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A remark on the homotopy type of certain gauge groups

By

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1. Introduction

Let P_k be the principal SU(2) bundle over a closed simply connected 4-manifold X with $c_2(P_k)=k$, $\mathfrak{g}_k$ its gauge group and $\mathfrak{g}_k^0$ its based gauge group consisting of bundle automorphisms of P_k which restrict to the identity on the fibre over a base point.

In [2], when $X=S^4$, it is shown that $\mathfrak{g}_k \simeq \mathfrak{g}_{k'}$ if and only if $(12, k)=(12, k')$ where $(12, k)$ is the GCD of 12 and k.

In this paper we show the similar results for closed simply connected 4-manifolds.

The homotopy type of X is determined by the intersection form Q. Define

$$d(X)=\begin{cases} 1 \text{ if } Q \text{ is even} \\ 2 \text{ if } Q \text{ is odd} \end{cases}$$

The purpose of this paper is to show following results.

Proposition 1. 1 $\mathfrak{g}_k^0 \simeq \mathfrak{g}_0^0$ *for any integer* k.

Theorem 1. 2 $\mathfrak{g}_k$ *is homotopy equivalent to* $\mathfrak{g}_{k'}$ *if and only if* $(12/d(X), k)=(12/d(X), k')$ *where* $(12/d(X), k)$ *denotes the GCD of* $12/d(X)$ *and* k *if* $k\neq 0$ *and* $12/d(X)$ *if* $k=0$.

Related results have been obtained by several authors, for example in [4].

2. Proof of Proposition 1. 1

In fact this is included in [4], essentially.

By [1], $B\mathfrak{g}_k^0 \simeq \mathrm{Map}_k^*(X, BSU(2))$. Fix $-k \in \mathrm{Map}_{-k}^*(S^4, BSU(2))$, for $f \in \mathrm{Map}_k^*(X, BSU(2))$ consider the map

$$f_{-k}: X \xrightarrow{p} X \vee S^4 \xrightarrow{f\vee -k} BSU(2) \vee BSU(2) \xrightarrow{\nabla} BSU(2)$$

where p is a pinching map and ∇ is a folding map. Then $f \to f_{-k}$ gives a

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homotopy equivalence between $\mathfrak{g}_k^0$ and $\mathfrak{g}_0^0$.

3. Proof of Theorem 1. 2

By [1], $B\mathfrak{g}_k \simeq \mathrm{Map}_k(X, \mathrm{BSU}(2))$ and therefore there are fiberings;

$$\mathrm{Map}_k^*(X, \mathrm{BSU}(2)) \to \mathrm{Map}_k^*(X, \mathrm{BSU}(2)) \to \mathrm{BSU}(2) \tag{3.1}$$

$$S^3 \simeq \Omega\mathrm{BSU}(2) \to \mathrm{Map}_k^*(X, \mathrm{BSU}(2)) \to \mathrm{Map}_k(X, \mathrm{BSU}(2)) \tag{3.2}$$

$$\mathfrak{g}_k \simeq \Omega\mathrm{Map}_k(X, \mathrm{BSU}(2)) \to S^3 \to \mathrm{Map}_k^*(X, \mathrm{BSU}(2)). \tag{3.3}$$

Let $b = \dim H_2(X; \mathbf{Q})$, then there is a cofibering;

$$S^3 \xrightarrow{\xi} \bigvee_b S^2 \xrightarrow{i} X \xrightarrow{q} S^4 \xrightarrow{\Sigma\xi} \bigvee_b S^3, \tag{3.4}$$

where ξ is an attaching map and i is an inclusion.

Note that the intersection form Q is even if and only if $\Sigma\xi = 0 \in \oplus_b \pi_4(S^3) \cong (\mathbf{Z}/2)^b$.

Consider the following diagram

$$\begin{array}{ccccc}
 & & \Pi_b \mathrm{Map}_k^*(S^3, \mathrm{BSU}(2)) & = & \Pi_b \Omega^3 \mathrm{BSU}(2) \\
 & & \downarrow{\scriptstyle \Sigma\xi^\#} & & \\
S^3 & \xrightarrow{m\varepsilon} & \mathrm{Map}_k^*(S^4, \mathrm{BSU}(2)) & \longrightarrow & \mathrm{Map}_k(S^4, \mathrm{BSU}(2)) \\
\| & & \downarrow{\scriptstyle q^\#} & & \downarrow{\scriptstyle q^\#} \\
S^3 & \longrightarrow & \mathrm{Map}_k^*(X, \mathrm{BSU}(2)) & \longrightarrow & \mathrm{Map}_k(X, \mathrm{BSU}(2)) \\
 & & \downarrow & & \\
 & & \Pi_b \Omega^2 \mathrm{BSU}(2) & &
\end{array} \tag{3.5}$$

where ε is a map which generates $\pi_3(\mathrm{Map}_k^*(S^4, \mathrm{BSU}(2))) = \pi_7(BS^3) = \pi_6(S^3) \cong \mathbf{Z}/12$, and $m \in \mathbf{Z}$. Since $(m, 12) = (k, 12)$ by [2], $(12/d(X), k) = (12/d(X), m)$.

Lemma 3. 1. $[q^\# \circ m\varepsilon] \in \pi_3(\mathrm{Map}_k^*(X, \mathrm{BSU}(2)))$ *is of order* $(12/d(X))/(12/d(X), m)$.

Remark. Since $\mathfrak{g}_k$ is the homotopy fibre of $q^\# \circ m\varepsilon$, if $\mathfrak{g}_k \simeq \mathfrak{g}_{k'}$, $(12/d(X), k) = (12/d(X), k')$.

Proof. By the homotopy sequence for (3.5);

$$\begin{array}{ccccccc}
\mathbf{Z} & & \oplus_b \pi_5(S^3) & = & (\mathbf{Z}/2)^b & & \\
\| & & \downarrow{\scriptstyle \Sigma\xi^*} & & & & \\
\pi_3(S^3) & \xrightarrow{m} & \mathbf{Z}/12 & \longrightarrow & \pi_3(\mathrm{Map}^*_k(S^4, \mathrm{BSU}(2))) & \longrightarrow & 0 \\
\| & & \downarrow{\scriptstyle q^*} & & \downarrow & & \\
\pi_3(S^3) & \longrightarrow & \pi_3(\mathrm{B}\mathfrak{g}^0_k) & \longrightarrow & \pi_3(\mathrm{B}\mathfrak{g}_k) & \longrightarrow & 0 \\
 & & \downarrow & & & & \\
 & & \oplus_b \pi_4(S^3) & = & (\mathbf{Z}/2)^b & &
\end{array}$$

and

$$d(X)=\begin{cases}1 & \text{if } \Sigma\xi^*=0\\ 2 & \text{if } \Sigma\xi^*\neq 0,\end{cases}$$

we get the short exact sequence

$$0\rightarrow \mathbf{Z}/(12/d(X)) \xrightarrow{q^*} \pi_3(\mathrm{B}\mathfrak{g}^0_k) \rightarrow (\mathbf{Z}/2)^{b-d(x)+1} \rightarrow 0.$$

Henceforth $q^*([m])$ is of order $(12/d(X))/(12/d(X), m)$.

Let F_m denote the homotopy fibre of $q^\# \circ m\varepsilon$.

Lemman 3. 2. *Assume F_n and $F_{n'}$ are H-spaces, then $F_n \simeq F_{n'}$ if and only if* $(12/d(X), n)=(12/d(X), n')$.

Proof. First of all, since S^3 admits a homotopy equivalence of degree -1, $F_n \simeq F_{-n}$ (3.6).

By an easy computation,

$$\pi_j(F_n)=\begin{cases}\mathbf{Z}^b+\text{finite} & j=1\\ \mathbf{Z}+\text{finite} & j=3\\ \text{finite} & \text{otherwise.}\end{cases}$$

Let $\{[a_i]\}_{i=1}^b$ be the generator of free part of $H_1(F_n;\mathbf{Z})$ and represent $[a_i]$ by a_i: $S^1\rightarrow F_n$. Choose $[u_i] \in [F_n, S^1] \cong H^1(F_n; \mathbf{Z})$ dual to a_i. Let $\widetilde{F}_n$ be the homotopy fibre of $\Pi\, u_i$: $F_n \rightarrow (S^1)^b$. Then we have a homotopy equivalence

$$F_n \simeq (S^1)^b \times \widetilde{F}_n$$

and $\pi_1(\widetilde{F}_n)$ is finite. Since F_n is an H-space, so is $\widetilde{F}_n$. Let $(\widetilde{F}_n)_{(2)}$, $(\widetilde{F}_n)_{(1/2)}$ denotes the $\widetilde{F}_n$ localized at 2, 1/2 respectively. Then we have homotopy equivalences

$$f:\ (\widetilde{F}_n)_{(2)} \simeq (\widetilde{F}_{5n})_{(2)}$$
$$\widetilde{g}:\ (\widetilde{F}_n)_{(1/2)} \simeq (\widetilde{F}_{2n})_{(1/2)}.$$

Moreover since $\pi_3(\mathrm{Map}_k^*(S^4, \mathrm{BSU}(2))) \cong \mathbf{Z}/12$, if $n \equiv n'$ (mod 3), $(\widetilde{F}_n)_{(1/2)} \simeq (\widetilde{F}_{n'})_{(1/2)}$, and therefore we have a homotopy equivalence

$$g\colon (\widetilde{F}_n)_{(1/2)} \simeq (\widetilde{F}_{2n})_{(1/2)} \simeq (\widetilde{F}_{5n})_{(1/2)}.$$

Let ε, ε' be generators of $\pi_3(\widetilde{F}_n)/\mathrm{Tor}$, $\pi_3(\widetilde{F}_{5n})/\mathrm{Tor}$ respectively, then we have

$$f_*\varepsilon = (l'/l)\varepsilon' \quad l, l'\colon \text{odd}$$

$$g_*\varepsilon = 2^r\varepsilon' \qquad r \in \mathbf{Z}_{\geq 0}$$

For an H-space Y and $l \in \mathbf{Z}$, define a map $\varphi_l\colon Y \to Y$ by

$$\varphi_l(y) = \underbrace{y \cdots y}_{l \text{ times}}$$

then we have homotopy equivalences

$$\varphi_{l'}\colon (\widetilde{F}_{5n})_{(2)} \to (\widetilde{F}_{5n})_{(2)}$$

$$\varphi_{2^r}\colon (\widetilde{F}_{5n})_{(1/2)} \to (\widetilde{F}_{5n})_{(1/2)}$$

Put

$$f' = \varphi_l \circ \varphi_{l'}^{-1} \circ f$$

$$g' = \varphi_{2^r}^{-1} \circ g.$$

Clearly f' and g' are homotopy equivalences and $f'_*\varepsilon = \varepsilon'$, $g'_*\varepsilon = \varepsilon'$ and so $f'_{(0)} = g'_{(0)}$ where $f'_{(0)}$ and $g'_{(0)}$ are rationalizations at 0. Hence there exists a homotopy equivalence

$$h\colon \widetilde{F}_n \to \widetilde{F}_{5n} \tag{3.7}$$

such that $h_{(2)} = f'$, $h_{(1/2)} = g'$.

The lemma follows from (3.6), (3.7) and the following table.

$(12, n)$	$n\,(0 \leq n < 12)$	$(6, n)$	$n\,(0 \leq n < 6)$
1	1,5,7,11	1	1,5
2	2,10	2	2,4
3	3,9	3	3
4	4,8	6	0
6	6		
12	0		

If $(12/d(X), k) = (12/d(X), k')$, since $(12, k) = (12, m)$ and $(12, k') = (12, m')$, $(12/d(X), m) = (12/d(X), m')$, therefore we have $\mathfrak{g}_k \simeq F_m \simeq F_{m'} \simeq \mathfrak{g}_{k'}$ and the theorem is proved.

4. Geometric view point

In this section we give a geometric interpretation of Theorem 1.2.

Consider X as $D^4 \bigcup_{S^3} \widetilde{X}$ where D^4 is a 4-disk, $X=X-D^4$ and $S^3=\partial D^4=\partial\widetilde{X}$. Choose a degree k map f_k: $(S^3; 1) \rightarrow (S^3, 1)$, then we have

$$P_k=D^4\times S \bigcup_{\widetilde{f}_k} \widetilde{X}\times S^3$$

where

$$\widetilde{f}_k\colon \partial D^4\times S^3\rightarrow S^3\times S^3 = \partial\widetilde{X}\times S^3$$

is defined by $\widetilde{f}_k(x, g)=(x, \widetilde{f}_k(x)\cdot g)$, $x, g\in S^3$. Clearly

$$AdP_k=D^4\times S^3 \bigcup_{A\widetilde{d}f_k} \widetilde{X}\times S^3$$

where

$$A\widetilde{d}f_k\colon S^3\times S^3\rightarrow S^3\times S^3$$

is defined by $\widetilde{A}df_k(x, g)=(x, Adf_k(x)(g))$ and

$$Adf_k\colon S^3\rightarrow \mathrm{Map}_1^*(S^3, S^3)$$

is $Adf_k(x)(g)=f_k(x)\cdot g\cdot f_k(x)^{-1}$. Define

$$F\colon \pi_3(\mathrm{Map}_1^*(S^3, S^3))\rightarrow [\mathrm{Map}_0(S^3, S^3), \mathrm{Map}_0(S^3, S^3)]$$

by $F(\eta)(\varphi)(x)=\eta(x)(\varphi(x))$, $\eta\in\pi_3(\mathrm{Map}_1^*(S^3, S^3))$, $\varphi\in \mathrm{Map}_0(S^3, S^3)$, $x\in S^3$. Note that F is a homomorphism i. e. $F(\eta+\xi)=F(\eta)\circ F(\xi)$ and $\pi_3(\mathrm{Map}_1^*(S^3, S^3))\cong\mathbf{Z}/12$ is generated by $\varepsilon=[Adf_1]$.

We can construct $\mathfrak{g}_k=\Gamma(AdP_k)$ as the fibre product of the following diagram;

$$\begin{array}{ccccc}
\mathrm{Map}(D^4, S^3) & & & & \mathfrak{g}_k \\
\downarrow & & & & \downarrow \\
\mathrm{Map}_0(S^3, S^3) & \xleftarrow{F(Adf_k)} & \mathrm{Map}_0(S^3, S^3) & \longleftarrow & \mathrm{Map}(\widetilde{X}, S^3)
\end{array} \tag{4.1}$$

where arrows except $F(Adf_k)$ are restrictions, and $\mathrm{Map}(D^4, S^3)\rightarrow\mathrm{Map}_0(S^3, S^3)$ is a fibration. Thus if $k\equiv k'$ (mod 12), $\mathfrak{g}_k\simeq\mathfrak{g}_{k'}$.

Next we see how $d(X)$ enters the story. Recall the cofibering (3.4);

$$\begin{array}{ccccc}
S^3 & \xrightarrow{\xi} & \bigvee_b S^2 & \xrightarrow{i} & X \\
\| & & \downarrow{\scriptstyle i} & & \| \\
\partial\widetilde{X} & \longrightarrow & \widetilde{X} & \longrightarrow & X.
\end{array}$$

Clearly $i\colon \bigvee_b S^2\rightarrow\widetilde{X}$ is a homotopy equivalence, therefore $\mathrm{Map}(\bigvee_b S^2, S^3)\simeq\mathrm{Map}(\widetilde{X}, S^3)$, and the diagram (4.1) becomes;

$$\begin{array}{ccccc} \mathrm{Map}(D^4, S^3) & & & & \mathfrak{g}_k \\ \downarrow & & & & \downarrow \\ \mathrm{Map}_0(S^3, S^3) & \xleftarrow{F(Adf_k)} & \mathrm{Map}_0(S^3, S^3) & \xleftarrow{\xi^{\#}} & \mathrm{Map}(\bigvee_b S^2, S^3). \end{array}$$

We have

Lemma 4. 1. *If $[Adf_k]$ is in the image of $\xi^{\#}$: $\oplus_b \pi_2(\mathrm{Map}_1^*(S^3, S^3)) \to \pi_3(\mathrm{Map}_1^*(S^3, S^3))$, then $\mathfrak{g}_k \simeq \mathfrak{g}_0$.*

Proof. If $[Adf_k] = \xi^{\#}[\eta]$, then for $\varphi \in \mathrm{Map}(\bigvee_b S^2, S^3)$ and $x \in S^3$,

$$\begin{aligned} \{F(Adf_k) \circ \xi^{\#}(\varphi)\}(x) = \{F(\xi^{\#}\eta)(\xi^{\#}\varphi)\}(x) &= \xi^{\#}\eta(x)(\xi^{\#}\varphi(x)) \\ &= \eta(\xi(x))(\varphi(\xi(x))) \\ &= \{F(\eta)(\varphi)\}(\xi(x)) \\ &= \{\xi^{\#} \circ F(\eta)(\varphi)\}(x). \end{aligned}$$

Therefore we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_0(S^3, S^3) & \xleftarrow{F(Adf_k) \circ \xi^{\#}} & \mathrm{Map}(\bigvee_b S^2, S^3) \\ \| & & \downarrow F(\eta) \\ \mathrm{Map}_0(S^3, S^3) & \xleftarrow{\xi^{\#}} & \mathrm{Map}(\bigvee_b S^2, S^3). \end{array}$$

Since $\pi_2(\mathrm{Map}_1^*(S^3, S^3)) \cong \mathbf{Z}/2$, $F(\eta)$ is a homotopy equivalence, henceforth $\mathfrak{g}_k \simeq \mathfrak{g}_0$.

As noted before F is a homomorphism, so if $k \equiv k'$ $(\mathrm{mod} 12/d(X))$, $\mathfrak{g}_k \simeq \mathfrak{g}_{k'}$.

Finally we show $\mathfrak{g}_k \simeq \mathfrak{g}_{-k}$. Consider the following maps; -1: $D^4 = \{q \in \mathbf{H}, |q| \leq 1\} \to D^4$ given by $-1(q) = \bar{q}$ and degree -1 map -1: $S^2 \to S^2$. It can be easily shown that the following diagram commutes up to homotopy and vertical arrows are homotopy equivalence,

$$\begin{array}{ccccccc} \mathrm{Map}(D^4, S^3) & \longrightarrow & \mathrm{Map}_0(S^3, S^3) & \xleftarrow{F(Adf_k)} & \mathrm{Map}_0(S^3, S^3) & \xleftarrow{\xi^{\#}} & \mathrm{Map}(\bigvee_b S^2, S^3) \\ \downarrow -1^{\#} & & \downarrow -1^{\#} & & \downarrow -1^{\#} & & \downarrow (\vee(-1))^{\#} \\ \mathrm{Map}(D^4, S^3) & \longrightarrow & \mathrm{Map}_0(S^3, S^3) & \xleftarrow{F(Adf_{-k})} & \mathrm{Map}_0(S^3, S^3) & \xleftarrow{\xi^{\#}} & \mathrm{Map}(\bigvee_b S^2, S^3) \end{array}$$

and we have $\mathfrak{g}_k \simeq \mathfrak{g}_{-k}$.

Thus if $k \equiv \pm k'$ $(\mathrm{mod} 12/d(X))$, $\mathfrak{g}_k \simeq \mathfrak{g}_{k'}$. Unfortunatly we cannot get the homotopy equivalence $\mathfrak{g}_k \simeq \mathfrak{g}_{5k}$ in this way, which is the crucial part of the proof of theorem 1. 2.

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