$

where l is a localization.

Lemma 2.2. *There exits a map $\alpha\colon A\to M_{k,S^4}$ such that $\pi\circ\alpha\simeq\lambda$ if and only if there exists a map $\beta\colon A\to M_{k,S^4}<2p>$ such that $\tilde{\pi}\circ\beta\simeq f\circ\lambda$.*

Proof. If there exists α, put $\beta=\tilde{f}\circ\alpha$.

Conversely if tehere exists β, since $\dim A=2p-2<2p-1$, we have a map $\beta'\colon A\to M_{k,S^4}$ such that $\tilde{f}\circ\beta'\simeq\beta$, and

$$f\circ\lambda\simeq\tilde{\pi}\circ\beta\simeq\tilde{\pi}\circ\tilde{f}\,\beta'.$$

Again since $\dim A<2p-1$, $\lambda\simeq\tilde{\pi}\circ\beta'$.

Lemma 2.3. *There exists α if and only if $(k,p)=p$.*

Proof. It is clear that there exists α if and only if there exists a map $\Phi\colon A\times S^4\to B$ such that the following commutes,

$$\begin{array}{ccc} A\vee S^4 & \xrightarrow{\lambda\vee k} & B\vee B \\ \downarrow & & \downarrow\nabla \\ A\times S^4 & \xrightarrow{\Phi} & B. \end{array}$$

Since the only obstruction to get Φ lives in the image of
$k=(1\times k)^*\colon H^{2p+2}(A\times S^4, A\vee S^4;\mathbf{Z}/p)\to H^{2p+2}(A\times S^4, A\vee S^4;\mathbf{Z}/p)$,
hence if $(k,p)=p$, there exists Φ.

Conversely if there exists Φ, we have a commutative diagram,

$$\begin{array}{ccc} H^4(B;\mathbf{Z}/p) & \xrightarrow{\mathcal{P}^1} & H^{2p+2}(B;Z/p) \\ \Phi^*\downarrow & & \downarrow\Phi^* \\ H^4(A\times S^4;Z/p) & \xrightarrow{\mathcal{P}^1} & H^{2p+2}(A\times S^4;\mathbf{Z}/p). \end{array}$$

There exists an element $u\in H^4(B;\mathbf{Z}/p)$ such that $\mathcal{P}^1(u)=2u^{p+1/2}\neq 0$.

$$\Phi^*\circ\mathcal{P}^1(u)=\Phi^*(2u^{p+1/2})=a^{p-1/2}\otimes kb,$$

where we write $\Phi^*(u)=a\otimes 1+1\otimes kb$.

On the other hand

$$\mathcal{P}^1(\Phi^*(u))=\mathcal{P}^1(a+kb)=0,$$

therefore $a^{p-1/2}\otimes kb=0$ and we have $(k,p)=p$.

Lemma 2.4. *There exists β if and only if the Postnikov invariant* $\mathbf{k}^{2p-2}(M_{k,S^4})=0\in H^{2p-2}(M_{k,S^4}<2p-3>,\mathbf{Z}/\mathrm{p})$.

Proof. By (2.1) and (2.2), we have a commutative diagram

$$\begin{array}{ccccc} M^*_{k,S^4} & \longrightarrow & M_{k,S^4} & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ K(\mathbf{Z}/p,2p-3)=M^*_{k,S^4}<2p> & \longrightarrow & M_{k,S^4}<2p> & \xrightarrow{\bar{\pi}} & B<2p> \\ & & \| & & \| \\ & & M_{k,S^4}<2p-2> & \longrightarrow & M_{k,S^4}<2p-3>. \end{array}$$

We have also an exact sequence

$$\begin{array}{ccccc} H^{2p-2}(M_{k,S^4}<2p>) & \xleftarrow{\bar{\pi}^*} & H^{2p-2}(B<2p>) & \xleftarrow{\tau} & H^{2p-3}(K(\mathbf{Z}/p,2p-3)) \\ & & (f\circ\lambda)^*\downarrow\cong & & \\ & & H^{2p-2}(A) & = & \mathbf{Z}/p \end{array}$$

where $H^*(\ )$ is understood to be the mod p cohomology.

Note that $\mathbf{k}^{2p-2}(M_{k,S^4})=\tau(1_{K(\mathbf{Z}/p,2p-3)})$.

If there exists β such that $\bar{\pi}\circ\beta\simeq f\circ\lambda$, then

$$(f\circ\lambda)^*\circ\tau(1)=\beta^*\circ\bar{\pi}^*\circ\tau(1)=0,$$

hence $\mathbf{k}^{2p-2}(M_{k,S^4})=0$.

Conversely if $\mathbf{k}^{2p-2}(M_{k,S^4})=0$, then

$$M_{k,S^4} < 2p > \simeq B < 2p > \times K(\mathbf{Z}/p, 2p-3)$$

and we can get β easily.

Lemma 2.5. *$\mathbf{k}^{2p-2}(M_{k,S^4}) = 0$ if and only if $\mathbf{k}^{2p-2}(M_{k,X}) = 0$*

Proof. From the cofibering

$$\bigvee_b S^2 \to X \xrightarrow{q} S^4$$

where q is the collapsing map, we have a fibering

$$M^*_{k,S^4} \to M^*_{k,X} \to \prod_b \Omega^2 B.$$

By (2.1) and (2.2), we have

$$\pi_j(M^*_{k,X}) = \begin{cases} 0 & 0 \le j < 2,\ 2 < j < 2p-3,\ j = 2p-2,\ 2p-3 < j < 4p-8 \\ \oplus_b \mathbf{Z}_{(p)} & j = 2 \\ \mathbf{Z}/p & j = 2p-3 \\ \oplus_b \mathbf{Z}/p & j = 2p-1. \end{cases}$$

Therefore the following commutative diagram

$$\begin{array}{ccccc} M^*_{k,S^4} & \longrightarrow & M_{k,S^4} & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ M^*_{k,X} & \longrightarrow & M_{k,X} & \longrightarrow & B \end{array}$$

induces a fibre map

$$\begin{array}{ccccc} K(\mathbf{Z}/p, 2p-3) & \longrightarrow & M_{k,S^4} < 2p-2 > & \longrightarrow & M_{k,S^4} < 2p-3 > \\ \| & & \downarrow & & \downarrow \\ K(\mathbf{Z}/p, 2p-3) & \longrightarrow & M_{k,X} < 2p-2 > & \longrightarrow & M_{k,X} < 2p-3 >, \end{array}$$

and a commutative diagram,

$$\begin{array}{ccc} H^{2p-2}(M_{k,S^4} < 2p-3 > ; \mathbf{Z}/p) & \xleftarrow{\tau} & H^{2p-3}(K(\mathbf{Z}/p, 2p-3); \mathbf{Z}/p) \\ q^* \downarrow & & \| \\ H^{2p-2}(M_{k,X} < 2p-3 > ; Z/p) & \xleftarrow{\tau} & H^{2p-3}(K(\mathbf{Z}/p, 2p-3); \mathbf{Z}/p). \end{array}$$

If $\mathbf{k}^{2p-2}(M_{k,X}) = 0$, then

$$\mathbf{k}^{2p-2}(M_{k,S^4}) = q^*\mathbf{k}^{2p-2}(M_{k,X}) = 0.$$

Since we have a commutative diagram

$$\begin{array}{ccc} M_{k,S^4} < 2p-3 > & = & B < 2p-3 > \\ \downarrow & & \| \\ M_{k,X} < 2p-3 > & \longrightarrow & B < 2p-3 >, \end{array}$$

we have a map $s\colon M_{k,X} < 2p-3 > \longrightarrow M_{k,S^4} < 2p-3 >$.

If $\mathbf{k}^{2p-2}(M_{k,S^4}) = 0$, as we saw in the proof of Lemma 2.4, there exists a section

$$M_{k,S^4} < 2p-3 > \longrightarrow M_{k,S^4} < 2p-2 > .$$

Therefore we get a fibre map

$$\begin{array}{ccccc} K(\mathbf{Z}/p,\ 2p-3) & \longrightarrow & M_{k,S^4} < 2p-2 > & \longrightarrow & M_{k,S^4} < 2p-3 > \\ \| & & \uparrow & & \uparrow s \\ K(\mathbf{Z}/p,\ 2p-3) & \longrightarrow & M_{k,X} < 2p-2 > & \longrightarrow & M_{k,X} < 2p-3 > . \end{array}$$

Hence

$$\mathbf{k}^{2p-2}(M_{k,X}) = s^*(\mathbf{k}^{2p-2}(M_{k,S^4})) = 0.$$

Thus we complete the proof of Proposition 1.2.

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