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Plane wave decomposition of odd-dimensional Brownian local times

By

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1. Introduction

The existence of multi-dimensional Brownian local times as generalized Wiener functionals has been shown by Imkeller and Weisz [5]. They have given adequate meaning to the following formal representation:

$$L(t,x)=\int_0^t \delta_x(B_s)ds, \tag{1.1}$$

where $L(t,x)$ denotes the local time for r-dimensional Brownian motion $\{B_t\}$ and δ_x the Dirac delta function at $x\in\mathbf{R}^r$.

On the other hand, it is well-known that δ_0 is decomposed as follows:

$$\delta_0(x)=\frac{1}{(-4\pi)^{(r-1)/2}}\frac{\Gamma(1/2)}{\Gamma(r/2)}\int_{\mathbf{S}^{r-1}}\delta_0^{(r-1)}(\langle x,\omega\rangle)\sigma(d\omega), \tag{1.2}$$

where δ_0 in the left hand side is the r-dimensional Dirac delta function at 0 and that in the right hand side 1-dimensional one, $\delta_0^{(r-1)}$ denoting $(r-1)$st derivative of δ_0. Moreover, $\mathbf{S}^{r-1}$ denotes the unit sphere in $\mathbf{R}^r$, $\sigma(d\omega)$ the uniform measure on $\mathbf{S}^{r-1}$ with total measure 1 and $\langle\star,*\rangle$ the Euclidean inner product on $\mathbf{R}^r$. This formula is called the plane wave decomposition of the δ function. Since this decomposition (1.2) is valid only in the case where r is odd, we restrict our investigation to the case where r is odd.

The purpose of this paper is to represent the r-dimensional Brownian local time by means of 1-dimensional ones; roughly speaking, (1.1) and (1.2) imply

$$L(t,x)=\frac{1}{(-4\pi)^{(r-1)/2}}\frac{\Gamma(1/2)}{\Gamma(r/2)}\int_{\mathbf{S}^{r-1}}L^{(r-1)}(t,x,\omega)\sigma(d\omega), \tag{1.3}$$

where

$$L^{(r-1)}(t,x,\omega)=\frac{d^{r-1}}{d\xi^{r-1}}L^{\omega}(t,\xi)\bigg|_{\xi=\langle x,\omega\rangle} \tag{1.4}$$

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and $L^{\omega}(t,\xi)$ denotes the local time for the 1-dimensional Brownian motion $\{\langle\omega, B_t\rangle\}$. We exactly establish the above equality (1.3) in the sense of generalized Wiener functionals.

Bass [1] has shown that every odd-dimensional Brownian additive functional associated with the measure which has the density function is represented by means of Brownian local times at hyperplanes under some conditions. This representation can also be obtained by the same argument which derives (1.3). It should also be noticed that Yamada [8] has obtained representations of a considerably wide class of continuous Brownian additive functionals of zero energy via Brownian local times at hyperplanes in the sense of distributions.

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2. Preliminaries and main theorem

We first introduce the multi-dimensional Brownian local time as a generalized Wiener functional due to Imkeller and Weisz [5]. We begin with preparing some notations.

Let (W_0^r, P) be the r-dimensional standard Wiener space: $W_0^r = \{B_t = (B_t^1, B_t^2, \dots, B_t^r) : [0,\infty) \to \mathbf{R}^r; B_t$ is continuous and $B_0 = 0\}$ and P is the standard Wiener measure. Let $I_n(f_n)$ be the n-ple Wiener-Itô integal with the kernel function f_n:

$$\begin{cases} f_n = (f_n(t_1, t_2, \dots, t_n)^{j_1, j_2, \dots, j_n})_{j_1, j_2, \dots, j_n = 1, 2, \dots, r} \\ I_n(f_n) = \displaystyle\sum_{j_1, j_2, \dots, j_n = 1, 2, \dots, r} \int_0^\infty \cdots \int_0^\infty f_n(t_1, t_2, \dots, t_n)^{j_1, j_2, \dots, j_n} dB_{t_1}^{j_1} \cdots dB_{t_n}^{j_n}, \end{cases} \tag{2.1}$$

where f_n belongs to $L^2([0,\infty)^n \to \mathbf{R}^{r^n})$, and is symmetric in the variables (j_1, t_1), $(j_2, t_2), \dots, (j_n, t_n)$ (see, for instance, Nualart [6]). We denote the totality of such functions by $L^2_{sym}([0,\infty)^n \to \mathbf{R}^{r^n})$. When $n = 0$, $I_0(f_0)$ represents a constant. Now we define some classes of (generalized) Wiener functionals $\mathbf{D}^{ser}$ and $\mathbf{D}_2^s$ as follows:

Definition 2.1. Let $s \in \mathbf{R}$. We set

$$\mathbf{D}^{ser} = \{\mathbf{I}(\mathbf{f}) = (I_0(f_0), I_1(f_1), \dots, I_n(f_n), \dots) : f_n \in L^2_{sym}([0,\infty)^n \to \mathbf{R}^{r^n}), \\ n = 1, 2, \dots\}$$

and

$$\mathbf{D}_2^s = \left\{\mathbf{I}(\mathbf{f}) \in \mathbf{D}^{ser} : \|\mathbf{I}(\mathbf{f})\|_{2,s}^2 \equiv \sum_{n=0}^{\infty} (1+n)^s n! \|f_n\|^2 < \infty\right\},$$

where $\|f\|$ denotes the L^2-norm of f.

Remark 2.1. Taking the Wiener-Itô decompsition into consideration, $\mathbf{D}_2^0$ can be identified with $L^2(P)$. Under this identification, $\mathbf{D}_2^s$ above coinsides with $\mathbf{D}_{2,s}$ in Ikeda and Watanabe [3] or $\mathbf{D}^{s,2}$ in Nualart [6].

We also introduce other classes of (generalized) Wiener functionals. Let $\gamma \in \mathbf{R}$. We set $\|f\|_{(\gamma)}$ by

$$\|f\|_{(\gamma)}^2 = \int \cdots \int_{[0,\infty)^n} ((s_1 \vee \cdots \vee s_n) \wedge 1)^{-\gamma} |f(s_1, \dots, s_n)|^2 ds_1 \dots ds_n$$

for a function $f \in L^2([0,\infty)^n; ((s_1 \vee \cdots \vee s_n) \wedge 1)^{-\gamma} ds_1 \cdots ds_n)$, where $x \vee y$ and $x \wedge y$ denote the maximum and the minimum of x and y, respectively.

Definition 2.2. Let $\gamma \in \mathbf{R}$, $c > 0$ and $\rho \in \mathbf{R}$. We set

$$\mathscr{D}_\gamma^{ser} = \{\mathbf{I(f)} = (I_0(f_0), I_1(f_1), \dots, I_n(f_n), \dots) : f_n \in L^2_{sym}([0,\infty)^n \to \mathbf{R}^{r^n}; \\ ((s_1 \vee \cdots \vee s_n) \wedge 1)^{-\gamma} ds_1 \cdots ds_n), n = 1, 2, \dots\}$$

and

$$\mathscr{D}_\gamma^{(c,\rho)} = \left\{ \mathbf{I(f)} \in \mathscr{D}_\gamma^{ser}; \|\mathbf{I(f)}\|_{(\gamma,c,\rho)}^2 \equiv \sum_{n=0}^\infty c^n (1+n)^\rho n! \|f_n\|_{(\gamma)}^2 < \infty \right\}.$$

Remark 2.2. (i) In the case where $\gamma < 0$, $I_n(f_n)$ in the definition of $\mathscr{D}_\gamma^{ser}$ is considered as a generalized Wiener functional satisfying $\langle I_n(f_n), I_m(g_m)\rangle_W = \delta_{n,m} n! \langle f_n, g_n\rangle_2$ for any $g_m \in L^2_{sym}([0,\infty)^m \to \mathbf{R}^{r^m}; ((s_1 \vee \cdots \vee s_m) \wedge 1)^\gamma ds_1 \cdots ds_m)$, where $\langle *, \star\rangle_W$ denotes the pairing of Wiener functionals and generalized ones, $\langle *, \star\rangle_2$ the $L^2([0,\infty)^n \to \mathbf{R}^{r^n}; ds_1 \cdots ds_n)$-inner product and $\delta_{n,m}$ Kronecker's δ.

(ii) It is a matter of course that $\mathscr{D}_\gamma^{(c,\rho)} \subset L^2(P)$ holds when $\gamma > 0$ and $c > 1$. Moreover, for $\gamma < 0$ and $c < 1$, $\mathscr{D}_\gamma^{(c,\rho)}$ can be identified with the dual space of $\mathscr{D}_{-\gamma}^{(1/c,-\rho)}$.

We introduce the multi-dimensional Brownian local time given by Imkeller and Weisz [5].

Lemma 2.1 ([5]). *Let $x(\neq 0) \in \mathbf{R}^r$ and $t > 0$ be given. Then there exists $L(t,x) \in \mathbf{D}_2^\alpha$ such that*

$$\int_0^t p_r(\varepsilon, B_s - x) ds \to L(t,x) \qquad \text{as } \varepsilon \to 0 \qquad \text{in } \mathbf{D}_2^\alpha$$

for all $\alpha < 1 - r/2$, where $p_r(s,x)$ denotes the r-dimensional Gaussian kernel:

$$p_r(s,x) = \frac{1}{(\sqrt{2\pi s})^r} e^{-|x|^2/2s}.$$

We call $L(t,x)$ above *the r-dimensional Brownian local time.*

Remark 2.3. Imkeller and Weisz [5] have proved the above theorem also for the multi-parameter Wiener process.

Remark 2.4. In the above theorem, $p_r(\varepsilon, \cdot - x)$ is used for a test function converging to δ_x. This can be replaced by $\varphi((\cdot - x)/\varepsilon)/\varepsilon^r$ where $\varphi \in C^\infty$ and is of compact support.

We next introduce the plane wave decomposition. For our purpose, the inversion formula for the Radon transform is rather useful, which is equivalent to the plane wave decomposition. Therefore we begin with an explanation of the Radon transform together with notations (see, for instance, Helgason [2]).

Let f be a function on $\mathbf{R}^r$, which is integrable on each hyperplane in $\mathbf{R}^r$. Let $\omega \in \mathbf{S}^{r-1}$ and $\xi \in \mathbf{R}$, where $\mathbf{S}^{r-1}$ denotes the unit sphere in $\mathbf{R}^r$. The Radon transform $\hat{R}[f]$ of f is defined by

$$\hat{R}[f](\omega, \xi) = \int_{\langle x, \omega \rangle = \xi} f(x)dx,$$

where $\langle \star, * \rangle$ denotes the Euclidean inner product on $\mathbf{R}^r$ and dx the Lebesgue measure on the hyperplane $\{x; \langle x, \omega \rangle = \xi\}$. The dual Radon transform is also defined as follows. Let φ be a locally integrable function on $\mathbf{S}^{r-1} \times \mathbf{R}$ such that $\varphi(\omega, \xi) = \varphi(-\omega, -\xi)$. The dual Radon transform $\check{R}[\varphi]$ is

$$\check{R}[\varphi](x) = \int_{\mathbf{S}^{r-1}} \varphi(\omega, \langle \omega, x \rangle)\sigma(d\omega),$$

$\sigma(d\omega)$ denoting the uniform measure on $\mathbf{S}^{r-1}$ with total measure 1.

We now introduce the inversion formula for the Radon transform. Suppose that r is odd. Let f be a Schwartz rapidly decreasing function on $\mathbf{R}^r$. Then we have the following inversion formula:

$$\begin{aligned} f &= \frac{1}{(-4\pi)^{(r-1)/2}} \frac{\Gamma(1/2)}{\Gamma(r/2)} \check{R}\left[\left(\frac{d^{r-1}}{d\xi^{r-1}} \hat{R}[f]\right)\right] \\ &= \frac{1}{(-4\pi)^{(r-1)/2}} \frac{\Gamma(1/2)}{\Gamma(r/2)} \int_{\mathbf{S}^{r-1}} \left(\frac{d^{r-1}}{d\xi^{r-1}} \int_{\langle y, \omega \rangle = \xi} f(y)dy\right)\Bigg|_{\xi = \langle x, \omega \rangle} \sigma(d\omega). \end{aligned} \tag{2.2}$$

We are now at the position to give our main theorem. For this purpose, we prepare the following two propositions, which ensure the differentiability of $L(t, x)$ and the integrability of $\tilde{L}^{(r-1)}(t, x, \omega)$ respectively, where

$$\tilde{L}^{(r-1)}(t, x, \omega) = L^{(r-1)}(t, x, \omega) - E[L^{(r-1)}(t, x, \omega)]$$

and $L^{(r-1)}(t, x, \omega)$ is defined in (1.4).

Proposition 2.1. *Suppose* $r \in \mathbf{N}$ *and* $x \neq 0$. *Then* $L(t, x)$ *is* k *times differentiable in* $\mathbf{D}_2^{\beta_1}$ *with respect to* x, *where* $\beta_1 < -r/2 - k$. *Moreover the kth derivative belongs to* $\mathbf{D}_2^{\beta_2}$ $(\beta_2 < 1 - r/2 - k)$.

Proposition 2.2. *Let $r \geq 2$ be any positive integer. Let $\gamma_1 < 2 - r$ and $\rho_1 < 1/2 - r$. Then $\tilde{L}^{(r-1)}(t, x, \omega)$ is $\sigma(d\omega)$-integrable in $\mathscr{D}_{\gamma_1}^{(1/r, \rho_1)}$.*

Our main theorem is as follows:

Theorem 2.1. *Suppose that r is odd. Let $L(t, x) \in \mathbf{D}_2^\alpha$ $(t > 0,\ x\ (\neq 0) \in \mathbf{R}^r,\ \alpha < 1 - r/2)$ be the r-dimensional Brownian local time. Then the following equality holds in $\mathbf{D}_2^\alpha$:*

$$L(t, x) = \frac{1}{(-4\pi)^{(r-1)/2}} \frac{\Gamma(1/2)}{\Gamma(r/2)} \int_{\mathbf{S}^{r-1}} L^{(r-1)}(t, x, \omega)\sigma(d\omega). \tag{2.3}$$

3. Proofs

In this section, we prove Propositions 2.1, 2.2 and Theorem 2.1.

To begin with, we state another representation of multiple Wiener-Itô integrals. Let $I_n(f_n)$ be the n-ple Wiener-Itô integral with kernel function f_n as in (2.1). Then, summing up again by every component of Brownian motion, we can easily get another representation for $I_n(f_n)$:

$$I_n(f_n) = \sum_{n_1 + n_2 + \cdots + n_r = n} I_{n_r}^r \cdots I_{n_1}^1 (f_{n_1, n_2, \ldots, n_r}), \tag{3.1}$$

where I_m^j denotes the m-ple Wiener-Itô integral with respect only to B_t^j. More precisely, $f_{n_1, n_2, \ldots, n_r} = f_{n_1, n_2, \ldots, n_r}(t_1^{(1)}, \ldots, t_{n_1}^{(1)}; \ldots; t_1^{(r)}, \ldots, t_{n_r}^{(r)})$ is determined by

$$f_{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!} f_n^{j_1, j_2, \ldots, j_n}$$

when $^\#\{k; j_k = i\} = n_i$ $(i = 1, 2, \ldots, r)$ and $n_1 + n_2 + \cdots + n_r = n$. Thus $f_{n_1, n_2, \ldots, n_r}$ belongs to $L^2([0, \infty)^n \to \mathbf{R})$, which is symmetric with respect to $t_1^{(j)}, \ldots, t_{n_j}^{(j)}$ for all fixed j $(j = 1, \ldots, r)$. When $f_n \in L^2([0, \infty)^n; ((s_1 \vee \cdots \vee s_n) \wedge 1)^{-\gamma} ds_1 \cdots ds_n)$ $(\gamma < 0)$, we can also get the same representation as (3.1) for $I_n(f_n)$; indeed, from Remark 2.2 (i), we can easily show that $I_{n_r}^r \cdots I_{n_1}^1 (f_{n_1, n_2, \ldots, n_r})$ can be considered as a generalized Wiener functional and that the equality (3.1) holds in the sense of generalized Wiener functionals. Noting that the following equality holds for any $\gamma \in \mathbf{R}$:

$$\|f_n\|_{(\gamma)}^2 = \sum_{n_1 + n_2 + \cdots + n_r = n} \frac{n_1! n_2! \cdots n_r!}{n!} \|f_{n_1, n_2, \ldots, n_r}\|_{(\gamma)}^2.$$

When we express the r-dimensional Brownian local time $L(t, x)$ by the fashion above, i.e. in the form

$$L(t, x) = \sum_{n=0}^{\infty} \sum_{n_1 + n_2 + \cdots + n_r = n} I_{n_r}^r \cdots I_{n_1}^1 (g_{n_1, n_2, \ldots, n_r}), \tag{3.2}$$

each $g_{n_1, n_2, \ldots, n_r}$ is exactly obtained as follows:

$$g_{n_1,n_2,\dots,n_r}(x;s_1,\dots,s_n)$$
$$= \frac{1}{n_1!n_2!\cdots n_r!}\int_{s_1\vee\cdots\vee s_n}^{t}\left(\frac{1}{\sqrt{s}}\right)^n \prod_{j=1}^{r} H_{n_j}\left(\frac{x_j}{\sqrt{s}}\right)p_r(s,x)ds$$
$$\times \mathbf{1}_{[0,t]}(s_1)\cdots\mathbf{1}_{[0,t]}(s_n), \tag{3.3}$$

where $x=(x_1,x_2,\dots,x_r)$ and $H_n(x)$ denotes the Hermite polynomial:

$$H_n(x) = (-1)^n e^{x^2/2}\left(\frac{d^n}{dx^n}e^{-x^2/2}\right) \qquad (n=0,1,2,\dots).$$

Remark 3.1. Although our representation (3.3) of $g_{n_1,n_2,\dots,n_r}$ above seems to be different from that in Imkeller and Weisz [5], it is completely the same. We can easily obtain (3.3) by considering the H-derivatives of sufficient orders of $\int_0^t \varphi(B_s)ds$, φ being a Schwartz rapidly decreasing function.

For the proof of Proposition 2.1, we introduce the following evaluation of Hermite polynomials, which has been given in Imkeller, Perez-Abreu and Vives [4]. Refer also to Szegö [7].

Lemma 3.1 ([4], [7]). *Let $\delta \in [1/4, 1/2]$. Then there exists a constant C independent of δ such that*

$$\sup_x |H_n(x)e^{-\delta x^2}| \le C\sqrt{n!}n^{-(8\delta-1)/12}. \tag{3.4}$$

Proof of Proposition 2.1. We prove only the case where $r=1$, as multidimensional cases can be done by the same way. First we observe the case where $k=1$. Set

$$L^{(1)}(t,x) = \sum_{n=0}^{\infty} I_n(g_n'(x)),$$

where $g_n'(x)$ is a derivative of $g_n(x)$ with respect to x, $g_n(x)$ being as in (3.3). Then the equality

$$\frac{L(t,x+h)-L(t,x)}{h} - L^{(1)}(t,x) = \frac{1}{h}\sum I_n\left(\int_x^{x+h}\int_x^y g_n''(z)dzdy\right)$$

holds. Therefore, to show that $L(t,x)$ is differentiable in $\mathbf{D}_2^{\beta_1}$ with respect to x, it is enough to verify

$$\sup_{z\in U(x)} \sum (1+n)^{\beta_1} n! \|g_n''(z)\|_2^2 < \infty$$

for some neighborhood $U(x)$ of x such that 0 is not included in its closure. In the same way, it is sufficient to verify

$$\sup_{z\in U(x)} \sum (1+n)^{\beta_1} n! \|g_n^{(\ell)}(z)\|_2^2 < \infty \qquad (\ell = 2,\dots,k+1) \tag{3.5}$$

for the proof of the former part of Proposition 2.1. Now $g_n^{(\ell)}(z)$ is exactly obtained as follows:

$$g_n^{(\ell)}(z) = \frac{(-1)^\ell}{n!}\int_{s_1 \vee \cdots \vee s_n}^t \left(\frac{1}{\sqrt{s}}\right)^{n+\ell} H_{n+\ell}\left(\frac{z}{\sqrt{s}}\right) p_1(s,z) ds\, \mathbf{1}_{[0,t]}(s_1)\cdots \mathbf{1}_{[0,t]}(s_n).$$

Thus it holds that

$$\|g_n^{(\ell)}(z)\|_2^2$$
$$= \frac{2}{(n!)^2}\int_0^t ds \int_0^s du \left(\frac{u}{s}\right)^{n/2}\left(\frac{1}{\sqrt{su}}\right)^\ell H_{n+\ell}\left(\frac{z}{\sqrt{s}}\right) H_{n+\ell}\left(\frac{z}{\sqrt{u}}\right) p_1(s,z) p_1(u,z). \tag{3.6}$$

Appealing to (3.4), for any $\delta_1 \in [1/2, 1)$ there exists a constant C_1 independent of δ_1 such that

$$\sup_z \left| H_{n+\ell}\left(\frac{z}{\sqrt{s}}\right) e^{-\delta_1 z^2/2s} H_{n+\ell}\left(\frac{z}{\sqrt{u}}\right) e^{-\delta_1 z^2/2u}\right| \le C_1 (n+\ell)!(n+\ell)^{-(4\delta_1-1)/6}.$$

Since 0 is not included in the closure of $U(x)$, there exists a constant M such that

$$\sup_{0\le s,u\le t}\ \sup_{z\in U(x)} \left|\left(\frac{1}{\sqrt{s}}\right)^{\ell+1} \mathrm{e}^{-(1-\delta_1)z^2/2s}\left(\frac{1}{\sqrt{u}}\right)^{\ell+1}\mathrm{e}^{-(1-\delta_1)z^2/2u}\right| \le M.$$

Substituting these inequalities into (3.6), we can easily show that there exists a constant C_2 such that

$$\|g_n^{(\ell)}(z)\|_2^2 \le C_2 \frac{1}{(n!)^2}(n+\ell)!(n+\ell)^{-(4\delta_1-1)/6}\frac{1}{n+2}.$$

Thus (3.5) holds if $\beta_1 < (4\delta_1 - 1)/6 - \ell$. Let $\delta_1 \to 1$ we have the former part of the claim.

In the proof above we also found that the k th derivative of $L(t,x)$ is equal to $\sum I_n(g_n^{(k)}(x))$, and it belongs to $\mathbf{D}_2^{\beta_1+1}$. Therefore the proof is completed. □

We next prove Proposition 2.2.

Proof of Proposition 2.2. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_r) \in \mathbf{S}^{r-1}$. Then we can expand $L^{(r-1)}(t,x,\omega)$ as a generalized functional of $\{B_s\}$ as follows:

$$L^{(r-1)}(t,x,\omega) = \sum_{n=0}^\infty \sum_{n_1+\cdots+n_r=n} I_{n_1,\ldots,n_r}(\tilde{g}_{n_1,\ldots,n_r}(\omega,x))$$

where

$$\tilde{g}_{n_1,\ldots,n_r}(\omega, x; s_1, \ldots, s_n)$$
$$= \frac{\omega_1^{n_1}\cdots\omega_r^{n_r}}{n_1!\cdots n_r!}\int_{s_1\vee\cdots\vee s_n}^t \left(\frac{1}{\sqrt{s}}\right)^{n+r-1} H_{n+r-1}\left(\frac{\langle x,\omega\rangle}{\sqrt{s}}\right) p_1(s,\langle x,\omega\rangle) ds$$
$$\mathbf{1}_{[0,t]}(s_1)\cdots\mathbf{1}_{[0,t]}(s_n). \tag{3.7}$$

Appealing to (3.4) for $\delta = 1/2$, we have

$$\|\tilde{g}_{n_1,\dots,n_r}(\omega, x)\|^2_{(\gamma_1)}$$

$$\leq \left(\frac{1}{n_1!\cdots n_r!}\right)^2 \int\cdots\int_{[0,t]^n} ((s_1 \vee \cdots \vee s_n) \wedge 1)^{-\gamma_1}$$

$$\times \left(\int_{s_1 \vee \cdots \vee s_n}^t \left(\frac{1}{\sqrt{s}}\right)^{n+r-1} H_{n+r-1}\left(\frac{\langle x, \omega\rangle}{\sqrt{s}}\right) p_1(s, \langle x, \omega\rangle) ds\right)^2 ds_1 \cdots ds_n$$

$$\leq C' \left(\frac{1}{n_1!\cdots n_r!}\right)^2 \frac{(n+r-1)!}{\sqrt{n+r-1}} \frac{n}{n-\gamma_1} \int_0^t s^{-\gamma_1 - r+1} ds \int_0^1 v^{(n-r-2\gamma_1)/2} dv$$

for some constant C', and moreover the last two integrals converge since $\gamma_1 < 2 - r$. Therefore

$$\|\tilde{L}^{(r-1)}(t, x, \omega)\|^2_{(\gamma_1, 1/r, \rho_1+1)}$$

$$\leq C_1' \sum_{n=1}^{\infty} \frac{1}{r^n}(1+n)^{\rho_1+1} \sum_{n_1+\cdots+n_r=n} \frac{1}{n_1!\cdots n_r!} \frac{(n+r-1)!}{\sqrt{n+r-1}} \frac{n}{n-\gamma_1} \frac{2}{n-r-2\gamma_1+2}$$

$$\leq C_2' \sum_{n=1}^{\infty} (1+n)^{\rho_1+1} n^{r-5/2} < \infty,$$

C_1' and C_2' being some constants. That is, $\tilde{L}^{(r-1)}(t, x, \omega) \in \mathscr{D}_{\gamma_1}^{(1/r, \rho_1+1)}$ for all $\omega \in \mathbf{S}^{r-1}$. Noting that

$$\left(\frac{1}{\sqrt{s}}\right)^k H_k\left(\frac{\xi}{\sqrt{s}}\right) p_1(s, \xi) - \left(\frac{1}{\sqrt{s}}\right)^k H_k\left(\frac{\tilde{\xi}}{\sqrt{s}}\right) p_1(s, \tilde{\xi})$$

$$= \int_{\tilde{\xi}}^{\xi} \left(\frac{1}{\sqrt{s}}\right)^{k+1} H_{k+1}\left(\frac{z}{\sqrt{s}}\right) p_1(s, z) dz,$$

we can easily show that $\tilde{L}^{(r-1)}(t, x, \omega)$ is continuous in $\mathscr{D}_{\gamma_1}^{(1/r, \rho_1)}$ with respect to ω by the same calculation as above. Since $\mathbf{S}^{r-1}$ is compact, this yields $\sigma(d\omega)$-integrability of $\tilde{L}^{(r-1)}(t, x, \omega)$ in $\mathscr{D}_{\gamma_1}^{(1/r, \rho_1)}$, which completes the proof. $\square$

Remark 3.2. Since the Riemann sum of $\tilde{g}_{n_1,\dots,n_r}(\omega, x)$ converges to $\int_{\mathbf{S}^{r-1}} \tilde{g}_{n_1,\dots,n_r}(\omega, x)\sigma(d\omega)$ in $L^2(((s_1 \vee \cdots \vee s_n) \wedge 1)^{-\gamma_1} ds_1 \cdots ds_n)$, we easily obtain that

$$\int_{\mathbf{S}^{r-1}} \tilde{L}^{(r-1)}(t, x, \omega)\sigma(d\omega) = \sum_{n=1}^{\infty} \sum_{n_1+\cdots+n_r=n} I_{n_1,\dots,n_r}\left(\int_{\mathbf{S}^{r-1}} \tilde{g}_{n_1,\dots,n_r}(\omega, x)\sigma(d\omega)\right).$$

Remark 3.3. Since $\int_0^t p_1(s, \xi) ds \in L^2(d\xi)$, we know that $\partial_\xi^{r-1} \int_0^t p_1(s, \xi) ds \in \mathscr{D}'$, where $\partial_\xi = \partial/\partial\xi$ and $\mathscr{D}'$ denotes the dual space of $\mathscr{D}$, the space of C^∞

functions of compact support. Therefore $\int_{\mathbf{S}^{r-1}} E[L^{(r-1)}(t,x,\omega)]\sigma(d\omega)$, which equals to the inverse Radon transform of $\partial_\xi^{r-1}\int_0^t p_1(s,\xi)ds$, exists in $\mathscr{D}'$.

We now prove the main theorem. We notice the following lemma concerned to Hermite polynomials.

Lemma 3.2. *Let $\boldsymbol{\omega}$ be an orthogonal matrix, whose first column is ${}^t\omega = {}^t(\omega_1,\dots,\omega_r)$. Let $n = n_1 + \cdots + n_r$. Then it holds that*

$$\int\cdots\int_{\mathbf{R}^{r-1}}\left(\frac{1}{\sqrt{s}}\right)^n H_{n_1}\left(\frac{(y\boldsymbol{\omega}^{-1})_1}{\sqrt{s}}\right)\cdots H_{n_r}\left(\frac{(y\boldsymbol{\omega}^{-1})_r}{\sqrt{s}}\right)p_r(s,y)dy_2\cdots dy_r$$

$$= \omega_1^{n_1}\cdots\omega_r^{n_r}\left(\frac{1}{\sqrt{s}}\right)^n H_n\left(\frac{y_1}{\sqrt{s}}\right)p_1(s,y_1).$$

Proof. Let f be a Schwartz rapidly decreasing function on $\mathbf{R}$. Set $\varphi(x) = f(\langle\omega,x\rangle)$. Then it is easy to show that

$$E[\partial_1^{n_1}\cdots\partial_r^{n_r}\varphi(B_s)] = \omega_1^{n_1}\cdots\omega_r^{n_r}E[f^{(n)}(\langle\omega,B_s\rangle)],$$

∂_i denoting $\partial/\partial x_i$. The left hand side above equals to

$$\int\cdots\int_{\mathbf{R}^r}\left(\frac{1}{\sqrt{s}}\right)^n H_{n_1}\left(\frac{x_1}{\sqrt{s}}\right)\cdots H_{n_r}\left(\frac{x_r}{\sqrt{s}}\right)p_r(s,x)f(\omega_1x_1+\cdots+\omega_rx_r)dx_1\cdots dx_r$$

and the right hand side

$$\omega_1^{n_1}\cdots\omega_r^{n_r}\int_{\mathbf{R}}\left(\frac{1}{\sqrt{s}}\right)^n H_n\left(\frac{y_1}{\sqrt{s}}\right)p_1(s,y_1)f(y_1)dy_1.$$

Therefore the assertion is easily obtained by the change of variables $x\boldsymbol{\omega} = y$ on the left hand side. □

Proof of Theorem 2.1. All we have to do is to show that

$$g_{n_1,n_2,\dots,n_r}(x) = \frac{1}{(-4\pi)^{(r-1)/2}}\frac{\Gamma(1/2)}{\Gamma(r/2)}\int_{\mathbf{S}^{r-1}}\tilde{g}_{n_1,\dots,n_r}(\omega,x)\sigma(d\omega), \tag{3.8}$$

$g_{n_1,n_2,\dots,n_r}(x)$ and $\tilde{g}_{n_1,\dots,n_r}(\omega,x)$ being as in (3.3) and as in (3.7), respectively. For $\omega\in\mathbf{S}^{r-1}$, let $\boldsymbol{\omega}$ be an orthogonal matix whose first column is ${}^t\omega$. Appealing to the change of variables $y = x\boldsymbol{\omega}$, we have

$$\hat{R}\left[H_{n_1}\left(\frac{x_1}{\sqrt{s}}\right)\cdots H_{n_r}\left(\frac{x_r}{\sqrt{s}}\right)p_r(s,x)\right](\omega,\xi)$$

$$= \int\cdots\int_{\mathbf{R}^{r-1}}\prod_j H_{n_j}\left(\frac{\sum_{i=2}^r y_i\omega_{ji}+\xi\omega_j}{\sqrt{s}}\right)\times p_1(s,\xi)\prod_{i=2}^r p_1(s,y_i)dy_2\cdots dy_r,$$

where ω_{ji} denotes the (j,i)-component of $\boldsymbol{\omega}$. Note that $\sum_{i=2}^r y_i\omega_{ji}+\xi\omega_j = (\tilde{y}\boldsymbol{\omega}^{-1})_j$ where $\tilde{y} = (\xi,y_2,\dots,y_r)$. Thus, applying Lemma 3.2 and the inversion formula for the Radon transform (2.2) to above, we obtain (3.8) in the case where $n = n_1 + \cdots + n_r > 0$. In the case where $n = 0$, let Φ be a function in $\mathscr{D}$ such that 0 is not included in the closure of its support. Then we easily have

$$\int_{\mathbf{R}^r} \Phi(x) \int_0^t p_r(s,x)dsdx = \frac{1}{(-4\pi)^{(r-1)/2}} \frac{\Gamma(1/2)}{\Gamma(r/2)} \int_{\mathbf{R}^r} \Phi(x)\check{R}\left[\partial_\xi^{r-1} \int_0^t p_1(s,\xi)ds\right] dx,$$

which ensures (3.8) for $n = 0$. Therefore the proof is completed. □

Remark 3.4. We can show that the following equality due to Bass [1],

$$\int_0^t f(B_s)ds = \frac{1}{(-4\pi)^{(r-1)/2}} \frac{\Gamma(1/2)}{\Gamma(r/2)} \int_{\mathbf{S}^{r-1}} \int_{-\infty}^{\infty} \left(\frac{\partial^{r-1}}{\partial\xi^{r-1}} \hat{R}[f]\right)(\omega,\xi) L^{\omega}(t,\xi)d\xi\sigma(d\omega), \tag{3.9}$$

holds almost surely for any Schwartz rapidly decreasing function f on $\mathbf{R}^r$ (The constant in Theorem 4.6 of Bass [1] should be corrected to $(-1)^{(d-1)/2}(2\pi)^{-(d-1)}/2$); indeed, by the proof of Proposition 2.2 we can easily show that $L^{\omega}(t,\xi)$ is continuous in $\mathscr{D}_{\gamma_2}^{(1/r,p_2)}$ ($\gamma_2 < 1, p_2 < -1/2$) with respect to (ω,ξ), and that

$$\int_{\mathbf{S}^{r-1}} \int_{-\infty}^{\infty} \left|\left(\frac{\partial^{r-1}}{\partial\xi^{r-1}} \hat{R}[f]\right)(\omega,\xi)\right| \|L^{\omega}(t,\xi)\|_{(\gamma_2,1/r,p_2)} d\xi\sigma(d\omega) < \infty.$$

Therefore the right hand side of (3.9) exists in $\mathscr{D}_{\gamma_2}^{(1/r,p_2)}$ and, by Lemma 3.2, its kernel functions coincide with those of the left hand side of (3.9). On the other hand, the left hand side of (3.9) belongs to $L^2(P)$. Thus the equality (3.9) holds in $L^2(P)$, which leads us the assertion. We should mention that Bass [1] has shown the equality (3.9) above under some mild assumption on f.

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