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On Chern numbers of homology planes of certain types

By

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1. Introduction

A nonsingular algebraic surface X defined over $\mathbf{C}$ is called a homology plane if homology groups $H_i(X; \mathbf{Z})$ vanish for all $i > 0$. We know how to construct a homology plane with Kodaira dimension $\kappa(X) \leq 1$ (cf. [2]). As for homology planes with $\kappa(X) = 2$, though plenty of examples of such homology planes have been constructed, we are still far from classifying them completely.

Since a homology plane X is an affine rational surface and X has a fiber space structure whose general fibers are isomorphic to $\mathbf{C}^{N*}$, where $\mathbf{C}^{N*}$ is the affine line minus N points, it seems natural to begin with a study in the case $N = 2$, that is, a homology plane with a $\mathbf{C}^{**}$-fibration. Note that the case $N = 1$ corresponds to $\kappa \leq 1$. In our previous paper [5], we treated this case $N = 2$ and classified homology planes with $\mathbf{C}^{**}$-fibrations. In [1], tom Dieck gave several examples in the case $N = 3$. In this context, the following problem seems interesting.

Problem 1. *Let X be a homology plane of general type. Define the number $F(X)$ by*

$$F(X) = \min \{N \mid \textit{there exists a } \mathbf{C}^{N*}\textit{-fibration on } X\}\,.$$

Is $F(X)$ then bounded or not? Namely, does there exist a constant A independent of X such that $F(X) \leq A$?

The Chern numbers and the Miyaoka-Yau inequality play an important role in the classification theory of projective surfaces. The inequality gives the first restriction to the existence area of surfaces in the (c_2, c_1^2)-plane and further precise research is made for the surfaces corresponding to values in this area. We would like to use Chern numbers in the study of homology planes. The Miyaoka-Yau inequality was extended to the open surfaces in [3, 4] and the inequality $c_1^2 \leq 3c_2$ holds also for open surfaces if c_1^2 and c_2 stand for logarithmic Chern numbers. We note that if X is an open surface, c_1^2 could be a rational number. (See below for the definition of c_1^2.) Since Betti numbers of a homology plane X are zero except for b_0, the Euler number c_2 of X equals one.

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In the second section we calculate c_1^2 for homology planes with $\mathbf{C}^{**}$-fibrations and obtain the following:

Theorem. *Let X be a homology plane of Kodaira dimension* 2 *with a* $\mathbf{C}^{**}$-*fibration. Then the second Chern number $c_1(X)^2$ of X is less than* 2. *Moreover there exists a sequence of homology planes with* $\mathbf{C}^{**}$-*fibrations whose c_1^2 converge to* 2.

This result is compared with Xiao's result for projective surfaces with fibrations of curves of genus 2 (cf. [8]). In the third section, we calculate c_1^2 for surfaces with $\mathbf{C}^{3*}$-fibrations given by tom Dieck. In several cases c_1^2 attains a value which is very close to 5/2 and it seems that there should be some relation between $F(X)$ and $c_1(X)^2$. So, we shall pose the following question:

Problem 2. *Does there exist a sequence of homology planes X_i whose Chern numbers $c_1(X_i)^2$ converge to* 3?

If there exist a sequence of surfaces X_i for which $c_1(X_i)^2$ converge to 3, it is more plausible that $F(X)$ is unbounded.

Now we recall several notions and terminologies from the open surface theory (cf. [6]). We embed X into a nonsingular projective surface V. The boundary divisor $D := V - X$ is called a *simple normal crossing divisor* if D satisfies the following three conditions:

1. every irreducible component of D is smooth,
2. no three irreducible components pass through a common point,
3. all intersections of the irreducible components of D are transverse.

Furthermore we say that D is a *minimal* normal crossing divisor if any (-1) curve in D intersects at least three other irreducible components of D. We choose below an embedding of X into V so that D is a minimal normal crossing divisor. A connected curve T consisting of irreducible components in D is called a *twig* if the dual graph of T is a linear chain and T meets $D - T$ in a single point at one of the end components of T. A connected component R (resp. F) of D is called a *rod* (resp. a *fork*) if the dual graph of R (resp. F) is a linear chain (resp. the dual graph of the exceptional curves of a minimal resolution of a non-cyclic quotient singularity, where the central component may have intersection ≥ -1).

A connected curve B contained in D is said to be *rational* if each irreducible component of B is rational. B is also said to be *admissible* if none of the irreducible components of B is a (-1) curve and the intersection matrix of B is negative definite. An admissible rational twig T is *maximal* if T is not extended to an admissible rational twig with more irreducible components.

Denote by K_V the canonical divisor of V. By the theory of peeling [6], we can decompose the divisor D uniquely into a sum of effective $\boldsymbol{Q}$-divisors $D = D^* + \mathrm{Bk}(D)$ such that

1. $\mathrm{Bk}(D)$ has the negative definite intersection form.

2. $(K_V + D^* \cdot Z) = 0$ for every irreducible component Z of all maximal twigs, rods and forks which are admissible and rational.
3. $(K_V + D^* \cdot Z) \geq 0$ for every irreducible component Y of D except for the *irrelevant* components of twigs, rods and forks which are non-admissible and rational.

Here we restrict our attention to the homology planes of general type. We know from [7] that a homology plane of general type is almost minimal. This implies $(K_V + D^* \cdot C) \geq 0$ for every irreducible curve C on V. We define the Chern number $c_1(X)^2$ of X by $(K_V + D^*)^2$, where $K_V + D^*$ is described also in the following way:

We contract all maximal twigs, rods and forks which are admissible and rational. We get a normal surface $\bar{S}$. Let $\rho: V \to \bar{S}$ be the contraction morphism. Then the total transform of the canonical divisor $K_{\bar{S}}$ of $\bar{S}$ plus $\rho_* D^*$ as a Q-divisor equals $K_V + D^*$. Thus we obtain $c_1(X)^2 := (K_V + D^*)^2 = (K_{\bar{S}} + \rho_* D^*)^2$. We use below the surface $\bar{S}$ to calculate $c_1(X)^2$.

2. Calculations of $c_1^2(X)$ for homology planes with C^{**}-fibrations

We use the notations of [5] freely. There are four types of homology planes with C^{**}-fibrations, which are types (UP_{3-1}), (UC_{2-1}), (TP_2), and (TC_{2-1}).

Type (UP_{3-1})

We start with a configuration of curves on $\mathbf{P}^1 \times \mathbf{P}^1$ given as follows:

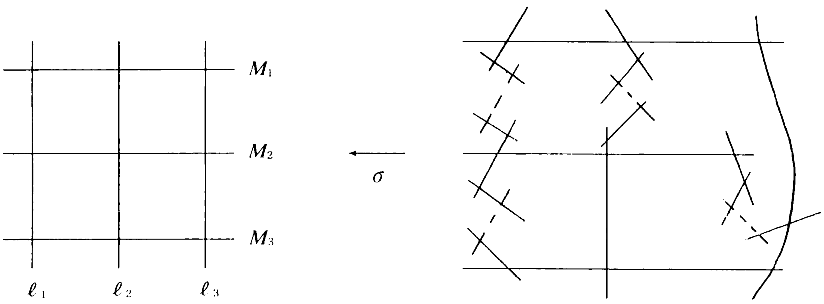

Figure 1

where l_1, l_2 and l_3 represent the fibers of the first projection $p_1: \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ and M_1, M_2 and M_3 the fibers of the second projection $p_2: \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$. Let $R_1 = M_1 \cap l_1$, $R_2 = M_3 \cap l_1$, $R_3 = M_1 \cap l_2$ and $R_4 = M_2 \cap l_3$. We perform oscillating sequences of blowing-ups $\sigma: V \to \mathbf{P}^1 \times \mathbf{P}^1$ with initial points R_1, R_2, R_3 and R_4 (cf. [5]). The projection p_1 induces a $\mathbf{P}^1$-fibration on V.

Let E_i $(1 \leq i \leq 4)$ be the unique (-1) curve contained in the exceptional set $\sigma^{-1}(R_i)$. Let T_{1a} be the connected component of $\sigma^{-1}(R_1) - E_1$ connecting with the fiber component l'_1 which is the proper transform of l_1. Let T_{1b} be the connected component of $\sigma^{-1}(R_1) - E_1$ connecting with the section M'_1 which is the proper transform of M_1. We define T_{ia}, T_{ib} $(2 \leq i \leq 4)$ in a similar way. We write the total transforms of l_i and M_i as follows:

$$\sigma^*(l_1) \sim l'_1 + a_1 E_1 + a_2 E_2 + \cdots$$

$$\sigma^*(l_2) \sim l'_2 + a_3 E_3 + \cdots$$

$$\sigma^*(l_3) \sim l'_3 + a_4 E_4 + \cdots$$

$$\sigma^*(M_1) \sim M'_1 + b_1 E_1 + b_3 E_3 + \cdots$$

$$\sigma^*(M_2) \sim M'_2 + b_4 E_4 + \cdots$$

$$\sigma^*(M_3) \sim M'_3 + b_2 E_2 + \cdots$$

We define the boundary divisor D on V by

$$D = \sum_{i=1}^{3} (l'_i + M'_i) + \sum_{i=1}^{4} (T_{ia} + T_{ib})$$

Then $X := V - D$ is a homology plane provided the following condition is satisfied:

$$a_3 a_4 b_1 b_2 + a_1 a_4 b_2 b_3 - a_2 a_3 b_1 b_4 - a_1 a_3 b_2 b_4 = \pm 1 .$$

The dual graph of D is given as follows:

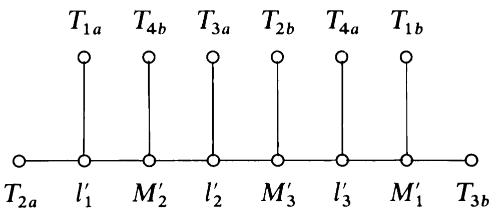

Figure 2

We remark that the branch T_{ia} is empty if and only if $b_i = 1$ and the branch T_{ib} is empty if and only if $a_i = 1$.

We denote by S the surface obtained by contracting all components of T_{ia} and T_{ib}. Let π: $V \to S$ be the contraction morphism. We denote by $\bar{S}$ the surface obtained by contracting all components of Supp Bk (D). Since T_{ia} and T_{ib} are contained in Supp Bk (D), the contraction morphism $V \to \bar{S}$ factors through S. Let ρ: $S \to \bar{S}$ be the natural factoring morphism. Put $\tilde{l}_i = \pi(l'_i)$ and $\tilde{M}_i = \pi(M'_i)$, put also $\bar{l}_i = \rho(\tilde{l}_i)$ and $\overline{M}_i = \rho(\tilde{M}_i)$ and put finally $\Delta = \pi(D)$ and $\bar{\Delta} = \rho(\Delta)$. If all T_{ia} and T_{ib} are not empty, Supp Bk $(D) = (\cup T_{ia}) \cup (\cup T_{ib})$ and $S = \bar{S}$. In this case the Chern number $c_1(X)^2$ of X equals $(K_S + \Delta)^2$. In any case we make use of the surface S in order to calculate the Chern number $c_1(X)^2$.

By symmetry we have to consider the following twelve cases separately, where the case 1 is a general case with all T_{ia} and T_{ib} not empty, while in the other cases some of T_{ia} and T_{ib} are empty and more components of D have to be contracted under $\rho: S \to \bar{S}$.

(Case 1) $\bar{\Delta} = \bar{l}_1 + \overline{M}_2 + \bar{l}_2 + \overline{M}_3 + \bar{l}_3 + \overline{M}_1$

(Case 2) $\bar{\Delta} = \overline{M}_2 + \bar{l}_2 + \overline{M}_3 + \bar{l}_3 + \overline{M}_1$

(Case 3) $\bar{\Delta} = \bar{l}_2 + \overline{M}_3 + \bar{l}_3 + \overline{M}_1$

(Case 4) $\bar{\Delta} = \overline{M}_3 + \bar{l}_3 + \overline{M}_1$

(Case 5) $\bar{\Delta} = \bar{l}_3 + \overline{M}_1$

(Case 6) $\bar{\Delta} = \overline{M}_1$

(Case 7) $\bar{\Delta} = \overline{M}_2 + \bar{l}_2 + \overline{M}_3 + \bar{l}_3$

(Case 8) $\bar{\Delta} = \bar{l}_2 + \overline{M}_3 + \bar{l}_3$

(Case 9) $\bar{\Delta} = \overline{M}_3 + \bar{l}_3$

(Case 10) $\bar{\Delta} = \bar{l}_3$

(Case 11) $\bar{\Delta} = \bar{l}_2 + \overline{M}_3$

(Case 12) $\bar{\Delta} = \overline{M}_3$

We shall look into each of the above cases separately.

(Case 1) $\bar{\Delta} = \bar{l}_1 + \overline{M}_2 + \bar{l}_2 + \overline{M}_3 + \bar{l}_3 + \overline{M}_1$

The configuration of the components of Δ and $\tilde{E}_i$ on the surface S is given as follows:

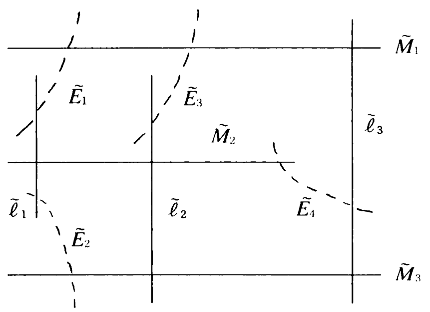

Figure 3

The linear equivalence relations $l_1 \sim l_2 \sim l_3$ and $M_1 \sim M_2 \sim M_3$ on $\mathbf{P}^1 \times \mathbf{P}^1$ give rise to the following relations on S:

$$\tilde{l}_1 + a_1\tilde{E}_1 + a_2\tilde{E}_2 \sim \tilde{l}_2 + a_3\tilde{E}_3 \sim \tilde{l}_3 + a_4\tilde{E}_4$$

$$\tilde{M}_1 + b_1\tilde{E}_1 + b_3\tilde{E}_3 \sim \tilde{M}_2 + b_4\tilde{E}_4 \sim \tilde{M}_3 + b_2\tilde{E}_2$$

Using these relations we get the intersection numbers of various curves on S as follows.

$$(\tilde{l}_1)^2 = -\left(\frac{a_1}{b_1} + \frac{a_2}{b_2}\right), \ (\tilde{l}_2)^2 = -\frac{a_3}{b_3}, \ (\tilde{l}_3)^2 = -\frac{a_4}{b_4},$$

$$(\tilde{M}_1)^2 = -\left(\frac{b_1}{a_1} + \frac{b_3}{a_3}\right), \ (\tilde{M}_2)^2 = -\frac{b_4}{a_4}, \ (\tilde{M}_3)^2 = -\frac{b_2}{a_2},$$

$$(\tilde{E}_1)^2 = -\frac{1}{a_1b_1}, \ (\tilde{E}_2)^2 = -\frac{1}{a_2b_2}, \ (\tilde{E}_3)^2 = -\frac{1}{a_3b_3}, \ (\tilde{E}_4)^2 = -\frac{1}{a_4b_4},$$

$$(\tilde{E}_1\cdot\tilde{M}_1) = \frac{1}{a_1}, \ (\tilde{E}_2\cdot\tilde{M}_3) = \frac{1}{a_2}, \ (\tilde{E}_3\cdot\tilde{M}_1) = \frac{1}{a_3}, \ (\tilde{E}_4\cdot\tilde{M}_2) = \frac{1}{a_4},$$

$$(\tilde{E}_1\cdot\tilde{l}_1) = \frac{1}{b_1}, \ (\tilde{E}_2\cdot\tilde{l}_1) = \frac{1}{b_2}, \ (\tilde{E}_3\cdot\tilde{l}_2) = \frac{1}{b_3}, \ (\tilde{E}_4\cdot\tilde{l}_3) = \frac{1}{b_4}.$$

Next we have to write down the canonical divisor K_S of S. We start with the canonical divisor $K_{\mathbf{P}^1\times\mathbf{P}^1} \sim -2M - 2l$ of $\mathbf{P}^1 \times \mathbf{P}^1$, where M is a section of $p_1: \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ and l is a general fiber. Then using an induction on the number of blowing-ups, it is not hard to obtain the following formula:

$$K_V \sim -2\sigma^*(M) - 2\sigma^*(l) + \sum_{i=1}^{4} (a_i + b_i - 1)E_i + (\text{components of } T_{i,a} \text{ and } T_{i,b}).$$

By construction, we obtain the following expression of K_S.

$$K_S \sim -2\pi_*\sigma^*(M) - 2\pi_*\sigma^*(l) + \sum_{i=1}^{4} (a_i + b_i - 1)\tilde{E}_i .$$

Then

$$K_S + \Delta \sim -2\pi_*\sigma^*(M) - 2\pi_*\sigma^*(l) + \sum_{i=1}^{4} (a_i + b_i - 1)\tilde{E}_i + \sum_{i=1}^{3} \{\tilde{l}_i + \tilde{M}_i\} .$$

We express $\tilde{l}_2$, $\tilde{l}_3$, $\tilde{M}_2$, $\tilde{M}_3$ by the rest of the curves and obtain:

$$\begin{aligned} K_S + \Delta \sim & -2(\tilde{M}_1 + b_1\tilde{E}_1 + b_3\tilde{E}_3) - 2(\tilde{l}_1 + a_1\tilde{E}_1 + a_2\tilde{E}_2) \\ & + \textstyle\sum_{i=1}^{4} (a_i + b_i - 1)\tilde{E}_i + \tilde{l}_1 + (\tilde{l}_1 + a_1\tilde{E}_1 + a_2\tilde{E}_2 - a_3\tilde{E}_3) \\ & + (\tilde{l}_1 + a_1\tilde{E}_1 + a_2\tilde{E}_2 - a_4\tilde{E}_4) + \tilde{M}_1 + (\tilde{M}_1 + b_1\tilde{E}_1 + b_3\tilde{E}_3 - b_4\tilde{E}_4) \\ & + (\tilde{M}_1 + b_1\tilde{E}_1 + b_3\tilde{E}_3 - b_2\tilde{E}_2) \\ = & \ \tilde{l}_1 + \tilde{M}_1 + (a_1 + b_1 - 1)\tilde{E}_1 + (a_2 - 1)\tilde{E}_2 + (b_3 - 1)\tilde{E}_3 - \tilde{E}_4 . \end{aligned}$$

Therefore we have

$$
\begin{aligned}
c_1(X)^2 &= (K_S + \Delta)^2 \\
&= \tilde{l}_1^2 + \tilde{M}_1^2 + (a_1 + b_1 - 1)^2\tilde{E}_1^2 + (a_2 - 1)^2\tilde{E}_2^2 + (b_3 - 1)^2\tilde{E}_3^2 + \tilde{E}_4^2 \\
&\quad + 2(a_1 + b_1 - 1)(\tilde{l}_1 \cdot \tilde{E}_1) + 2(a_2 - 1)(\tilde{l}_1 \cdot \tilde{E}_2) \\
&\quad + 2(a_1 + b_1 - 1)(\tilde{M}_1 \cdot \tilde{E}_1) + 2(b_3 - 1)(\tilde{M}_1 \cdot \tilde{E}_3) \\
&= 2 - \left(\frac{1}{a_1 b_1} + \frac{1}{a_2 b_2} + \frac{1}{a_3 b_3} + \frac{1}{a_4 b_4}\right) \\
&< 2.
\end{aligned}
$$

(Case 2) $\bar{\Delta} = \bar{M}_2 + \bar{l}_2 + \bar{M}_3 + \bar{l}_3 + \bar{M}_1$

In this case T_{1a} or T_{2a} is empty and l_1' is contained in Supp Bk (D). We have to perform the peeling of the bark of $\tilde{l}_1$ on S, and obtain $c_1(X)^2$ as $c_1(X)^2 = (K_S + \Delta + \alpha\tilde{l}_1)^2$, where the number α is determined by the condition:

$$(K_S + \Delta + \alpha\tilde{l}_1 \cdot \tilde{l}_1) = 0\,,$$

i.e.,

$$\alpha = -\frac{(K_S + \Delta \cdot \tilde{l}_1)}{(\tilde{l}_1)^2}\,.$$

We use the expression of $K_S + \Delta$ and the intersection numbers obtained in (Case 1) to compute

$$
\begin{aligned}
(K_S + \Delta \cdot \tilde{l}_1) &= \tilde{l}_1^2 + (a_1 + b_1 - 1)(\tilde{l}_1 \cdot \tilde{E}_1) + (a_2 - 1)(\tilde{l}_1 \cdot \tilde{E}_2) \\
&= -\left(\frac{a_1}{b_1} + \frac{a_2}{b_2}\right) + \frac{a_1 + b_1 - 1}{b_1} + \frac{a_2 - 1}{b_2} \\
&= 1 - \left(\frac{1}{b_1} + \frac{1}{b_2}\right),
\end{aligned}
$$

which entails $\alpha = \dfrac{b_1 b_2 - b_1 - b_2}{a_1 b_2 + a_2 b_1}$ and

$$
\begin{aligned}
c_1(X)^2 &= (K_S + \Delta + \alpha\tilde{l}_1)^2 \\
&= (K_S + \Delta)^2 + \alpha(K_S + \Delta \cdot \tilde{l}_1) \\
&= 2 - \left(\frac{1}{a_1 b_1} + \frac{1}{a_2 b_2} + \frac{1}{a_3 b_3} + \frac{1}{a_4 b_4}\right) + \frac{(b_1 b_2 - b_1 - b_2)^2}{b_1 b_2(a_1 b_2 + a_2 b_1)} \\
&< 2\,.
\end{aligned}
$$

Here we note that $b_1 = 1$ or $b_2 = 1$.

(Case 3) $\bar{\Delta} = \bar{l}_2 + \bar{M}_3 + \bar{l}_3 + \bar{M}_1$

In this case T_{1a} (or T_{2a}) and T_{4b} are empty and l_1' and M_1' are contained in Supp Bk (D). This case occurs when $b_1 = 1$ (or $b_2 = 1$) and $a_4 = 1$. We determine the numbers α and β by the following conditions:

$$\begin{cases} (K_S + \varDelta + \alpha \tilde{l}_1 + \beta \tilde{M}_2 \cdot \tilde{l}_1) = 0 \\ (K_S + \varDelta + \alpha \tilde{l}_1 + \beta \tilde{M}_2 \cdot \tilde{M}_2) = 0 \,, \end{cases}$$

i.e.,

$$\begin{cases} \alpha(\tilde{l}_1)^2 + \beta(\tilde{M}_2 \cdot \tilde{l}_1) = -(K_S + \varDelta \cdot \tilde{l}_1) = -1 + \left(\dfrac{1}{b_1} + \dfrac{1}{b_2}\right) \\ \alpha(\tilde{l}_1 \cdot \tilde{M}_2) + \beta(\tilde{M}_2)^2 = -(K_S + \varDelta \cdot \tilde{M}_2) = 0 \end{cases}$$

Hence we have

$$\alpha = \frac{b_4\{b_1 b_2 - (b_1 + b_2)\}}{b_4(a_1 b_2 + a_2 b_1) - b_1 b_2}, \quad \beta = \frac{\alpha}{b_4}$$

and

$$\begin{aligned} c_1(X)^2 &= (K_S + \varDelta + \alpha \tilde{l}_1 + \beta \tilde{M}_2)^2 \\ &= (K_S + \varDelta)^2 + \alpha(K_S + \varDelta \cdot \tilde{l}_1) + \beta(K_S + \varDelta \cdot \tilde{M}_2) \\ &= 2 - \left(\frac{1}{a_1 b_1} + \frac{1}{a_2 b_2} + \frac{1}{a_3 b_3} + \frac{1}{a_4 b_4}\right) + \frac{b_4(b_1 b_2 - b_1 - b_2)^2}{b_1 b_2\{b_4(a_1 b_2 + a_2 b_1) - b_1 b_2\}} \\ &< 2 \,. \end{aligned}$$

(Case 4) $\bar{\varDelta} = \bar{M}_3 + \bar{l}_3 + \bar{M}_1$

In this case T_{1a} (or T_{2a}), T_{4b} and T_{3a} are empty, and l_1', M_2' and l_2' are contained in Supp Bk (D). This case occurs when $b_1 = 1$ (or $b_2 = 1$), $a_4 = 1$ and $b_3 = 1$. We determine the numbers α, β and γ by the following conditions:

$$\begin{cases} \alpha(\tilde{l}_1)^2 + \beta = -(K_S + \varDelta \cdot \tilde{l}_1) = -1 + \left(\dfrac{1}{b_1} + \dfrac{1}{b_2}\right) \\ \alpha + \beta(\tilde{M}_2)^2 + \gamma = -(K_S + \varDelta \cdot \tilde{M}_2) = 0 \\ \beta + \gamma(\tilde{l}_2)^2 = -(K_S + \varDelta \cdot \tilde{l}_2) = 0 \,. \end{cases}$$

As seen from the former case, we need only the value of α. We consider the case $b_1 = 1$. Then

$$\alpha = -\frac{a_3 b_4 - 1}{(a_3 b_4 - 1)(a_1 b_2 + a_2) - a_3 b_2}$$

and

$$c_1(X)^2 = (K_S + \Delta + \alpha\tilde{l}_1 + \beta\tilde{M}_2 + \gamma\tilde{l}_2)^2$$
$$= 2 - \left(\frac{1}{a_1} + \frac{1}{a_2 b_2} + \frac{1}{a_3} + \frac{1}{b_4}\right) + \frac{a_3 b_4 - 1}{b_2\{(a_3 b_4 - 1)(a_1 b_2 + a_2) - a_3 b_2\}}$$
$$< 2.$$

(Case 5) $\bar{\Delta} = \bar{l}_3 + \bar{M}_1$

In this case T_{1a} (or T_{2a}), T_{4b}, T_{3a} and T_{2b} are empty and l'_1, M'_1, l'_2 and M'_3 are contained in Supp Bk (D). This case occurs when $b_1 = 1$ (or $b_2 = 1$), $a_4 = 1$, $b_3 = 1$ and $a_2 = 1$. Here we assume $b_1 = 1$. We can treat the case $b_2 = 1$ in a similar way. We determine the numbers α, β, γ and δ by the following conditions:

$$\begin{cases} \alpha(\tilde{l}_1)^2 + \beta = -(K_S + \Delta \cdot \tilde{l}_1) = \dfrac{1}{b_2} \\ \alpha + \beta(\tilde{M}_2)^2 + \gamma = -(K_S + \Delta \cdot \tilde{M}_2) = 0 \\ \beta + \gamma(\tilde{l}_2)^2 + \delta = -(K_S + \Delta \cdot \tilde{l}_2) = 0 \\ \gamma + \delta(\tilde{M}_3)^2 = -(K_S + \Delta \cdot \tilde{M}_3) = 0\,. \end{cases}$$

The only value we need is α, which is given as follows:

$$\alpha = -\frac{b_4(a_3 b_2 - 1) + b_2}{\{b_4(a_3 b_2 - 1) + b_2\}(a_1 b_2 + 1) - b_2(a_3 b_2 - 1)} < 0\,.$$

Hence we have

$$c_1(X)^2 = (K_S + \Delta + \alpha\tilde{l}_1 + \beta\tilde{M}_2 + \gamma\tilde{l}_2 + \delta\tilde{M}_3)^2$$
$$= 2 - \left(\frac{1}{a_1} + \frac{1}{b_1} + \frac{1}{a_3} + \frac{1}{b_4}\right)$$
$$+ \frac{b_4(a_3 b_2 - 1) + b_2}{b_2[\{b_4(a_3 b_2 - 1) + b_2\}(a_1 b_2 + 1) - b_2(a_3 b_2 - 1)]}$$
$$< 2\,.$$

(Case 6) $\bar{\Delta} = \bar{M}_1$

In this case T_{1a} (or T_{2a}), T_{4b}, T_{3a}, T_{2b} and T_{4a} are empty and l'_1, M'_1, l'_2, M'_3 and l'_3 are contained in Supp Bk (D). This case occurs when $b_1 = 1$ (or $b_2 = 1$) and $a_4 = b_3 = a_2 = b_4 = 1$. Here we assume $b_1 = 1$. We determine the numbers α, β, γ, δ and ε by the following conditions.

$$\begin{cases} \alpha(\tilde{l}_1)^2 + \beta = -(K_S + \Delta \cdot \tilde{l}_1) = \dfrac{1}{b_2} \\ \alpha + \beta(\tilde{M}_2) + \gamma = -(K_S + \Delta \cdot \tilde{M}_2) = 0 \\ \beta + \gamma(\tilde{l}_2)^2 + \delta = -(K_S + \Delta \cdot \tilde{l}_2) = 0 \\ \gamma + \delta(\tilde{M}_3)^2 + \varepsilon = -(K_S + \Delta \cdot \tilde{M}_3) = 0 \\ \delta + \varepsilon(\tilde{l}_3)^2 = -(K_S + \Delta \cdot \tilde{l}_3) = 0 \end{cases}$$

Then we obtain the following:

$$\alpha = -\frac{(a_3-1)(b_2-1)-1}{\{(a_3-1)(b_2-1)-1\}(a_1b_2+1)-b_2\{a_3(b_2-1)-1\}}$$

$$\begin{aligned} c_1(X)^2 &= (K_S + \Delta + \alpha\tilde{l}_1 + \beta\tilde{M}_2 + \gamma\tilde{l}_2 + \delta\tilde{M}_3 + \varepsilon\tilde{l}_3)^2 \\ &= 2 - \left(\frac{1}{a_1} + \frac{1}{b_2} + \frac{1}{a_3} + 1\right) \\ &\quad + \frac{(a_3-1)(b_2-1)-1}{b_2[\{(a_3-1)(b_2-1)-1\}(a_1b_2+1)-b_2\{a_3(b_2-1)-1\}]} \\ &< 2. \end{aligned}$$

We can calculate $c_1(X)^2$ for the remaining cases (7) $\sim$ (12) by combining the former cases. For example we consider the (case 8). In this case l'_1, M'_2 and M'_1 are contained in Supp Bk (D). We obtain $c_1(X)^2$ by combining the (case 1) and the (case 2). We give the result when $b_1 = a_4 = a_1 = 1$. Namely, we have

$$c_1(X)^2 = 2 - \left(1 + \frac{1}{a_2b_2} + \frac{1}{a_3b_3} + \frac{1}{b_4}\right) + \frac{b_4}{b_2\{b_4(b_2+a_2)-b_2\}} + \frac{1}{a_3(b_3+a_3)}\,.$$

We note finally that the Ramanujam surface is obtained by this construction. The corresponding values of a_i and b_i are as follows:

$$a_1 = a_2 = a_3 = a_4 = 1,\ b_1 = 1,\ b_2 = 3,\ b_3 = 2,\ b_4 = 2\,.$$

We thus obtain $c_1(X)^2 = \frac{2}{15}$ for the Ramanujam surface.

Type (UC_{2-1})

We start with a configuration of curves on $\mathbf{P}^1 \times \mathbf{P}^1$ given as follows.

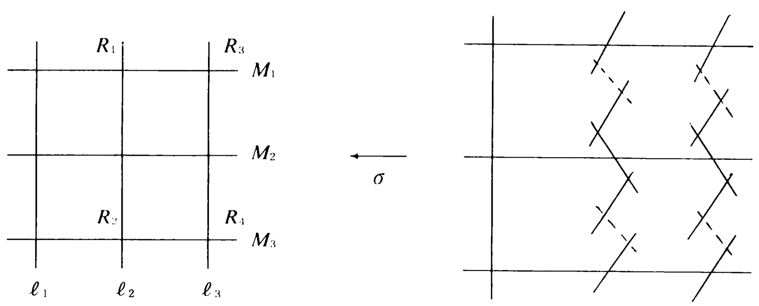

Figure 4

We perform oscillating sequences of blowing-ups $\sigma\colon V \to \mathbf{P}^1 \times \mathbf{P}^1$ with initial points R_1, R_2, R_3 and R_4, where $R_1 = M_1 \cap l_2$, $R_2 = M_3 \cap l_2$, $R_3 = M_1 \cap l_3$, $R_4 =$

$M_3 \cap l_3$. We use the notations l_i, M_i, E_i, T_{ia} and T_{ib} in the same way as in the former case. We write the total transforms of l_i and M_i as follows:

$$\sigma^*(l_2) \sim l_2' + a_1 E_1 + a_2 E_2 + \cdots$$

$$\sigma^*(l_3) \sim l_3' + a_3 E_3 + a_4 E_4 + \cdots$$

$$\sigma^*(M_1) \sim M_1' + b_1 E_1 + b_3 E_3 + \cdots$$

$$\sigma^*(M_3) \sim M_3' + b_2 E_2 + b_4 E_4 + \cdots$$

We define the boundary divisor D on V by

$$D = \sum_{i=1}^{3} (l_i' + M_i') + \sum_{i=1}^{4} (T_{ia} + T_{ib})$$

Then $X := V - D$ is a homology plane provided the following condition is satisfied:

$$a_2 a_3 b_1 b_4 - a_1 a_4 b_2 b_3 = \pm 1 .$$

The dual graph of D is given as follows.

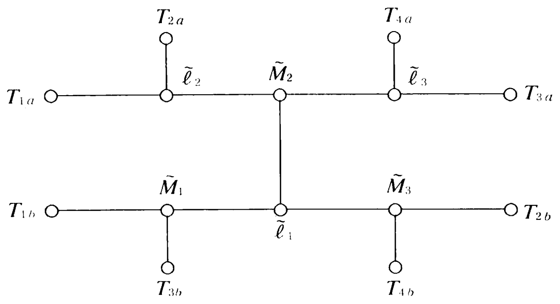

Figure 5

We also use the notations such as S, Δ, $\bar{S}$ and $\bar{\Delta}$.

By symmetry we have to consider the following six cases.

(Case 1) $\bar{\Delta} = \bar{l}_2 + \bar{l}_3 + \bar{M}_2 + \bar{l}_1 + \bar{M}_1 + \bar{M}_3$

(Case 2) $\bar{\Delta} = \bar{l}_3 + \bar{M}_2 + \bar{l}_1 + \bar{M}_1 + \bar{M}_3$

(Case 3) $\bar{\Delta} = \bar{M}_2 + \bar{l}_1 + \bar{M}_1 + \bar{M}_3$

(Case 4) $\bar{\Delta} = \bar{l}_3 + \bar{M}_2 + \bar{l}_1 + \bar{M}_1$

(Case 5) $\bar{\Delta} = \bar{M}_2 + \bar{l}_1 + \bar{M}_1$

(Case 6) $\bar{\Delta} = \bar{M}_2 + \bar{l}_1$

The last four cases are treated by combining the other cases. Therefore we have to consider the first two cases. The computation of $c_1(X)^2$ in these cases are similar to the former case of type (UP_{3-1}). We simply give the values of $c_1(X)^2$:

(Case 1)

$$c_1(X)^2 = (K_S + \Delta)^2 = 2 + \sum_{i=1}^{4} (\tilde{E}_i)^2 = 2 - \sum_{i=1}^{4} \frac{1}{a_i b_i} < 2 .$$

(Case 2)

$$\begin{aligned} c_1(X)^2 &= (K_S + \Delta + \alpha \tilde{l}_2)^2 \\ &= (K_S + \Delta)^2 + \alpha(K_S + \Delta \cdot \tilde{l}_2) \\ &= 2 - \sum_{i=1}^{4} \frac{1}{a_i b_i} + \frac{(b_1 b_2 - b_1 - b_2)^2}{b_1 b_2 (a_1 b_2 + a_2 b_1)} \\ &< 2 . \end{aligned}$$

Here we note that $b_1 = 1$ or $b_2 = 1$. For the remaining cases, we only list up the results in the cases 3 and 6.

(Case 3)

$$\begin{aligned} c_1(X)^2 &= 2 - \sum_{i=1}^{4} \frac{1}{a_i b_i} + \frac{(b_1 b_2 - b_1 - b_2)^2}{b_1 b_2 (a_1 b_2 + a_2 b_1)} + \frac{(b_3 b_4 - b_3 - b_4)^2}{b_3 b_4 (a_3 b_4 + a_4 b_3)} \\ &< 2 . \end{aligned}$$

(Case 6)

$$\begin{aligned} c_1(X)^2 = 2 &- \frac{1}{a_1 b_1} + \frac{(b_1 b_2 - b_1 - b_2)^2}{b_1 b_2 (a_1 b_2 + a_2 b_1)} - \frac{1}{a_2 b_2} + \frac{(a_2 a_4 - a_2 - a_4)^2}{a_2 a_4 (a_2 b_4 + a_4 b_2)} \\ &- \frac{1}{a_3 b_3} + \frac{(a_1 a_3 - a_1 - a_3)^2}{a_1 a_3 (a_1 b_3 + a_3 b_1)} - \frac{1}{a_4 b_4} + \frac{(b_3 b_4 - b_3 - b_4)^2}{b_3 b_4 (a_3 b_4 + a_4 b_3)} \\ < 2 . & \end{aligned}$$

Here we note that one of the integers equals to one for each pair (b_1, b_2), (a_2, a_4), (a_1, a_3) and (b_3, b_4).

Type (TP_2)

We start with the ruled surface Σ_1. Let M_1 be the minimal section of Σ_1 and let p_1 be the morphism from Σ_1 to $\mathbf{P}^1$ which gives the natural $\mathbf{P}^1$-bundle structure on Σ_1. Let C be a 2-section of Σ_1 disjoint from M_1 and let l_1 and l_2 be fibers of p_1 containing ramification points of $p_1|_C$ and l_3 be a fiber of p_1 other than l_1 and l_2. We blow-up $l_1 \cap C$, $l_2 \cap C$ and their infinitely near points on C. We call this surface Σ_1' and let $\sigma_0 \colon \Sigma_1' \to \Sigma_1$ be the composition of the above four blowing-ups. Thus we obtain the following configuration of curves on Σ_1', where M_2 is the proper transform of C and G_2, G_3, H_2 and H_3 are the exceptional curves of σ_0.

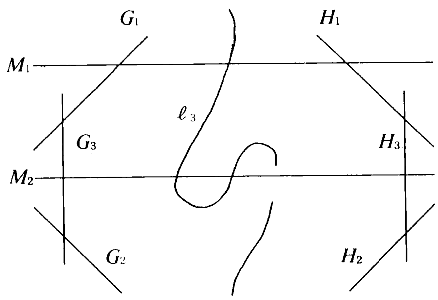

Figure 6

Next we perform oscillating sequences of blowing-ups. Let R_3 be one of two intersection points of l_3 and M_2. There are following six possibilities for choosing initial points R_1 and R_2 of these oscillating sequences. We shall consider first three cases since calculations of the remaining cases are similar.

(Case 1) $R_1 = M_1 \cap G_1$ and $R_2 = M_1 \cap H_1$

(Case 2) $R_1 = G_1 \cap G_3$ and $R_2 = M_1 \cap H_1$

(Case 3) $R_1 = G_3 \cap M_2$ and $R_2 = M_1 \cap H_1$

(Case 4) $R_1 = G_1 \cap G_3$ and $R_2 = H_1 \cap H_3$

(Case 5) $R_1 = G_3 \cap M_2$ and $R_2 = H_1 \cap H_3$

(Case 6) $R_1 = G_3 \cap M_2$ and $R_2 = H_3 \cap M_2$

(Case 1)

Put $R_1 = M_1 \cap G_1$ and $R_2 = M_1 \cap H_1$. Now let $\sigma_1 \colon V \to \Sigma_1'$ be the composition of oscillating blowing-ups with initial points R_1, R_2 and R_3. Let $E_i (1 \leq i \leq 3)$ be the unique (-1) curve contained in $\sigma_1^{-1}(R_1)$. Let T_{1b} be the connected component of $\sigma_1^{-1}(R_1) - E_1$ connecting with the section M_1 and T_{1a} be the connected component of $\sigma_1^{-1}(R_1) - E_1$ connecting with the fiber component G_1'. We define T_{ia}, T_{ib} $(i = 2, 3)$ in a similar way. We write the total transforms of l_i, M_i and C as follows, where $\sigma = \sigma_0 \sigma_1$.

$$\sigma^*(l_1) \sim G_1' + G_2' + 2G_3' + a_1 E_1 + \cdots$$

$$\sigma^*(l_2) \sim H_1' + H_2' + 2H_3' + a_2 E_2 + \cdots$$

$$\sigma^*(l_3) \sim l_3' + a_3 E_3 + \cdots$$

$$\sigma^*(M_1) \sim M_1' + b_1 E_1 + b_2 E_2 + \cdots$$

$$\sigma^*(C) \sim M_2' + G_2' + 2G_3' + H_2' + 2H_3' + b_3 E_3 + \cdots$$

We define the boundary divisor D on V by

$$D = \sum_{i=1}^{2} M_i' + l_3' + \sum_{i=1}^{4} (G_i' + H_i' + T_{ia} + T_{ib}) .$$

Then $X := V - D$ is a homology plane provided the following condition is satisfied:

$$a_1 a_2(2a_3 - b_3) + 2a_2 a_3 b_1 + 2a_1 a_3 b_2 = \pm 1 .$$

The dual graph of D is given as follows.

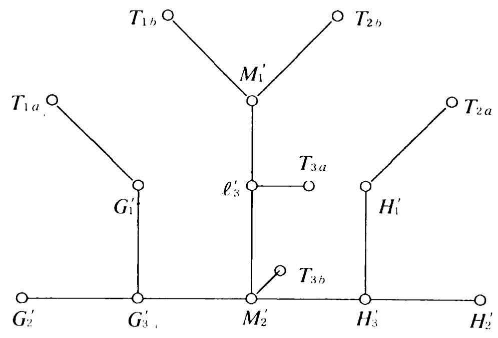

Figure 7

In this case Supp Bk (D) contains not only T_{ia}, T_{ib} but also G_1', G_2', H_1' and H_2'. Let π: $V \to S$ be the contraction of T_{ia}, T_{ib}, G_1', G_2', H_1' and H_2'. Let $\bar{S}$ be the surface obtained by contracting all the components of Supp Bk (D) and let ρ: $S \to \bar{S}$ be the natural factoring morphism. We use the notation like $\tilde{M}_i$ (with tilde) for the curves on S and $\overline{M}_i$ (with bar) for the curves on $\bar{S}$. Put $\Delta = \pi(D)$ and $\bar{\Delta} = \rho(\Delta)$.

In this case the above equality shows that b_3 cannot be equal to 1, that is, the branch T_{3a} is not empty. We have to consider the following two cases depending on whether or not T_{1b} or similarly T_{2b} is empty.

(Case 1-1) $\bar{\Delta} = \bar{G}_3 + \bar{H}_3 + \overline{M}_2 + \bar{l}_3 + \overline{M}_1$

(Case 1-2) $\bar{\Delta} = \bar{G}_3 + \bar{H}_3 + \overline{M}_2 + \bar{l}_3$

Consider first

(Case 1-1) $\bar{\Delta} = \bar{G}_3 + \bar{H}_3 + \overline{M}_2 + \bar{l}_3 + \overline{M}_1$

The configuration of the components of Δ and $\tilde{E}_i$ on the surface S is given as follows:

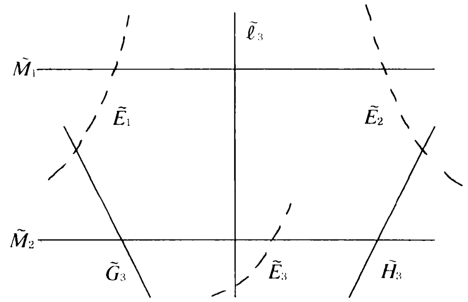

Figure 8

The linear equivalence relations $l_1 \sim l_2 \sim l_3$ and $C \sim 2M_1 + 2l_1$ on the surface Σ_1 give the following relations on S:

$$a_1\tilde{E}_1 + 2\tilde{G}_3 \sim a_2\tilde{E}_2 + 2\tilde{H}_3 \sim \tilde{l}_3 + a_3\tilde{E}_3$$

$$\tilde{M}_2 + 2\tilde{G}_3 + 2\tilde{H}_3 + b_3\tilde{E}_3 \sim 2\tilde{M}_1 + 2b_1\tilde{E}_1 + 2b_2\tilde{E}_2 + 4\tilde{G}_3 + 2a_1\tilde{E}_1 \,.$$

Using these relations, we compute $c_1(X)^2$ as follows:

$$c_1(X)^2 = (K_S + \Delta)^2 = 2 - \frac{2}{a_1(a_1 + 2b_1)} - \frac{2}{a_2(a_2 + 2b_2)} - \frac{1}{a_3 b_3} < 2 \,.$$

(Case 1-2) $\bar{\Delta} = \bar{G}_3 + \bar{H}_3 + \bar{M}_2 + \bar{l}_3$

In this case T_{1b} or T_{2b} is empty and it occurs when $a_1 = 1$ or $a_2 = 1$, respectively. Peeling the bark of $\tilde{M}_1$ as in the case (2) of Type (UP_{3-1}), we obtain

$$c_1(X)^2 = 2 - \frac{2}{a_1(a_1 + 2b_1)} - \frac{2}{a_2(a_2 + 2b_2)} - \frac{1}{a_3 b_3} + \frac{(a_1a_2 - a_1 - a_2)^2}{a_1a_2(a_1a_2 + a_1b_2 + a_2b_1)}$$

$$< 2 \,,$$

where $a_1 = 1$ or $a_2 = 1$.

(Case 2)

Put $R_1 = G_1 \cap G_3$ and $R_2 = M_1 \cap H_1$. We use the same notation as before. First we write the total transforms of G_1 and G_3 by σ_1

$$\sigma_1^*(G_1) \sim G_1' + a_1 E_1 + \cdots$$

$$\sigma_1^*(G_3) \sim G_3' + b_1 E_1 + \cdots$$

and next we write the total transforms of l_i, M_i and C by σ as follows, where $\sigma = \sigma_0\sigma_1$

$$\sigma^*(l_1) \sim G'_1 + G'_2 + 2G'_3 + (a_1 + 2b_1)E_1 + \cdots$$

$$\sigma^*(l_2) \sim H'_1 + H'_2 + 2H'_3 + a_2 E_2 + \cdots$$

$$\sigma^*(l_3) \sim l'_3 + a_3 E_3 + \cdots$$

$$\sigma^*(M_1) \sim M'_1 + b_2 E_2 + \cdots$$

$$\sigma^*(C) \sim M'_2 + G'_2 + 2G'_3 + H'_2 + 2H'_3 + 2b_1 E_1 + b_3 E_3 + \cdots .$$

We define the boundary divisor D in the similar way. Then $X := V - D$ is a homology plane when the following condition is satisfied

$$(a_1 + 2b_1)a_2(2a_3 - b_3) - a_2 a_3 b_2 + 2(a_1 + 2b_1)a_3 b_2 = \pm 1 .$$

The Chern number c_1^2 is given by the same formula as in the case 1 when all T_{ia} and T_{ib} are non-empty. The calculation of the case when some of T_{ia} or T_{ib} are empty is similar to the former case and we always obtain the inequality $c_1^2 < 2$.

(Case 3)

Put $R_1 = G_3 \cap M_2$ and $R_2 = M_1 \cap H_1$. We use the same notation as before. First we write the total transforms of G_3 and M_2 by σ_1

$$\sigma_1^*(G_3) \sim G'_3 + a_1 E_1 + \cdots$$

$$\sigma_1^*(M_2) \sim M'_2 + b_1 E_1 + \cdots$$

and next we write the total transforms of l_i, M_i and C by σ as follows, where $\sigma = \sigma_0 \sigma_1$:

$$\sigma^*(l_1) \sim G'_1 + G'_2 + 2G'_3 + 2a_1 E_1 + \cdots$$

$$\sigma^*(l_2) \sim H'_1 + H'_2 + 2H'_3 + a_2 E_2 + \cdots$$

$$\sigma^*(l_3) \sim l'_3 + a_3 E_3 + \cdots$$

$$\sigma^*(M_1) \sim M'_1 + b_2 E_2 + \cdots$$

$$\sigma^*(C) \sim M'_2 + G'_2 + 2G'_3 + H'_2 + 2H'_3 + (2a_1 + b_1)E_1 + b_3 E_3 + \cdots .$$

We define the boundary divisor D in the similar way. Then $X := V - D$ is a homology plane when the following condition is satisfied:

$$2a_1 a_2(2a_3 - b_3) - a_2 a_3(2a_1 + b_1) + 4a_1 a_3 b_2 = \pm 1 .$$

The Chern number c_1^2 is given by

$$c_1^2 = 2 - \frac{2}{a_1 b_1} - \frac{1}{a_2(a_2 + 2b_2)} - \frac{1}{a_3 b_3}$$

when all T_{ia} and T_{ib} are non-empty. The calculation of the case when some of T_{ia} or T_{ib} are empty is similar to the former case and we always obtain the inequality $c_1^2 < 2$.

Type (TC_{2-1})

We use the same surface Σ_1 and the same configuration of curves on Σ_1' obtained in the former case Type (TP_2). We perform oscillating sequences of blowing-ups as before. The order of $H_1(X, \mathbf{Z})$ is given in the previous paper [5] and the formula given there shows that we have to choose $M_1 \cap G_1$ and $M_2 \cap l_3$ (consisting of two points) as initial points. Put $R_1 = M_1 \cap G_1$ and $M_2 \cap l_3 = \{R_2, R_3\}$.

Now let $\sigma_1 \colon V \to \Sigma_1'$ be the composition of oscillating blowing-ups with initial points R_1, R_2 and R_3. We use the notations like E_i, T_{ia}, and T_{ib} for the same meaning as before. We write the total transforms of l_i, M_i and C as follows, where $\sigma = \sigma_0\sigma_1$

$$\sigma^*(l_1) \sim G_1' + G_2' + 2G_3' + a_1E_1 + \cdots$$

$$\sigma^*(l_2) \sim H_1' + H_2' + 2H_3'$$

$$\sigma^*(l_3) \sim l_3' + a_2E_2 + a_3E_3 + \cdots$$

$$\sigma^*(M_1) \sim M_1' + b_1E_1 + \cdots$$

$$\sigma^*(C) \sim M_2' + G_2' + 2G_3' + H_2' + 2H_3' + b_2E_2 + b_3E_3 + \cdots .$$

We define the boundary divisor D on V by

$$D = \sum_{i=1}^{2} M_i' + l_3' + \sum_{i=1}^{3} (G_i' + H_i' + T_{ia} + T_{ib}) .$$

Then $X := V - D$ is a homology plane provided the following condition is satisfied:

$$a_1a_2b_3 - a_1a_3b_2 = \pm 1 .$$

Here we note that the above condition implies that $a_1 = 1$ and T_{1b} is always empty. The dual graph of D is given as follows.

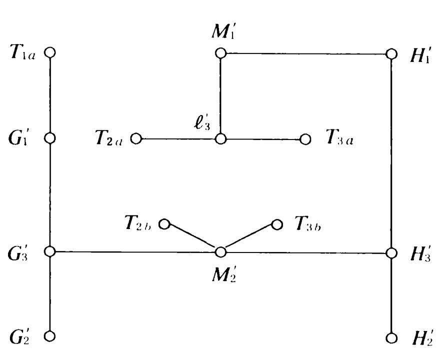

Figure 9

In this case Supp Bk (D) contains T_{ia}, T_{ib}, G'_1, G'_2 and H'_2. Let $\pi: V \to S$ be the contraction of T_{ia}, T_{ib}, G'_1, G'_2 and H'_2. Let $\bar{S}$, $\rho: S \to \bar{S}$, $\Delta = \pi(D)$ and $\bar{\Delta} = \rho(\Delta)$ be the same as before.

We have to consider the following two cases depending on whether or not T_{2b} (similarly T_{3b}) is empty.

(Case 1) $\bar{\Delta} = \bar{G}_3 + \bar{M}_2 + \bar{H}_3 + \bar{H}_1 + \bar{M}_1 + \bar{l}_3$

(Case 2) $\bar{\Delta} = \bar{G}_3 + \bar{M}_2 + \bar{H}_3$

Consider first

(Case 3) $\bar{\Delta} = \bar{G}_3 + \bar{M}_2 + \bar{H}_3 + \bar{H}_1 + \bar{M}_1 + \bar{l}_3$

The configuration of the components of Δ and $\tilde{E}_i$ on the surface S is given as follows.

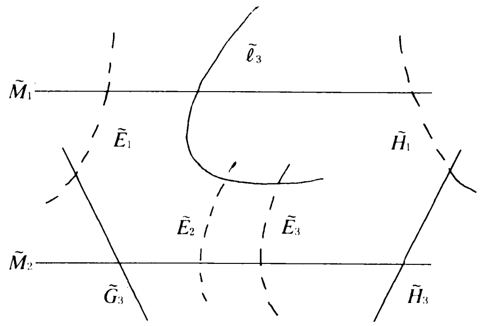

Figure 10

The linear equivalence relations $l_1 \sim l_2 \sim l_3$ and $C \sim 2M_1 + 2l_2$ on the surface Σ_1 give the following relations on S:

$$\tilde{E}_1 + 2\tilde{G}_3 \sim \tilde{l}_3 + a_2\tilde{E}_2 + a_3\tilde{E}_3 \sim \tilde{H}_1 + 2\tilde{H}_3 ,$$

$$\tilde{M}_2 + 2\tilde{G}_3 + 2\tilde{H}_3 + b_2\tilde{E}_2 + b_3\tilde{E}_3 \sim 2\tilde{M}_1 + 2b_1\tilde{E}_1 + 2\tilde{H}_1 + 4\tilde{H}_3$$

Using these relations we compute $c_1(X)^2$ as follows:

$$c_1(X)^2 = 2 - \frac{2}{1+2b_1} - \frac{1}{a_2b_2} - \frac{1}{a_3b_3} < 2 .$$

(Case 2) $\bar{\Delta} = \bar{G}_3 + \bar{M}_2 + \bar{H}_3$

We peel the barks of $\tilde{l}_2$, $\tilde{M}_1$ and $\tilde{H}_1$, and obtain

$$c_1(X)^2 = 2 - \left(\frac{2}{1+2b_1} + \frac{1}{a_2b_2} + \frac{1}{a_3b_3}\right) + \frac{(b_2b_3 - b_2 - b_3)}{b_2b_3\{(1+2b_1)(a_2b_3 + a_3b_2) - 2b_2b_3\}}$$
$$< 2 .$$

Finally from the calculations made in this section, we conclude that:

Theorem. *Let X be a homology plane of Kodaira dimension 2 with a C^{**}-fibration. Then the second Chern number $c_1(X)^2$ of X is less than 2. Furthermore, for every type of X, there exists a sequence of homology planes whose $c_1(X)^2$ converge to 2.*

Remark. For each type (UP_{3-1}), (UC_{2-1}), (TP_2) and (TC_{2-1}), we can find a sequence of homology planes whose Chern numbers c_1^2 converge to 2. First we recall the following fact [2, Lemma 3.5].

Lemma. *Let S be a nonsingular surface and let M and l be smooth curves on S. We assume that M and l intersect transversally at a point P on S. Then for each pair of coprime integers (a, b), there exists a composition of blowing-ups $\sigma: T \to S$ which satisfies*

$$\sigma^{-1}(M) \sim M' + aE + \cdots, \quad \sigma^{-1}(l) \sim l' + bE + \cdots,$$

where E is the exceptional curve of the last blowing-up.

Type (UP_{3-1}). We consider the case $b_1 = 1$ and rewrite the condition on a_i and b_i as follows:

$$a_4 b_2(a_3 + a_1 b_3) - b_4 a_3(a_2 + a_1 b_2) = \pm 1 .$$

First we choose a_1, a_2 and b_3, then choose b_2 which is relatively prime to a_2. Next we choose a_3 such that $(a_3, a_1 b_3) = 1$, $(a_3 + a_1 b_3, a_2 + a_1 b_2) = 1$ and $(a_3, b_2) = 1$. Because $b_2(a_3 + a_1 b_3)$ and $a_3(a_2 + a_1 b_2)$ are relatively prime under these choices, there exist a_4 and b_4 which satisfy the above equation. Since we can choose a_i and b_i to be arbitrarily large numbers, there exists a sequence of homology planes of this type whose Chern numbers c_1^2 converge to 2.

Type (UC_{2-1}). Since the condition on a_i and b_i is $a_2 a_3 b_1 b_4 - a_1 a_4 b_2 b_3 = \pm 1$, it is easy to see that there exists a sequence of homology planes of this type whose Chern numbers c_1^2 converge to 2.

Type (TP_2). We rewrite the condition on a_i and b_i as follows:

$$2a_3(a_1 a_2 + a_2 b_1 + a_1 b_2) - a_1 a_2 b_3 = \pm 1 .$$

First we choose a_1, a_2, b_1 and b_2 such that $(a_1, a_2) = 1$, $(a_1, b_1) = 1$ and $(a_2, b_2) = 1$ and that a_1 and a_2 are odd. Because $2(a_1 a_2 + a_2 b_1 + a_1 b_2)$ and $a_1 a_2$ are relatively prime under these choices, there exist a_3 and b_3 which satisfy the above equation. Since we can choose a_i and b_i to be arbitrarily large numbers, there exists a sequence of homology planes of this type whose Chern numbers c_1^2 converge to 2.

Type (TC_{2-1}). Since the condition on a_i and b_i is $a_1 a_2 b_3 - a_1 a_3 b_2 = \pm 1$. It is easy to see that there exists a sequence of homology planes of this type whose Chern numbers c_1^2 converge to 2.

3. $c_1^2(X)$ of homology planes with C^{3*}-fibrations

In this section, we shall calculate $c_1(X)^2$ of homology planes with a C^{3*}-fibration which are described in a paper of tom Dieck [1]. We make use of his notations in [1] with slight modifications.

1. Case of cubic with two lines

Let $C \subset \mathbf{P}^2$ be a cubic with a cusp s. There is a unique flex $p \in C$ whose tangent we denote by L. Let T be an ordinary tangent to C in a regular point r. C intersects T in another regular point q. The points q, r are different from p. First we blow up s to obtain Σ_1. The exceptional curve M gives a unique minimal section of the natural $\mathbf{P}^1$-fibration on Σ_1. We use the same symbols to denote the proper transform of curves on Σ_1. Let $t = M \cap C$. Then $M \cdot C = 2t$. Let l_1, l_2 and l_3 be the fibers of the $\mathbf{P}^1$-fibration passing through t, p and r, respectively. Let ρ: $W \rightarrow \Sigma_1$ be a minimal sequence of blowing-ups with initial centers t, p and r which makes the total transform of the divisor $M + C + L + T$ a simple normal crossing divisor. We exhibit the configuration of curves on W and its dual graph as below:

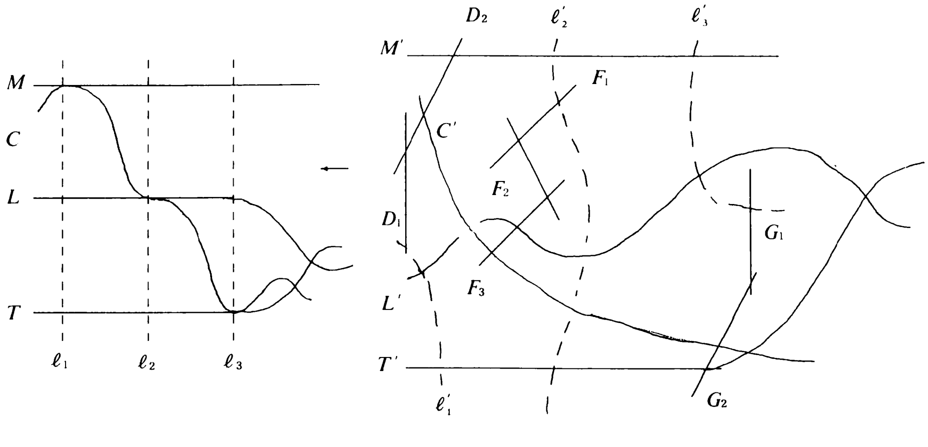

Figure 11

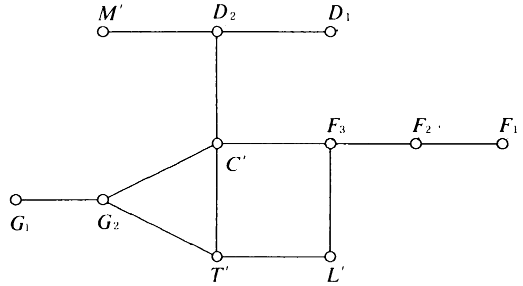

Figure 11 bis

Let $C' \cap F_3 = R_2$ and $T' \cap G_2 = R_3$. We perform oscillating sequences of blowing-ups $\sigma\colon V \to W$ with initial points R_2 and R_3. For a curve A on W, we denote by $\tilde{A}$ the proper transform of A by σ. Let E_i ($i = 2, 3$) be a unique (-1) curve contained in $\sigma^{-1}(R_i)$. Let T_{2a} be the connected component of $\sigma^{-1}(R_2) - E_2$ connecting with the fiber component $\tilde{F}_3$ and let T_{2b} be the connected component of $\sigma^{-1}(R_2) - E_2$ connecting with the section $\tilde{C}$. Similarly, let T_{3a} be the connected component of $\sigma^{-1}(R_3) - E_3$ connecting with the fiber component $\tilde{G}_2$ and let T_{3b} be the remaining connected component of $\sigma^{-1}(R_3) - E_3$ connecting with the section $\tilde{T}$. We write the total transforms of F_3, C', G_2 and T' as follows:

$$\sigma^*(F_3) \sim \tilde{F}_3 + a_2 E_2 + \cdots$$

$$\sigma^*(C') \sim \tilde{C} + b_2 E_2 + \cdots$$

$$\sigma^*(G_2) \sim \tilde{G}_2 + a_3 E_3 + \cdots$$

$$\sigma^*(T') \sim \tilde{T} + b_3 E_3 + \cdots .$$

We define the boundary divisor Δ on V by

$$\Delta = \tilde{M} + \tilde{C} + \tilde{T} + \tilde{L} + \sum_{i=1}^{2} (\tilde{G}_i + \tilde{D}_i) + \sum_{i=1}^{3} \tilde{F}_i + \sum_{i=2}^{3} (T_{ia} + T_{ib}) .$$

Then our homology plane is $X := V - \Delta$ provided the following condition is satisfied:

$$(6a_2 - b_2)(6a_3 + b_3) - 6a_2 a_3 = \pm 1 .$$

The total transforms by ρ of curves on Σ_1 are written as follows:

$$\rho^*(T) \sim T' + G_1 + 2G_2$$

$$\rho^*(L) \sim L' + F_1 + 2F_2 + 3F_3$$

$$\rho^*(M) \sim M' + D_1 + 2D_2$$

$$\rho^*(C) \sim C' + F_1 + 2F_2 + 3F_3 + G_1 + 2G_2 + D_1 + 2D_2$$

$$\rho^*(l_1) \sim l_1' + D_1 + D_2$$

$$\rho^*(l_2) \sim l_2' + F_1 + F_2 + F_3$$

$$\rho^*(l_3) \sim l_3' + G_1 + G_2 .$$

We can show that Supp Bk (Δ) consists of $\tilde{G}_1$, $\tilde{F}_1$, $\tilde{F}_2$, $\tilde{D}_1$, $\tilde{M}$, T_{2a}, T_{2b}, T_{3a} and T_{3b}. Let $\tau\colon V \to \bar{S}$ be the contraction of these curves except for $\tilde{M}$. Write $\bar{A} = \tau(A)$ for a curve A on V and let $\bar{\Delta} = \tau(\Delta)$. First, we calculate $(K_{\bar{S}} + \bar{\Delta})^2$ and then make a necessary modification due to the peeling of M.

The linear equivalence relations $l_1 \sim l_2 \sim l_3$, $L \sim T \sim M + l_1$ and $C \sim M + 3l_1$ on Σ_1 give the following relations on $\bar{S}$:

$$\bar{l}_1 + \bar{D}_2 \sim \bar{l}_2 + \bar{F}_3 + a_2\bar{E}_2 \sim \bar{l}_3 + \bar{G}_2 + a_3\bar{E}_3$$

$$\bar{T} + 2\bar{G}_2 + b_3\bar{E}_3 + 2a_3\bar{E}_3 \sim \bar{L} + 3\bar{F}_3 + 3a_2\bar{E}_2 \sim \bar{M} + \bar{l}_1 + 3\bar{D}_2$$

$$\bar{C} \sim \bar{M} + 3\bar{D}_2 - 3\bar{F}_3 - 2\bar{G}_2 + 3\bar{l}_1 - (3a_2 + b_2)\bar{E}_2 - 2a_3\bar{E}_3 .$$

Starting with the canonical divisor $K_{\Sigma_1} \sim -2M - 3l_1$ of Σ_1, we can write K_W and $K_{\bar{S}}$ as follows:

$$K_W \sim -2\rho^*(M) - 3\rho^*(l_1) + D_1 + 2D_2 + F_1 + 2F_2 + 3F_3 + G_1 + 2G_2$$

$$K_{\bar{S}} \sim -2(\bar{M} + 2\bar{D}_2) - 3(\bar{l}_1 + \bar{D}_2) + 2\bar{D}_2 + 3\bar{F}_3 + 3a_2\bar{E}_2 + 2\bar{G}_2 + 2a_3\bar{E}_3$$
$$+ (a_2 + b_2 - 1)\bar{E}_2 + (a_3 + b_3 - 1)\bar{E}_3 .$$

Since

$$\bar{\Delta} = \bar{M} + \bar{D}_2 + \bar{F}_3 + \bar{G}_2 + \bar{C} + \bar{L} + \bar{T},$$

we have

$$K_{\bar{S}} + \bar{\Delta} \sim 2\bar{M} + 5\bar{D}_2 + 2\bar{l}_1 - 2\bar{F}_3 - \bar{G}_2 - (2a_2 + 1)\bar{E}_2 - (a_3 + 1)\bar{E}_3 .$$

So, by computing the intersection numbers of curves on $\bar{S}$ as in the case 1 of the section 2, we obtain

$$(K_{\bar{S}} + \bar{\Delta})^2 = 2 - \frac{1}{3} - \frac{1}{a_2b_2} - \frac{1}{a_3b_3} .$$

We omit the details of the calculations. In order to obtain $c_1(X)^2$, we have to peel the bark of $\bar{M}$. Namely, determine the number α by the condition:

$$(K_{\bar{S}} + \bar{\Delta} + \alpha\bar{M} \cdot \bar{M}) = 0 ,$$

from which results $\alpha = -\frac{1}{3}$ because $\bar{M}^2 = -3$. We therefore have

$$\begin{aligned} c_1(X)^2 &= (K_{\bar{S}} + \bar{\Delta} - \tfrac{1}{3}\bar{M})^2 \\ &= (K_{\bar{S}} + \bar{\Delta})^2 - \tfrac{1}{3}(K_{\bar{S}} + \bar{\Delta} \cdot \bar{M}) \\ &= 2 - \frac{1}{a_2b_2} - \frac{1}{a_3b_3} . \end{aligned}$$

2. Case of four sections in Σ_2

Let $\pi: \Sigma_2 \to \mathbf{P}^1$ be the natural $\mathbf{P}^1$-fibration, let F be a general fiber and let M be the minimal section. We choose curves Q_1, Q_2 and L on Σ_2 such that

$$Q_1 \sim M + 2F , \qquad Q_2 \sim M + 3F , \qquad L \sim M + 2F$$

and that the intersection pattern is given as follows:

$$Q_2 \cdot M = x , \qquad Q_2 \cdot Q_1 = 3y , \qquad Q_1 \cdot L = 2v , \qquad Q_2 \cdot L = 2z + u$$

where x, y, z, u and v are five different points. Let l_1, l_2 and l_3 be the fibers of the $\mathbf{P}^1$-fibration passing through y, v and z, respectively. Let $\rho\colon W \to \Sigma_2$ be a minimal sequence of blowing-ups with initial centers y, v and z which makes the total transform of the divisor $M + Q_1 + Q_2 + L$ a simple normal crossing divisor. We exhibit the configuration of curves on W and its dual graph as below.

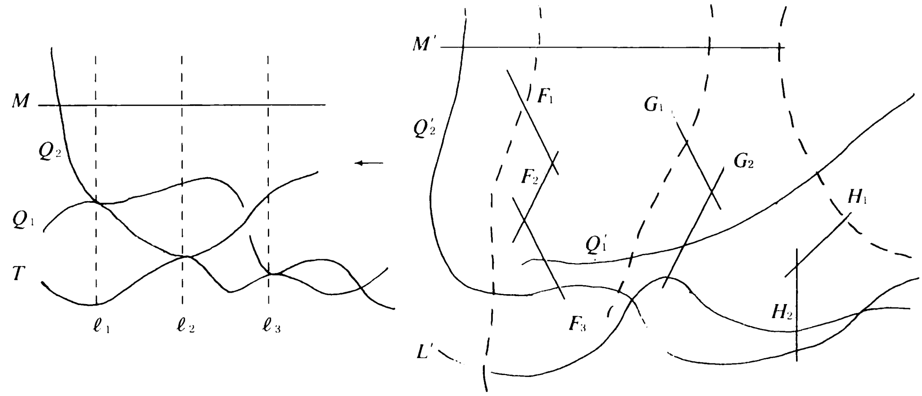

Figure 12

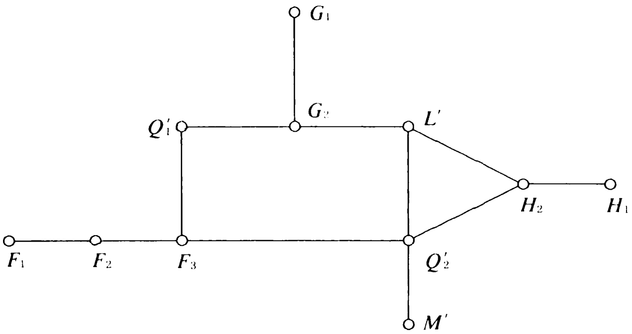

Figure 12 bis

Let $R_1 := Q'_1 \cap F_3$ and $R_3 := L' \cap H_2$. We perform oscillating sequences of blowing-ups $\sigma\colon V \to W$ with initial points R_1 and R_3. We make use of the same notations as before. For example, T_{2a} connects with $\tilde{F}_3$ and T_{2b} connects with the section $\tilde{Q}_1$. We write the total transforms of F_3, Q'_1, H_2 and L' as follows:

$$\sigma^*(F_3) \sim \tilde{F}_3 + a_1 E_1 + \cdots$$
$$\sigma^*(Q'_1) \sim \tilde{Q}_1 + b_1 E_1 + \cdots$$
$$\sigma^*(H_2) \sim \tilde{H}_2 + a_3 E_3 + \cdots$$
$$\sigma^*(L') \sim \tilde{L} + b_3 E_3 + \cdots$$

and define the boundary divisor Δ on V by

$$\Delta = \tilde{M} + \tilde{L} + \sum_{i=1}^{3} \tilde{F}_i + \sum_{i=1}^{2} (\tilde{Q}_i + \tilde{G}_i + \tilde{H}_i) + \sum_{i=2}^{3} (T_{ia} + T_{ib}) .$$

Then our homology plane is $X := V - \Delta$ provided the following condition is satisfied:

$$(3a_1 - b_1)(2a_3 - b_3) - 4b_1 b_3 = \pm 1 .$$

The total transforms by ρ of curves on Σ_2 are written as follows:

$$\rho^*(Q_1) \sim Q'_1 + F_1 + 2F_2 + 3F_3 + G_1 + 2G_2$$
$$\rho^*(L) \sim L' + G_1 + 2G_2 + H_1 + 2H_2$$
$$\rho^*(Q_2) \sim Q'_2 + F_1 + 2F_2 + 3F_3 + H_1 + 2H_2$$
$$\rho^*(l_1) \sim l'_1 + F_1 + F_2 + F_3$$
$$\rho^*(l_2) \sim l'_2 + G_1 + G_2$$
$$\rho^*(l_3) \sim l'_3 + H_1 + H_2$$

and Supp Bk (Δ) consists of $\tilde{F}_1$, $\tilde{F}_2$, $\tilde{G}_1$, $\tilde{H}_1$, $\tilde{M}$, $\tilde{Q}_1$, T_{2a}, T_{2b}, T_{3a} and T_{3b}. Let $\tau\colon V \to \bar{S}$ be the contraction of these curves except for $\tilde{M}$ and $\tilde{Q}_1$. We write $\bar{A} = \tau(A)$ for a curve A on V. Making use of the linear equivalence relations on Σ_2, we obtain

$$(K_{\bar{S}} + \bar{\Delta})^2 = 2 - \frac{1}{3} - \frac{1}{a_1 b_1} - \frac{1}{a_3 b_3} ,$$

where

$$\bar{\Delta} = \bar{M} + \bar{F}_3 + \bar{G}_2 + \bar{H}_2 + \bar{Q}_1 + \bar{Q}_2 + \bar{L}$$

and

$$K_{\bar{S}} + \bar{\Delta} \sim 2\bar{M} + 3\bar{l}_2 - 2\bar{F}_3 + 2\bar{G}_2 - \bar{H}_2 - (2a_1 + 1)\bar{E}_1 + (a_3 + 1)\bar{E}_3 .$$

Now we have to peel the barks of $\bar{M}$ and $\bar{Q}_1$ in order to calculate $c_1(X)^2$. Namely, determine the numbers α and β by conditions,

$$\begin{cases} (K_{\bar{S}} + \bar{\Delta} + \alpha\bar{M} + \beta\bar{Q}_1 \cdot \bar{M}) = 0 \\ (K_{\bar{S}} + \bar{\Delta} + \alpha\bar{M} + \beta\bar{Q}_1 \cdot \bar{Q}_1) = 0 \end{cases}$$

and compute $c_1(X)^2$ as follows:

$$\begin{aligned} c_1(X)^2 &= (K_{\bar{S}} + \bar{\Delta} + \alpha\bar{M} + \beta\bar{Q}_1)^2 \\ &= \frac{5}{2} - \left\{\frac{1}{a_1 b_1} + \frac{b_1}{3(3a_1 + b_1)} + \frac{1}{a_3 b_3}\right\} . \end{aligned}$$

We note that $c_1(X)^2 > 2$ for a suitable choice of the integers a_i, b_i.

3. Case of quartic and bitangent

Let Q be a quartic on $\mathbf{P}^2$ with three cusps $\{x, y, z\}$ and let T be a bitangent of Q.

First we blow-up $\mathbf{P}^2$ at the center u which is one of the intersection points of T and Q to obtain Σ_1. The exceptional curve M gives a unique minimal section of the natural $\mathbf{P}^1$-fibration on Σ_1. Let l_1, l_2 and l_3 be the fibers of the $\mathbf{P}^1$-fibration π passing through x, y and z, respectively. Let $\rho: W \to \Sigma_1$ be the minimal sequence of blowing-ups with initial centers x, y and z which makes the total transform of the divisor $M + T + Q$ a simple normal crossing divisor. We exhibit the configuration of curves on W and its dual graph as below.

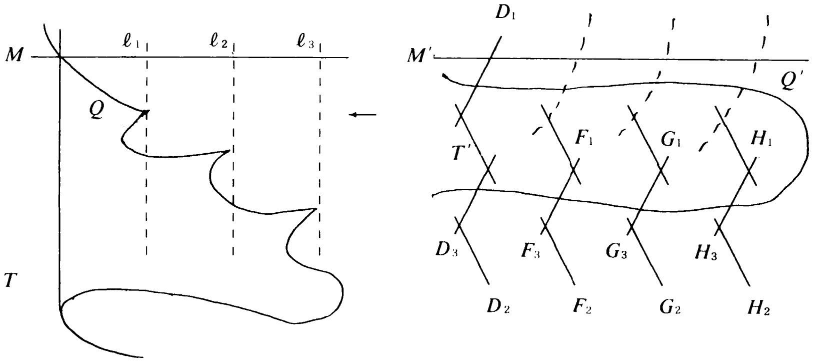

Figure 13

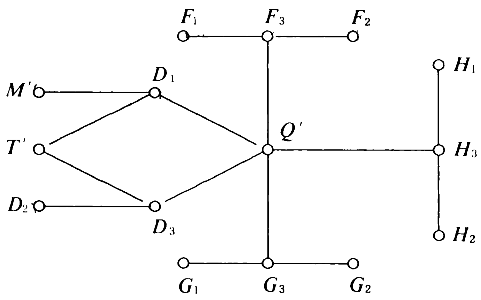

Figure 13 bis

Let $R := Q' \cap D_1$. We perform an oscillating sequence of blowing-ups $\sigma: V \to W$ with the initial point R_1. We make use of the same notations as before. We write the total transforms of D_1 and Q' as follows:

$$\sigma^*(D_1) \sim \tilde{D}_1 + aE + \cdots$$

$$\sigma^*(Q') \sim \tilde{Q} + bE + \cdots$$

and define the boundary divisor $\varDelta$ on V by

$$\varDelta = \tilde{M} + \tilde{Q} + \tilde{T} + \sum_{i=1}^{3} (\tilde{D}_i + \tilde{F}_i + \tilde{G}_i + \tilde{H}_i) + T_a + T_b .$$

Then our homology plane is $X := V - \varDelta$ when the following condition is satisfied:

$$b - 6a = \pm 1 .$$

The total transforms by ρ of the curves on Σ_1 are written as follows:

$$\begin{aligned} \rho^*(Q) &\sim Q' + D_1 + D_2 + 2D_3 + 2F_1 + 3F_2 + 6F_3 \\ &\qquad + 2G_1 + 3G_2 + 6G_3 + 2H_1 + 3H_2 + 6H_3 \\ \rho^*(T) &\sim T' + D_1 + D_2 + 2D_3 \\ \rho^*(M) &\sim M' + D_1 \\ \rho^*(l_1) &\sim l_1' + F_1 + F_2 + F_3 \end{aligned}$$

and $\operatorname{Supp} \operatorname{Bk}(D)$ consists of $\tilde{F}_1$, $\tilde{F}_2$, $\tilde{G}_1$, $\tilde{G}_2$, $\tilde{H}_1$, $\tilde{H}_2$, $\tilde{D}_2$, $\tilde{M}$, T_a and T_b. Let $\tau: V \to \bar{S}$ be the contraction of these curves except for $\tilde{M}$. Making use of the linear equivalence relations on Σ_1 and $\bar{S}$, we obtain

$$(K_S + \bar{\varDelta})^2 = 2 - \frac{1}{ab} ,$$

where

$$\bar{\varDelta} = \bar{M} + \bar{Q} + \bar{T} + \bar{D}_1 + \bar{D}_3 + \bar{F}_3 + \bar{G}_3 + \bar{H}_3 ,$$

and

$$K_{\bar{S}} + \bar{\varDelta} \sim 2\bar{M} + 2\bar{T} + 3\bar{D}_1 + 3\bar{D}_3 - 3\bar{F}_3 - \bar{G}_3 - \bar{H}_3 + (3a - 1)\bar{E} .$$

After peeling the bark of $\bar{M}$, finally we obtain

$$c_1(X)^2 = \left(K_S + \bar{\varDelta} - \frac{1}{2}\bar{M}\right)^2 = \frac{5}{2} - \frac{1}{ab} .$$

In this case the solutions of the equation $b - 6a = \pm 1$ are given by $a = n$ and $b = 6n \pm 1$, where n is a natural number. Therefore there exists a sequence of homology planes for which $c_1(X)^2$ converge to $\frac{5}{2}$.

4. Other cases

By similar arguments as above, we can obtain homology planes X with the chern numbers $c_1(X)^2$ as given below.

4.1. Four sections in Σ_2. We start with the configulation of curves on Σ_2 such that

$$C \sim M + 3F , \qquad L_1 \sim L_2 \sim M + 2F ,$$

where M is a minimal section and F is a general fiber, and that

$$C \cdot M = x\,, \qquad C \cdot L_1 = 3y\,, \qquad C \cdot L_2 = 3z\,, \qquad L_1 \cdot L_2 = u + v\,,$$

where x, y, z, u and v are five different points. We obtain a homology plane X with

$$c_1(X)^2 = 2 - \frac{1}{3} - \frac{1}{a_1 b_1} - \frac{1}{a_2 b_2}$$

provided that a_i and b_i satisfy the following equality:

$$(b_2 - a_2)(2b_1 - a_1) - 9a_1 a_2 = \pm 1\,.$$

4.2 Cubic, quadric, line. Starting with a cuspidal cubic C, a regular quadric Q and a line L on $\mathbf{P}^2$ such that

$$C \cdot L = 3z\,, \qquad C \cdot Q = 3x + 2z + y\,, \qquad Q \cdot L = 2z\,,$$

we obtain a homology plane X with

$$c_1(X)^2 = \frac{5}{2} - \frac{1}{a_1 b_1} - \frac{1}{a_2 b_2}$$

provided a_i and b_i satisfy the following equality:

$$(2a_1 - b_1)(3a_2 + b_2) - 12a_1 a_2 = \pm 1\,.$$

4.3. 2-Section and two sections in Σ_2. Starting with the configulation of curves on Σ_2 such that

$$C \sim 2M + 5F\,, \qquad T \sim M + 2F$$

and that

$$C \cdot M = x\,, \qquad C \cdot T = 4y + z\,,$$

where M is the minimal section and F is a general fiber, we obtain a homology plane X with

$$c_1(X)^2 = 2 - \frac{1}{12} - \frac{1}{ab}$$

provided that a and b satisfy $2a - 5b = \pm 1$.

4.4. A quintic with cuspidal tangent. We start with a quintic Q on $\mathbf{P}^2$ with three cusps $\{x, y, z\}$ whose multiplicity sequences are (2, 2) and the tangent line T to Q at x. We obtain a homology plane X with

$$c_1(X)^2 = \frac{5}{2} - \frac{1}{ab}\,.$$

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