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On the Newton's method for transcendental functions

By

Hartje KRIETE

Abstract

The family of polynomials $P_n : \mathbb{C} \times \mathbb{C} \to \mathbb{C}\,; (\lambda, z) \mapsto \lambda(1 + z/n)^n$ converges uniformly on compact subsets of the complex plane to the family of the complex exponentials $E : \mathbb{C} \times \mathbb{C} \to \mathbb{C}\,; (\lambda, z) \mapsto \lambda e^z$, as n tends to infinity. Due to this convergence certain dynamical properties of the polynomials $P_n(\lambda, \cdot)$ carry over to the exponentials $E(\lambda, \cdot)$. Thus it possible to study entire transcendental maps, the exponentials, by considering polynomials for which the theory is well-known. Two particular problems have received attraction:

(1) For a fixed parameter $\lambda \in \mathbb{C}$ do the Julia sets of the polynomials $P_n(\lambda, \cdot)$ converge to the Julia set of $E(\lambda, \cdot)$?

(2) Do the hyperbolic components in the parameter space of P_n converge to hyperbolic components of the family E?

In the present paper we study the Newton's method associated with the entire transcendental functions $f(z) = p(z)e^{q(z)} + az + b$, with complex numbers a and b, and complex polynomials p and q. These functions N_f can be approximated by the Newton's method associated with $f_m(z) = p(z)(1 + q(z)/m)^m + az + b$. In this paper we study the convergence of the Julia sets $\mathcal{J}(N_{f_m}) \to \mathcal{J}(N_f)$ and the Hausdorff convergence of hyperbolic components in the families $\{N_{f_m}\}$ to the hyperbolic components of the family $\{N_f\}$.

1. Introduction

In the last decades, the interest in the iteration of transcendental holomorphic mappings has substantially increased, and a number of papers has appeared, e.g. [6], [2], [3], [4], [8], [10], [9], [11], [12]. Iteration theory for transcendental functions develops along the line of the rational case. However, there is a variety of phenomena and problems that do not occur in the rational case. This leads to the

Question. *Which results carry over from the rational case and which do not?*

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A first approach to this question has been suggested in [10] and was illustrated in [9], [14], [18], [17].

The family of polynomials $P_n : \mathbb{C}\times\mathbb{C} \to \mathbb{C}\,; (\lambda, z) \mapsto \lambda(1+z/n)^n$ converges uniformly on compact subsets of the complex plane to the family of the complex exponentials $E : \mathbb{C} \times \mathbb{C} \to \mathbb{C}\,; (\lambda, z) \mapsto \lambda e^z$, as n tends to infinity. Due to this convergence certain dynamical properties of the polynomials $P_n(\lambda, \cdot)$ carry over to the exponentials $E(\lambda, \cdot)$. Thus it possible to study *entire* transcendental maps, the exponentials, by considering polynomials for which the theory is well-known. Two particular problems have received attraction:

1. *For a fixed parameter $\lambda \in \mathbb{C}$ do the Julia sets of the polynomials $P_n(\lambda, \cdot)$ converge to the Julia set of $E(\lambda, \cdot)$?*

2. *Do the hyperbolic components in the parameter space of P_n converge to hyperbolic components of the family E?*

Contributions to the first question can be found in [14], [18], [20]. The publications [10], [9], [16], [15] deal with the second question. In the present paper we want to extend the methods presented in [16] to a wide class of *meromorphic* transcendental functions.

Let a and b complex numbers, and p and q complex polynomials. We approximate the entire transcendental functions given by $f(z) := p(z)e^{q(z)} + az + b$ by the polynomials $f_m(z) := p(z)(1 + q(z)/m)^m + az + b$, where m runs through the non-negative integers $\mathbb{N}$. Note that both, the f and the f_m can be parameterized over some parameter space $\mathcal{T}$, whose dimension depends on the degrees of p and q, cf. Section 3 for the details. We study the convergence of the Newton's method N_{f_m} associated with f_m to the Newton's method N_f associated with f. In particular, we establish a sufficient (and in a certain sense optimal) condition which assures the Haussdorf convergence $\mathcal{J}(N_{f_m}) \to \mathcal{J}(N_f)$, cf. Theorem 9. The main result of this paper, Theorem 19, deals with the Kernel convergence of the hyperbolic components in the parameter space. In short, the Main Theorem says, that each hyperbolic component H of the limit family $\{N_f\}_{f\in\mathcal{T}}$ is Carathéodory kernel of a sequence of hyperbolic components H_m of the approximating families $\{N_{f_m}\}_{\{f_m\in\mathcal{T}\}}$.

2. Notions and notations

Consider a function F meromorphic in the complex plane and let $F^{\circ n}$ denote its n-th iterate. If F is transcendental, then F and all iterates $F^{\circ n}$ have an essential singularity at ∞. In addition, if F has a pole in $\mathbb{C}$ which is *not* a Picard exceptional value, then each iterate is not well defined at infinitely many points. Thus one has to redefine some notions. For example, some $\zeta \in \overline{\mathbb{C}}$ is called a *periodic point of period n of F*, if for some integer $n > 0$

(i) $F^{\circ n}$ is holomorphic on some neighborhood $U \subset \overline{\mathbb{C}}$ of ζ, and

(ii) $F^{\circ n}(\zeta) = \zeta$

holds.

Note that, if F is transcendental, then $\zeta = \infty$ cannot be a periodic point. A periodic point $\zeta \in \mathbb{C}$ of F is called attracting (or repelling) if $|(F^{\circ n})'(\zeta)|$ is smaller (respectively larger) than one. If $(F^{\circ n})'(\zeta) = 0$ then ζ is called a superattracting periodic point. The *Julia set* $\mathcal{J}(F)$ *of* F is the closure (with respect to the Riemann sphere) of the set of all repelling periodic points of F, cf. [1], and the *Fatou set* $\mathcal{F}(F)$ *of* F is its complement (with respect to the Riemann sphere). Again, the Julia set is a non-empty perfect set, cf. [5], and one has

Lemma 1. *The Julia set is completely invariant, i.e.* $z \in \mathcal{J}(F)$ *implies* $F(z) \in \mathcal{J}(F)$ *(unless* F *is transcendental and* $z = \infty$*), and* $F(z) = w \in \mathcal{J}(F)$ *implies* $z \in \mathcal{J}(F)$.

We view the Fatou set and Julia set as subsets of the Riemann sphere $\overline{\mathbb{C}}$. For some subset $S \subset \overline{\mathbb{C}}$ we write $F(S) := \{F(z) \mid z \in S$ and F is defined in $z\}$, and $F^{-1}(S) := \{z \in \overline{\mathbb{C}} \mid F(z) \in S$ and F is defined in $z\}$. We are interested in the singularities of F. Roughly speaken, these are the points $v \in \overline{\mathbb{C}}$ such that for every neighborhood U of v there exists a branch of the inverse of F which is not holomorphic on U. We recall the precise definition.

Definition 2 (Singular value). Let F be a function meromorphic on the complex plane. A point $v_0 \in \overline{\mathbb{C}}$ is called a *singular value of* F, if for *every* neighborhood U of v_0 there exists some $z_0 \in F^{-1}(v_0)$ such that no continuous function $\phi : U \to \overline{\mathbb{C}}$ satisfying $F \circ \phi \equiv \mathrm{id}$ and $\phi(v_0) = z_0$ exists. Let $\mathrm{sing}(F)$ denote the set of all *finite* singular values of F.

For transcendental functions $\mathrm{sing}(F^{-1})$ is the closure of the set of all finite *critical values of* F and all finite *asymptotic values of* F. If $\zeta \in \overline{\mathbb{C}}$ is not contained in $(F^{\circ n})^{-1}(\infty)$ (for all $n \in \mathbb{N}$) then $F^{\circ n}(\zeta)$ is defined for all $n \in \mathbb{N}$ and we call $O^+(\zeta) := \{F^{\circ n}(\zeta) \mid n \in \mathbb{N}\}$ the (forward) orbit of ζ. To every attracting periodic point ζ there belongs an *attracting cycle* $\mathcal{Z}$. As usual, the *basin of attraction* $A(\mathcal{Z})$ of this orbit is defined as the set of all points $z \in \mathbb{C}$ such that ζ is an accumulation point of $O^+(z)$, that is to say, the ω-limit set $\omega(z)$, i.e. the set of all accumulation points of the sequence $\{F^{\circ n}(z)\}_{n \in \mathbb{N}}$, is equal to $\mathcal{Z}$. Clearly, $\mathcal{A}(O^+(\zeta))$ is a subset of the Fatou set $\mathcal{F}(F)$. Furthermore, each basin of attraction contains at least one singular value. In the sequel, we are concerned with *hyperbolic* Julia sets. However, no canonical definition of the hyperbolicity of meromorphic transcendental functions exists. We refer the reader to [16, Section 3] for a discussion of the problem. In the next section we shall introduce a special class of meromorphic transcendental functions, and we shall introduce a notion of hyperbolicity which will turn out to be appropriate for our purposes. For example, hyperbolicity will be a robust property, that is to say, the set of hyperbolic functions is an open set of the parameter space. Some authors use the term 'structurally stable' instead of 'robust'.

Definition 3 (Robust property). Let $\{F_\lambda\}_{\lambda \in \mathcal{T}}$ be family of functions with some topological space $\mathcal{T}$ as parameter space. A property is said to be *robust* or *structurally stable* if and only if the set of parameters λ such that f_λ has this property is an open subset of $\mathcal{T}$.

A further difficulty will arise when we deal with the limit of (open) sets, because several different notions of convergence can be used. Throughout this chapter we work with the so-called kernel convergence. Recall that some authors use the term ‘Caratheodory convergence’ instead of ‘kernel convergence’.

Definition 4 (Kernel)**.** Let $\mathcal{M}$ some complex manifold and $\{H_n\}_{n\in\mathbb{N}}$ be a sequence of open sets $H_n \subset \mathcal{M}$ and $H \subset \mathcal{M}$ another open set. H is called the *kernel* of the sequence $\{H_n\}_{n\in\mathbb{N}}$, if it is connected, and if every compact subset of H is contained in all but finitely many H_n and this is not true for any open set $\widetilde{H}$ satisfying $H \subsetneqq \widetilde{H}$.

Remark. Note that a sequence $\{H_n\}_{n\in\mathbb{N}}$ might have more than one kernel. However, if both, H and $\widetilde{H}$, are kernels of the same sequence, then either $H = \widetilde{H}$ or $H \cap \widetilde{H} = \emptyset$ holds.

Note that polynomials or rational functions have a finite number of singular values, only, and that the singular values continuously depend on the function in question. This is the reason, that for this class hyperbolicity is robust. However, many transcendental functions have an infinite number of singular values, e.g. $F(z) = z + \sin(z)$. Further below we will deal with examples of meromorphic (but not entire) transcendental functions with infinitely many singular values. Consequently, one cannot expect hyperbolicity to be a robust property. In fact, this holds in special classes, only.

Further difficulties are caused by the fact, that neither Sullivan's nonwandering Theorem nor his classification of periodic domains hold for transcendental functions. In other words, we have to be aware of the existence of wandering domains and Baker domains. Since this type of components of the Fatou set do not occur in the case of rational functions, we briefly recall their definitions. To this end we choose a transcendental function F. For each component U of its Fatou set $\mathcal{F}(F)$ and each $n \in \mathbb{N}$ there exists another component U_n of $\mathcal{F}(F)$ satisfying $F^{\circ n}(U) \subset U_n$. If $U_n \cap U_k = \emptyset$ for all distinct $n, k \in \mathbb{N}$, then U is called a *wandering domain.* A Baker domain is a periodic component, that is to say $U = U_k$ holds for some $k \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$, such that $\lim_{n\to\infty} F^{\circ nk+l} \equiv \infty$ uniformly on compact subsets of U for some integer $l \in \mathbb{N}$.

3. Transcendental functions and the Newton's method

The starting point is the class $\mathcal{T}$ of entire functions of the form

$$f(z) = p(z)e^{q(z)} + az + b\,. \tag{3.1}$$

Here and throughout this chapter we make the following assumptions

- p is a polynomial of degree $\widetilde{p} \geq 0$ satisfying $p \not\equiv 0$,
- q is a non-constant polynomial of degree $\widetilde{q} \geq 1$, and
- $a, b \in \mathbb{C}$.

Since adding a constant to q is the same as multiplying p by a constant, we may and will assume

- $q(0) = 0$.

We are interested in the roots of f, and therefore we apply the (standard) Newton's method N_f associated with f:

$$N_f(z) := z - \frac{f(z)}{f'(z)}\,.$$

According to the introductory remarks, we look for 'nice' approximations f_m of f, hoping that dynamical properties of the Newton's method N_{f_m} associated with f_m carry over to the Newton's method N_f associated with f. We choose

$$f_m := p(z) \cdot \left(1 + \frac{q(z)}{m}\right)^m + az + b\,, \tag{3.2}$$

where m runs through $\mathbb{N}^*$, and p, q, a and b are as in the equation (3.1) defining f. The following properties of the Newton's method are well known:

Proposition 5. *Let f be as above. Then*

- *every simple root of f is a superattracting fixed point of N_f,*
- *every multiple root of f is an attracting but not superattracting fixed point of N_f, and*
- *every critical point c of N_f is a root of f or a root of f''.*

Later we shall learn that N_f has at most one finite asymptotic value, namely $-b/a$. Note that f'' has finitely many roots, only. Hence, N_f should be viewed as a function with a finite number of 'free' singular values. As usual a singular value is called 'free' if it is not an attracting fixed point. One can readily establish a bijective correspondence between the free singular values of N_f and those of the N_{f_m}. This elementary fact is the key to the results of this chapter, and we shall come back to this point later.

The first (and natural) question is for the convergence of the Julia sets $\mathcal{J}(N_{f_m})$ as m tends to ∞. As a corollary to Theorem 1 in [18] we obtain:

Corollary 6. *If f is a meromorphic function of the form* (3.1) *with a sequence $\{f_m\}_{m\in\mathbb{N}}$ of approximants of the form* (3.2), *and $\mathcal{F}(N_f)$ is the union of attracting periodic basins, then $\mathcal{J}(N_{f_m})$ converges to $\mathcal{J}(N_f)$ with respect to the Hausdorff metric as m tends to ∞.*

The next result deals with the convergence of the corresponding sets $\mathcal{G}(f)$ and $\mathcal{G}(f_m)$ of good initial values of the Newton's method N_f respectively N_{f_m}, where $m \in \mathbb{N}^*$.

Definition 7. Let g be a function meromorphic on the complex plane $\mathbb{C}$, and $N_g(z) := z - g(z)/g'(z)$ the Newton's method associated with g. A point $z_0 \in \mathbb{C}$ is called *good initial value* if all the iterates $z_n := N_g^n(z_n)$ are well defined, the limit $\zeta := \lim_{n\to\infty} z_n$ exists, and $g(\zeta) = 0$ holds.

In other words, a good initial value is an initial guess which makes the Newton's method to converge to a root of the function in question. As a corollary to Theorem 3.1 in [20] we obtain

Corollary 8. *Let f be a meromorphic function of the form* (3.1) *with a sequence $\{f_m\}_{m\in\mathbb{N}}$ of approximants of the form* (3.2)*. If N_f has no wandering domain, Baker domains and rationally indifferent cycle, then $\mathcal{G}(f_m)$ converges to $\mathcal{G}(f)$ with respect to the Hausdorff metric as m tends to ∞.*

Later we shall learn, that by Theorem 14 due to Bergweiler and Terglane, N_f cannot have wandering domains. Thus we can reformulate Corollary 8:

Theorem 9. *Let f be a meromorphic function of the form* (3.1) *with a sequence $\{f_m\}_{m\in\mathbb{N}}$ of approximants of the form* (3.2)*. If N_f has no rationally indifferent cycle and no cycle of Baker domains, then $\mathcal{G}(f_m)$ converges to $\mathcal{G}(f)$ with respect to the Hausdorff metric as m tends to ∞.*

Note that the assumptions on N_f in Theorem 9 do not imply that $\mathcal{F}(N_f)$ consists of attracting basins, only. Hence, Corollary 6 does not yield the Theorem.

4. Hyperbolicity

In iteration theory for rational function, the following definition is standard.

Definition 10. A polynomial or rational function g is called *hyperbolic* if the forward orbit $O^+(CV(g))$ of the set $CV(g)$ of its critical values is relatively compact in the Fatou set $\mathcal{F}(g)$:

$$O^+(CV(g)) \subset\subset \mathcal{F}(g) .$$

For self mappings of some compact space, e.g. polynomials or rational functions, it is well-known that hyperbolicity is a structurally stable property. However, since in our setting we are not dealing with self mappings of some compact space, the situation is more complicated. One can prove the following lemma, see [19], which is well-known in the category of holomorphic self mappings of the Riemann sphere, that is to say, in the category of rational maps.

Lemma 11. *Let $g : \mathbb{C} \to \overline{\mathbb{C}}$ be a transcendental function and A the union of the basin of attraction of all its attracting periodic orbits. For every sequence $\{g_n\}_{n\in\mathbb{N}}$ of transcendental functions $g_n : \mathbb{C} \to \overline{\mathbb{C}}$ converging to g uniformly on compact subsets of $\mathbb{C}$ let A_n denote the union of the basin of attraction of all attracting cycles of g_n. Then for every compact set $K \subset\subset A$ there exists some number $n_0 \in \mathbb{N}$ such that $K \subset\subset A_n$ for every $n \geq n_0$.*

Recall that if g is a polynomial or a rational function then the set of its singular values is a compact set depending continuously on g. Thus Lemma 11 yields that for every $d \geq 2$ the set of all hyperbolic polynomials or rational functions of degree d forms an open subset in the space of all polynomials respectively rational functions of degree d. We would like to have an analogue for transcendental functions. However, the set of (finite) singular values of some transcendental function g need not need to be compact. In addition, $\mathrm{sing}(g)$ need not to depend continuously on g. Consequently, for meromorphic functions we have to redefine the notion of hyperbolicity and we have to restrict to suitable subclasses of transcendental functions. This will be done later.

Now we return to the dynamics of the Newton's method N_f, where f is of the form (3.1). Since the case $a = 0$ has intensively been studied by Jankowski, cf. [13], in the sequel we assume

$$\boxed{a \neq 0}$$

Later we shall need the first two derivatives of f. A short calculation gives

$$f'(z) = S_1(z)e^{q(z)} + a \tag{4.1}$$

$$f''(z) = S_2(z)e^{q(z)} \tag{4.2}$$

with polynomials

$$S_1(z) := p'(z) + p(z) \cdot q'(z) \tag{4.3}$$

$$S_2(z) := S_1'(z) + S_1(z) \cdot q'(z)\,. \tag{4.4}$$

We will prove that f is the solution of an inhomogeneous linear differential equation of second order with non-constant coefficients. Accordingly, we expect N_f to satisfy a 'nice' differential equation, too. In fact, we will show that N_f satisfies a Riccati differential equation: We then can apply a result of Bergweiler and Terglane for showing that N_f has one asymptotic value, only, and that is has no wandering domain.

Lemma 12. *Let $f(z) = p(z)e^{q(z)} + az + b$ as above. Then f satisfies an inhomogeneous linear differential equation of second order with non-constant rational coefficient functions.*

Proof. Using equations (4.1) and (4.2) one readily calculates

$$\begin{aligned} f(z) - zf' &= (p(z)e^{q(z)} + b) - zS_1e^{q(z)} \\ &= (p(z) - zS_1(z))e^{q(z)} + b \end{aligned}$$

$$\iff f(z) - zf'(z) + \frac{zS_1(z) - p(z)}{S_2(z)}f''(z) - b = 0.$$ □

Lemma 13. *Let $f(z) = p(z)e^{q(z)} + az + b$ as above. Then N_f satisfies the Riccati differential equation*

$$N_f'(z) + \frac{S_2(z)}{p(z) - \frac{b}{a}S_1(z) - zS_1(z)}\,(N_f(z) - z)\left(N_f(z) + \frac{b}{a}\right) \equiv 0\,. \tag{4.5}$$

Proof. We begin by recalling $N_f' \equiv ff''/(f')^2$. We are interested in finding some rational function X such that the equation

$$N_f'(z) + X(z)\,(N_f(z) - z)\left(N_f(z) + \frac{b}{a}\right) \equiv 0\,.$$

holds. A straightforward calculation gives

$$\begin{aligned}
& N_f' + X(N_f - \mathrm{id})\left(N_f + \frac{b}{a}\right) \equiv 0 \\
\iff\quad & \frac{ff''}{(f')^2} + X \cdot \frac{f}{f'}\left(N_f + \frac{b}{a}\right) \equiv 0 \\
\iff\quad & \frac{f''}{f'} + X\left(N_f + \frac{b}{a}\right) \equiv 0 \\
\iff\quad & X\left(N_f + \frac{b}{a}\right) \equiv -\frac{f''}{f'} \\
\iff\quad & X \equiv \frac{-f''}{f'\left(\mathrm{id} - \frac{f}{f'} + \frac{b}{a}\right)} \\
& \equiv \frac{-f''}{\mathrm{id}\, f' - f + \frac{b}{a} f'} \\
& \equiv \frac{-S_2 e^q}{(\mathrm{id} \cdot S_1 e^q + a\,\mathrm{id}) - (pe^q + a\,\mathrm{id} + b) + \frac{b}{a}\left(S_1 e^q + a\right)} \\
& \equiv \frac{-S_2}{\mathrm{id}\, S_1 - p + \frac{b}{a} S_1} \qquad \square
\end{aligned}$$

Then, due to a result of Bergweiler and Terglane, cf. [21, Lemma 6.2], N_f has $-b/a$ as the only asymptotic value. Furthermore, they have shown, that N_f cannot have wandering domains, cf. [7, Theorem 3].

Theorem 14 (Bergweiler, Terglane)**.** *Let $f(z) = p(z)e^{q(z)} + az + b$ as above. Then N_f has $-b/a$ as the only asymptotic value, and it has no wandering domain.*

In order to find the critical values of N_f we first look for the critical points, i.e. the roots of N_f'. It is well known that

$$N_f'(c) = 0 \Longrightarrow f(c) = 0 \quad \text{or} \quad f''(c) = 0\,.$$

Clearly, the solutions of $f(c) = 0$ are fixed points of N_f. Thus we need to look at the roots of f'', only, that is to say, the roots of the polynomial S_2. Altogether we obtain the *finite* set

$$\mathcal{S} := N_f\left(S_2^{-1}(0)\right) \cup \left\{\frac{-b}{a}\right\}$$

as the set of free singular values. Consequently, we define

Definition 15 (Hyperbolicity)**.** Let f be an entire transcendental function of the form (3.1). If $\omega(v) \subset\subset \mathcal{F}(N_f)$ or, equivalently, $O^+(v) \subset\subset \mathcal{F}(N_f)$ holds for every $v \in \mathcal{S}$, then the Newton's method N_f is called *hyperbolic.*

Note that the space $\mathcal{T}$ of all entire functions of the form (3.1) with $a \in \mathbb{C}^*$ can naturally been identified with $\mathbb{C}^M \times (\mathbb{C}^*)^3$, where $M := \widetilde{p} + \widetilde{q}$ and $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. For a moment we now turn our attention to the family of appoximants f_m. Note that the space $\mathcal{T}_m$ of all polynomials f_m of the form (3.2) can also be identified with $\mathbb{C}^M \times (\mathbb{C}^*)^3$, where $M := \widetilde{p} + \widetilde{q}$:

$$\mathcal{T} \cong \mathcal{T}_m \cong \mathbb{C}^M \times (\mathbb{C}^*)^3 .$$

Since the mappings $f \mapsto -b/a$ and $f \mapsto S_2$ are continuous, one can use Lemma 11 to readily prove, that hyperbolicity is robust in the space $\mathcal{T}$. Note that the set of roots of the polynomials continuously depend on S_2 and, consequently, continuously depend on f.

Proposition 16. *The set $\mathcal{H}$ of all $f \in \mathcal{T}$ such that N_f is hyperbolic is open.*

We are now prepared to define 'hyperbolic components'.

Definition 17. The components of $\mathcal{H}$ are called *hyperbolic components.*

Note that for every $f \in \mathcal{H}$ the Newton's method does not have rationally indifferent cycles, Siegel disc or Herman rings. Can N_f have Baker domains? By a result of Bergweiler and Terglane, cf. [21, Satz 3.2] and [21, Lemma 6.1], each cycle of Baker domains has to contain a singular value. Altogether we obtain

Proposition 18. *If $f \in \mathcal{H}$, then the Fatou set of the Newton's method N_f associated with f consists of attracting basins, only.*

Remark. Note that the statement of this proposition assures that every $f \in \mathcal{H}$ satisfies the assumption of Corollaries 6, 8 and Theorem 9.

5. Convergence of hyperbolic components

We now turn our attention to the convergence of hyperbolic components. We are concerned with the family

$$G_\infty : \mathcal{T} \times \mathbb{C} \to \overline{\mathbb{C}}\,;\ (f, z) \mapsto N_f(z)$$

of meromorphic transcendental functions N_f and a sequence

$$G_m : \mathcal{T}_m \times \mathbb{C} \to \overline{\mathbb{C}}\,;\ (f_m, z) \mapsto N_{f_m}(z)\,,$$

where $m \in \mathbb{N}^*$, of families of rational functions N_{f_m}, such that the functions G_m converge to G_∞ uniformly on compact subsets of $\mathcal{T} \times \mathbb{C} \cong \mathcal{T}_m \times \mathbb{C}$. The main result of this paper is the kernel convergence of hyperbolic components. Krauskopf and the author have proved a similar result for certain classes of entire transcendental functions. The proof presented here bases on the paper [16] but many changes have had to be made.

Theorem 19 (Main Theorem). *Let G_∞ and $\{G_m\}_{m\in\mathbb{N}}$ be as above. For every hyperbolic component $H \subset \mathcal{T}$ of G_∞ there exists a sequence $\{H_m\}_{m\in\mathbb{N}^*}$ of hyperbolic components $H_m \subset \mathcal{T}_m$ of G_m having H as a kernel.*

Recall that the N_{f_m} are rational functions, hence we may use Definition 10 for defining hyperbolicity of these functions. In analogy to the transcendental case, we obtain:

Proposition 20. *If f_m is as above, then for every singular value v of N_{f_m} there exists a point $c \in \mathbb{C}$ such that $N'_{f_m}(c) = 0$ and $N_{f_m}(c) = v$, in particular $f_m(c) = 0$ or $f''_m(c) = 0$.*

Again we ignore the roots of f_m since they are attracting fixed points of N_{f_m}. Thus we need to compute f''_m. Recall $f_m = p(1 + q/m)^m + a \cdot \mathrm{id} + b$. With S_1 as defined in (4.3) we obtain

$$\begin{aligned} f'_m &= p'\left(1+\frac{q}{m}\right)^m + pq'\left(1+\frac{q}{m}\right)^{m-1} + a \\ &= \left[p'\left(1+\frac{q}{m}\right) + pq'\right]\left(1+\frac{q}{m}\right)^{m-1} + a \\ &= \left[S_1 + \frac{q}{m}\right]\left(1+\frac{q}{m}\right)^{m-1} + a \end{aligned}$$

and finally

$$\begin{aligned} f''_m &= \left[S'_1 + \frac{q'}{m}\right]\left(1+\frac{q}{m}\right)^{m-1} + \left[S_1 + \frac{q}{m}\right]\frac{(m-1)q'}{m}\left(1+\frac{q}{m}\right)^{m-2} \\ &= \left(1+\frac{q}{m}\right)^{m-2}\left[S'_1 + S'_1\frac{q}{m} + \frac{q'}{m} + S_1 q' - \frac{S_1 q'}{m} + \frac{qq'}{m}\right] \\ &= \left(1+\frac{q}{m}\right)^{m-2}\Bigg[\underbrace{S_2 + \frac{1}{m}\left(S'_1 q + q' - S_1 q' + qq'\right)}_{S_{2,m}:=}\Bigg]. \end{aligned}$$

We note $\deg(S_{2,m}) = \deg(S_2)$ and that the $S_{2,m}$ converge to S_2 uniformly on compact subsets of $\mathbb{C}$ as m tends to ∞. Hence, $S_{2,m}^{-1}(0)$ converges to S_2^{-1} as m tends to ∞, and this carries over to the corresponding singular values of the N_{f_m}. f''_m has further roots, namely the roots of $1 + q(z)/m$. We determine the corresponding singular value. Clearly, $(1 + q(z_0)/m) = 0$ for some $z_0 \in \mathbb{C}$ implies $f_m(z_0) = az_0 + b$ and $f'_m(z_0) = a$. This yields

$$\left(1 + \frac{q(z_0)}{m}\right) = 0 \Longrightarrow N_{f_m}(z_0) = \frac{-b}{a}\,.$$

Thus, the singular value corresponding to the roots of $1 + q/m$ again is $-b/a$. As above we define the set of free singular values of N_{f_m}:

$$\mathcal{S}_m := \left(S_{2,m}^{-1}(0)\right) \cup \left\{\frac{-b}{a}\right\} .$$

We summarize.

Proposition 21. *For an entire transcendental function of the form* (3.1) *with approximants f_m as defined in* (3.2)*, let $\mathcal{S}$ and $\mathcal{S}_m$ denote the set of free singular values of the corresponding Newton's method N_f respectively N_{f_m}. Then*

$$\mathrm{dist}_{HD}(\mathcal{S}_m, \mathcal{S}) = 0 .$$

Combining this statement with Corollary 8 we obtain.

Theorem 22. *Let $f_\infty \in \mathcal{T}$ such that N_{f_∞} is hyperbolic.*

(a) *There exists a neighborhood $U \subset \mathbb{C}^M \times (\mathbb{C}^*)^3 \cong \mathcal{T}$ of f_∞ such that N_f is hyperbolic for every $f \in U$.*

(b) *There exist a neighborhood $U \subset \mathbb{C}^M \times (\mathbb{C}^*)^3 \cong \mathcal{T}_m \cong \mathcal{T}$ of f_∞ and some $n_0 \in \mathbb{N}$ such that N_{f_m} is hyperbolic for every $f_m \in U$ and every $m \in \mathbb{N}$ satisfying $m \geq n_0$.*

We rephrase this in terms of the convergence of hyperbolic components.

Corollary 23. *Every hyperbolic component H of the family G_∞ is contained in a kernel of a sequence $\{H_m\}_{m\in\mathbb{N}}$ of hyperbolic components H_m of the families G_m.*

It is our aim to prove the kernel convergence of hyperbolic components, the main result of this chapter. Recall, that in general one can not expect a hyperbolic component of the limit family to be a limit with respect to the Hausdorff metric of hyperbolic components of the approximating families. Counterexamples are given in [16]. Clearly, it is sufficient to prove the reverse of Corollary 23. The key ingredient is

Theorem 24. *Let $\widetilde{H} \subset \mathcal{T}$ be kernel of a sequence $\{H_m\}_{m\in\mathbb{N}}$ of hyperbolic components $H_m \subset \mathcal{T}_m$ of the families G_m. Then either*

(i) *$G_\infty(f, \cdot)$ is not hyperbolic for any $f \in \widetilde{H}$ or*

(ii) *$\widetilde{H} \subset H$ for some hyperbolic component H of the family G_∞.*

Proof. The idea of the proof is to show that case (ii) holds if case (i) does not. To this end we assume that case (i) does not hold. That means that there is a hyperbolic component H of G_∞ with $\widetilde{H} \cap H \neq \emptyset$ and a domain $B \subset\subset \widetilde{H} \cap H_m$ for all but finitely many $m \in \mathbb{N}$. The idea is now to show that $B \subset H$. This is done by repeatedly using the Theorem of Montel for the parameterizations of the singular values.

Note that $\mathcal{J}(N_f)$ and $\mathcal{J}(N_{f_m})$ are equal to the closure of the set of repelling periodic points of N_f respectively N_{f_m}. By the Implicit Function Theorem, repelling periodic points locally admit an holomorphic parameterization. After assuming B to be sufficiently small we may assume that there are two repelling periodic points of N_f respectively N_{f_m} which in fact have a holomorphic parameterization on B. After applying a suitable affine change of coordinates, we thus may and will assume $\{0,1\} \subset \mathcal{J}(N_f), \mathcal{J}(N_{f_m})$ for all $f \in B$ respectively $f_m \in B$ and every $m \in \mathbb{N}$. By definition, $\infty \notin \mathcal{S}(\cdot)$ holds. Hence, we have $\mathcal{S}(f), \mathcal{S}(f_m) \subset \mathbb{C} \setminus \{0,1\}$ provided $f \in B$.

Note that a singular value of N_f is either $-b/a$ or the preimage (with respect to N_f) of a root of S_2. Clearly, the mapping $(b,a) \mapsto -b/a$ is holomorphic on $\mathbb{C} \times \mathbb{C}^*$. Since S_2 is a polynomial, its roots depend holomorphically on its coefficients for an open and dense subset of the parameter space. After choosing B small enough, we may and will assume the roots of S_2 to depend holomorphically on f in some open neighborhood of $\overline{B}$. Recall that the polynomials $S_{2,m}$ have the same degree as S_2 and that $S_{2,m} \to S_2$ uniformly on compact subset. This yields that for all but finitely many $m \in \mathbb{N}$ the roots of $S_{2,m}$ also admit a holomorphically parameterization on B.

Let $s_1, \dots, s_N \in \mathcal{O}(B)$ be the parameterizations of the $N := \sharp\mathcal{S}$ free singular values of N_f. Let $s_{m,1}, \dots, s_{m,N} \in \mathcal{O}(B)$ be the parameterizations of the N singular values of N_{f_m}. From now on we fix $\mu \in \{1, \dots, N\}$. Note that by construction $s_{m,\mu} \to s_\mu$ uniformly on compact sets (after renumbering the $s_{\nu,\mu}$ if necessary).

For $f \in B$ we have $s_{m,\mu}(f_m) \notin \mathcal{J}(N_{f_m})$ and the invariance of Julia sets yields $N_{f_m}^{\circ\nu}(s_{m,\mu}(f_m)) \notin \mathcal{J}(N_{f_m})$ and, in particular, $N_{f_m}^{\circ\nu}(s_{m,\mu}(f_m)) \in \mathbb{C}\setminus\{0,1\}$ for all $f_m \in B$. In other words $\left\{N_{f_m}^{\circ\nu}(s_{m,\mu}(f_m)) \mid m \in \mathbb{N}\right\}$ is a normal family for each fixed ν. A limit function is either some constant $c_0 \in \{0,1,\infty\}$ or some holomorphic function with values in $\mathbb{C}\setminus\{0,1\}$. Due to the uniform convergence on compact sets of the N_{f_m} to N_f and the $s_{m,\mu}$ to s_μ the limit

$$S_{\nu,\mu} : B \to \overline{\mathbb{C}}; f \mapsto S_{\nu,\mu}(f) := \lim_{m\to\infty} N_{f_m}^{\circ\nu}(s_{m,\mu}(f_m))$$

exists and

$$S_{\nu,\mu}(f) = \lim_{m\to\infty} N_{f_m}^{\circ\nu}(s_{m,\mu}(f_m)) = N_f^{\circ\nu}(s_\mu(f))$$

holds. If $N_f^{\circ\nu}(s_\mu(f)) \equiv 0,1,\infty$ for some smallest ν, then N_f would not be hyperbolic on $B \cap H$, a contradiction. We conclude that indeed

$$S_{\nu,\mu}(f) = N_f^{\circ\nu}(s_\mu(f)) \in \mathbb{C} \setminus \{0,1\}$$

for all $f \in B$ and every $\nu \in \mathbb{N}$. In other words $\{S_{\nu,\mu}(f) \mid \nu \in \mathbb{N}\}$ is a normal family. Let S_μ denote a limit function. For simplicity we assume $S_\mu := \lim_{\nu\to\infty} S_{\nu,\mu}(f)$.

For the limit function we have either $S_\mu \equiv 0,1,\infty$ or $S_\mu(B) \in \mathbb{C} \setminus \{0,1\}$. Since H is a hyperbolic component of G_∞ we have on $B \cap H$ for the limit $S_\mu(f) = \alpha(f)$, where $\alpha(f)$ is an attracting periodic point of, say, period k of N_f. Consequently, $S_\mu \not\equiv 0,1$.

In case $S_\mu \equiv \infty$ the Identity Theorem yields that ∞ is an attracting periodic point of N_f for every $f \in B$. In particular, N_f has to be a polynomial. This contradicts the fact, that N_f is a transcendental function.

We now assume $S \not\equiv 0, 1, \infty$. Then we have $\alpha_m(f_m) \to \alpha(f)$ for all $f \in B \cap H$, where $\alpha_m(f_m)$ is an attracting periodic point of period k of N_{f_m}. But since $B \subset\subset H_n$ we conclude that $\alpha_m(f)$ is an attracting periodic point of period k of N_{f_m} for all $f \in B$. In particular, $\alpha_m(B) \in \mathbb{C} \setminus \{0,1\}$ and $\{\alpha_m \mid m \in \mathbb{N}^*\}$ is a normal family, which converges on $B \cap H$ to the well-defined function α. Hence, we can extend α to the whole of B. Due to the uniform convergence of the G_m to G on compact subsets of $\mathcal{T} \times \mathbb{C}$ each point $\alpha(f)$ is a non-repelling point of $G_\infty(f,\cdot)$ for all $f \in B$. Since $\alpha(f)$ is attracting on $B \cap H$ we conclude from the Maximum Modulus Principle that it is even attracting on the whole of $B \subset\subset H_n$. Finally we conclude

$$S_\mu(f) \in \mathcal{F}(N_f)$$

for all $f \in B$. Since μ was arbitrary we have shown that B is in a hyperbolic component of G_∞ and, hence, $B \subset H$. Since $B \subset\subset \widetilde{H}$ was arbitrary we conclude $\widetilde{H} \subset H$ and the proof is completed. □

Proof of the Main Theorem (*Theorem* 19). According to Corollary 23 every compact set in H is contained in some hyperbolic components H_n of the G_n. Let $\widetilde{H}$ be the kernel of the sequence $\{H_m\}_{m\in\mathbb{N}}$, in particular, $H \subset \widetilde{H}$. We have to show $H = \widetilde{H}$, for which it suffices to show $\widetilde{H} \subset H$. According to Theorem 24 this is the case or $G_\infty(f,\cdot)$ is not hyperbolic for all $f \in \widetilde{H}$. The latter contradicts the assumption that the hyperbolic component H is a subset of $\widetilde{H}$. □

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Mathematische Fakultät
Georg-August Universität
Bunsenstr. 3-5
37073 Göttingen, Germany

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