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A new proof of a theorem of Ramanujam-Morrow

By

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Abstract

Morrow [9] classified all weighted dual graphs of the boundary of the minimal normal compactifications of the affine plane $\mathbf{A}^2$ by using a result of Ramanujam [10] that any minimal normal compactification of $\mathbf{A}^2$ has a linear chain as the graph of the boundary divisor. In this article, we give a new proof of the above-mentioned results of Ramanujam-Morrow [9] from a different point of view and by the purely algebro-geometric arguments. Moreover, we show that the affine plane $\mathbf{A}^2$ is characterized by the weighted dual graph of the boundary divisor.

1. Introduction

All algebraic varieties considered in the present article are defined over the field of complex numbers $\mathbf{C}$. For any nonsingular quasi-projective variety X, a *normal compactification* of X is a nonsingular projective variety $\overline{X}$ such that $\overline{X}$ contains X as a Zariski open subset and the boundary $D = \overline{X} - X$ is a divisor with simple normal crossings. In the two-dimensional case, we say that $\overline{X}$ is a *minimal normal compactification* (or *m.n.c.* for short) of X if $\overline{X}$ is a normal compactification of X and, moreover, if any (-1)-curve in D meets at least three other irreducible components of D. A normal compactification of the affine line $\mathbf{A}^1$ is unique, and it a smooth rational curve $\mathbf{P}^1$. But there exist infinitely many normal compactifications of the affine plane $\mathbf{A}^2$. Let $\mathbf{A}^2 \hookrightarrow V$ be any (minimal) normal compactification of $\mathbf{A}^2$ and $D := V - \mathbf{A}^2$ be the boundary divisor. Let $C_1, \ldots, C_r$ denote the irreducible components of D. Then it is easy to see that $H_1(\cup_{i=1}^{r} C_i; \mathbf{Z}) = 0$, namely,

(i) each irreducible component C_i of D is isomorphic to $\mathbf{P}^1$ and D contains no cycles.

The $\mathbf{R}$-vector space $F := \oplus_{i=1}^{r} \mathbf{R} \cdot C_i$ with basis $C_1, \ldots, C_r$ has a quadratic form on it defined by intersection multiplicity. Let b^+ be the number of positive eigenvalues of this quadratic form on F. Note that the Picard group Pic (V)

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of V is a free abelian group generated by the classes of $C_1, \ldots, C_r$ because the coordinate ring $\Gamma(\mathcal{O}_{\mathbf{A}^2})$ of $\mathbf{A}^2$ is factorial and $\Gamma(\mathcal{O}_{\mathbf{A}^2})^* = \mathbf{C}^*$. Therefore, $\mathrm{NS}(V) \otimes_{\mathbf{Z}} \mathbf{R}$ is isomorphic to F, where $\mathrm{NS}(V)$ is the Néron-Severi group of V. Thus, by Hodge Index Theorem, we have:

(ii) $b^+ = 1$.

In the course of the proof of the main result in the famous paper of Ramanujam [10] which gives a topological characterization of the affine plane $\mathbf{A}^2$, he shows that for any m.n.c. of $\mathbf{A}^2$ the dual graph of the boundary divisor D is a linear chain. Ramanujam, in fact, proved this result using the properties (i), (ii) and the following topological condition:

(iii) the fundamental group of the boundary of a tubular neighborhood of D is trivial.

(The fact that any m.n.c. of $\mathbf{A}^2$ has a linear chain as the graph of the boundary divisor is obtained by a more general result of Daigle [2, Lemma 5.11] which asserts that the weighted graph equivalent to a linear chain contracts to a linear chain. But we can not find a printed proof for this result in [2].) Ramanujam showed, moreover, that there exist at least one and at most two irreducible components of D having non-negative self-intersection number and that if D contains exactly two components having non-negative self-intersection number, then they meet each other and at least one of them has self-intersection number zero. Using these results of Ramanujam concerning the algebro-topological properties of the boundary of $\mathbf{A}^2$, Morrow [9] gave a complete classification of all wighted dual graphs of the boundary divisor for m.n.c.'s of $\mathbf{A}^2$. Once we admit the results of Ramanujam, Morrow's argument in [9] is a detailed observation for the weighted linear chain to be equivalent to a graph consisting of only one vertex with weight 1.

In this article, we shall reprove a theorem of Morrow as to the classification of the boundary of m.n.c.'s of $\mathbf{A}^2$ from a different point of view. Our method is purely algebro-geometric. Our argument is, roughly speaking, stated as follows: Let $\mathbf{A}^2 \hookrightarrow V$ be any m.n.c. of the affine plane $\mathbf{A}^2$. Since $\mathbf{A}^2$ has a structure of an $\mathbf{A}^1$-fibration over $\mathbf{A}^1$, the closures in V of the fibers of this $\mathbf{A}^1$-fibration generate an irreducible linear pencil Λ on V. If Λ is already base point free, then the observation is considerably simple and we can easily determine the weighted dual graph of the boundary. Whereas, if $\mathrm{Bs}\,\Lambda$ is not empty, then it consists of only a single point, say p_0. We then consider the shortest succession of blowing-ups $\sigma : \widetilde{V} \to V$ with center p_0 including its infinitely near points such that the proper transform $\widetilde{\Lambda} := \sigma'(\Lambda)$ of Λ is base point free. This process σ is, in fact, a composite of *Euclidean transformations* and *equi-multiplicity transformations* (see Section 2) determined by the datum at the singularity p_0 of a general member of Λ. (In order to determine the weighted dual graph of the boundary, essential is an explicit description of the shortest process σ to resolve $\mathrm{Bs}\,\Lambda$. In fact, the heart of this paper is devoted to a concrete description of the process σ as a composite of Euclidean transformations and equi-multiplicity transformations. This is done in Section 3.) The base point free linear pencil $\widetilde{\Lambda}$ defines a $\mathbf{P}^1$-fibration on $\widetilde{V}$. By observing the singular fibers of this $\mathbf{P}^1$-fibration, we obtain the result of Morrow. As a result, we prove also the result of Ramanujam as to the linearity of the boundary graph. To state the results of Ramanujam and Morrow, we have the following result:

Theorem 1.1. *Let $\mathbf{A}^2 \hookrightarrow V$ be a minimal normal compactification of the affine plane. Then the weighted dual graph of the boundary $D := V - \mathbf{A}^2$ is given by one of the graphs (a)–(f) in Figure 1, where $n, m_1, \ldots, m_{\beta-1}$ are arbitrary positive integers and every vertex stands for a smooth rational curve. These graphs actually occur as the boundary graphs of minimal normal compactifications of $\mathbf{A}^2$.*

Furthermore, we shall prove that the converse of Theorem 1.1 holds. In fact, we have the following result:

Theorem 1.2. *Let X be a smooth affine surface and let $X \hookrightarrow V$ be a normal compactification of X. (This compactification is not necessarily minimal.) Suppose that each irreducible component of the boundary $D = V - X$ is a rational curve and the weighted dual graph of D is the same as that of the boundary divisor of a suitable normal compactification of the affine plane $\mathbf{A}^2$. Then X is isomorphic to $\mathbf{A}^2$.*

Theorem 1.2 means that an affine plane $\mathbf{A}^2$ is determined by the weighted dual graph of the boundary divisor. But, as remarked in Section 4, if we replace the affine plane $\mathbf{A}^2$ by another $\mathbf{Q}$-homology plane in the statement of Theorem 1.2, then a similar result fails to hold in general. Here, a $\mathbf{Q}$-homology plane is, by definition, a smooth affine surface S with $H_i(S; \mathbf{Q}) = 0$ for $i > 0$ (see Miyanishi and Sugie [8] for the relevant results of $\mathbf{Q}$-homology planes).

In view of the importance of the result of Ramanujam-Morrow in affine algebraic geometry, or more specifically, in the theory of open algebraic surfaces, it is useful to reprove the result of Ramanujam-Morrow by a purely algebro-geometric method and from a different point of view. In Section 2, we give some preliminary results with respect to the singular fibers of a $\mathbf{P}^1$-fibration. Further, we define the notions of Euclidean transformations and equi-multiplicity transformations which will replace the topological and combinatorial arguments of Ramanujam-Morrow. In Section 3, we prove Theorem 1.1 by making use of the theory of an $\mathbf{A}^1$-fibration and a $\mathbf{P}^1$-fibration and the notions of an Euclidean transformation and an equi-multiplicity transformation. We prove Theorem 1.2 in Section 4. The author would like to express his hearty thanks to Prof. M. Miyanishi for helpful discussions.

2. Preliminaries

We denote by $\mathbf{A}^n$ the affine space of dimension n. A smooth projective, rational curve with self-intersection number $-n$ on a smooth algebraic surface is called a $(-n)$-curve. A morphism φ from a smooth algebraic surface V to a smooth algebraic curve B is called a $\mathbf{P}^1$-fibration if a general fiber of φ is isomorphic to $\mathbf{P}^1$. Similarly, an $\mathbf{A}^1$-fibration is defined.

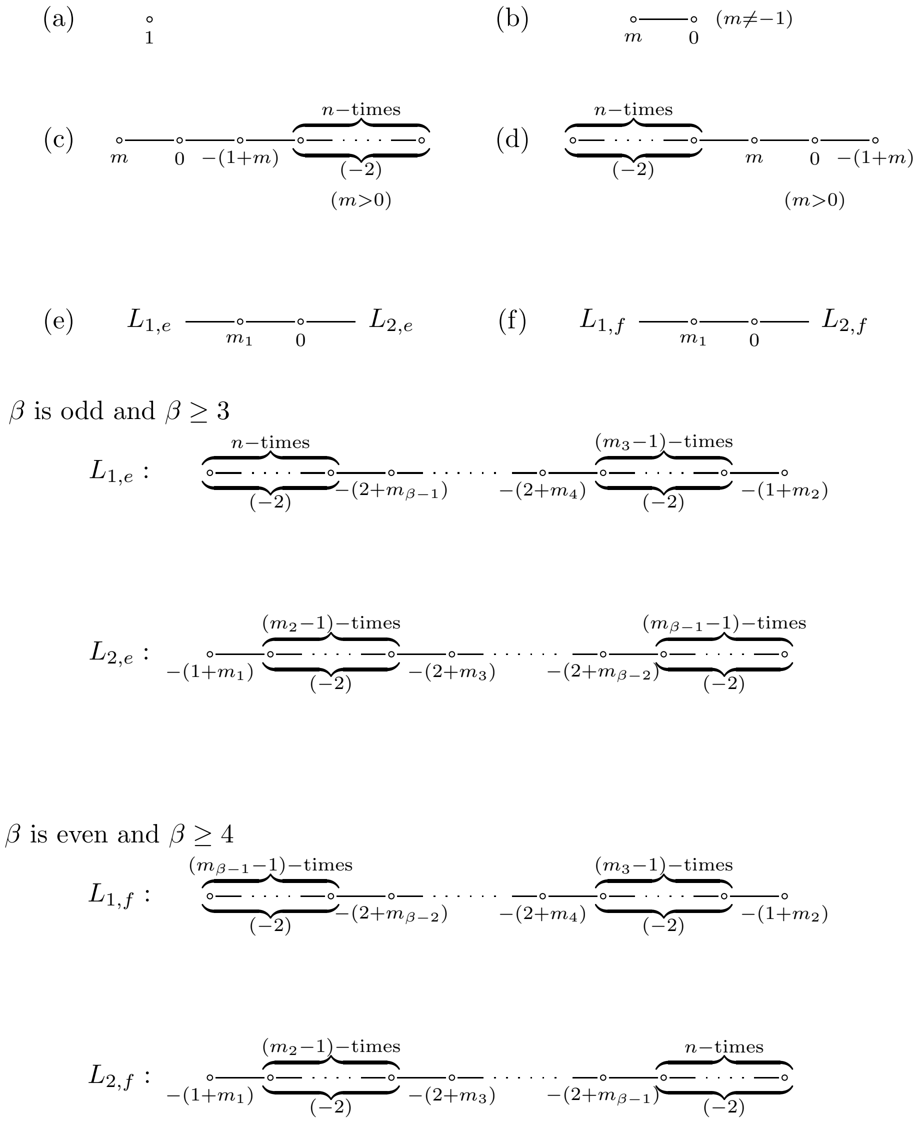
(a)
1
(b)
m 0 (m≠−1)
(c)
n−times
m 0 −(1+m) (−2)
(m>0)
(d)
n−times
(−2) m 0 −(1+m)
(m>0)
(e) L1,e m1 0 L2,e
(f) L1,f m1 0 L2,f
β is odd and β ≥ 3
L1,e :
n−times (−2) −(2+mβ−1) −(2+m4) (m3−1)−times (−2) −(1+m2)
L2,e :
−(1+m1) (m2−1)−times (−2) −(2+m3) −(2+mβ−2) (mβ−1−1)−times (−2)
β is even and β ≥ 4
L1,f :
(mβ−1−1)−times (−2) −(2+mβ−2) −(2+m4) (m3−1)−times (−2) −(1+m2)
L2,f :
−(1+m1) (m2−1)−times (−2) −(2+m3) −(2+mβ−1) n−times (−2)

Figure 1

The following elementary result about singular fibers of a $\mathbf{P}^1$-fibration on a smooth projective surface is useful in the various situations (cf. Miyanishi [7, Chapter I, 4.4.1]).

Lemma 2.1. *Let $f : V \to B$ be a $\mathbf{P}^1$-fibration on a smooth projective surface, where B is a smooth complete algebraic curve. Let F be a singular fiber of f, i.e., F is not isomorphic to $\mathbf{P}^1$. Then the following assertions are true.*

(1) *The reduced curve F_{red} is a divisor with simple normal crossings and each irreducible component of F_{red} is isomorphic to $\mathbf{P}^1$. Furthermore, the dual graph of F is a tree.*

(2) *At least one of the irreducible components of F is a (-1)-curve.*

(3) *If a (-1)-curve E occurs with multiplicity 1 in the fiber F, then F contains another (-1)-curve.*

(4) *Any (-1)-curve in F meets at most two other irreducible components in F.*

Remark 2.2. Suppose that the support of a fiber F of f is written as

$$\mathrm{Supp}\,(F) = C + \Gamma_1 + \Gamma_2 + \Gamma_3,$$

where C is an irreducible component, $\mathrm{Supp}\,(F) - C$ is a disjoint union of three connected parts Γ_i's and Γ_1, Γ_2 contain no (-1) components. Then Γ_3 is contractible to a smooth point. For otherwise, after the contraction, say τ, of all possible contractible components in Γ_3, the part $\tau(F) - \tau(C)$ contains no (-1)-curves. So, $\tau(C)$ is a unique (-1)-curve in the fiber $\tau(F)$ by Lemma 2.1 (2). Then $\tau(C)$ meets three distinct fiber components. This is a contradiction to Lemma 2.1 (4).

We shall recall the definitions of *Euclidean transformation* and *EM-transformation*, which will play very important roles in the subsequent arguments of Section 3 to prove Theorem 1.1. Let V_0 be a smooth projective surface, let p_0 be a point on V_0 and let l_0 be an irreducible curve on V_0 such that p_0 is a simple point of l_0. Let d_0 and d_1 be positive integers such that $d_1 < d_0$. By the Euclidean algorithm with respect to $d_1 < d_0$, we find positive integers $d_2, \ldots, d_\alpha$ and $q_1, \ldots, q_\alpha$:

$$\left\{\begin{array}{rcll} d_0 & = & q_1 d_1 + d_2, & d_2 < d_1, \\ d_1 & = & q_2 d_2 + d_3, & d_3 < d_2, \\ \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ d_{\alpha-2} & = & q_{\alpha-1} d_{\alpha-1} + d_\alpha, & d_\alpha < d_{\alpha-1}, \\ d_{\alpha-1} & = & q_\alpha d_\alpha, & q_\alpha > 1. \end{array}\right.$$

Set $N := \sum_{s=1}^{\alpha} q_s$. Define the infinitely near points p_i's of p_0 for $1 \le i < N$ and the blowing-up $\sigma_i : V_i \to V_{i-1}$ with center at p_{i-1} for $1 \le i \le N$ inductively as follows:

(i) p_i is an infinitely near point of order one of p_{i-1} for $1 \le i < N$.

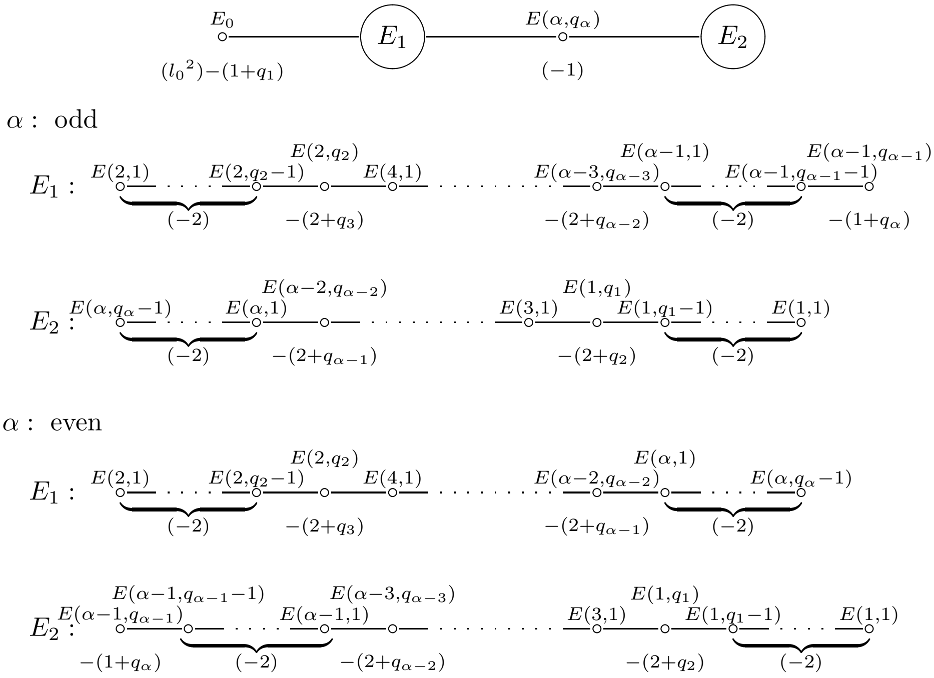

Figure 2

(ii) Let $E_i := \sigma_i^{-1}(p_{i-1})$ for $1 \le i \le N$ and let $E(s,t) := E_i$ if $i = q_1 + \cdots + q_{s-1} + t$ with $1 \le s \le \alpha$ and $1 \le t \le q_s$, where we set $q_0 := 0$ and $E(0,0) := l_0$. The point p_i is an intersection point of the proper transform of $E(s-1, q_{s-1})$ on V_i and the exceptional curve $E(s,t)$ if $i = q_1 + \cdots + q_{s-1} + t$ with $1 \le s \le \alpha$ and $1 \le t \le q_s$ ($1 \le t < q_\alpha$ if $s = \alpha$).

Then a composite $\sigma := \sigma_1 \cdots \sigma_N$ is called an *Euclidean transformation* associated with the datum $\{p_0, l_0, d_0, d_1\}$ (cf. Miyanishi [5], [6, p. 92]). The weighted dual graph of Supp $(\sigma^{-1}(l_0))$ is given in Figure 2, where $E_0 := \sigma'(l_0)$ which denotes the proper transform of l_0 by σ and where we denote the proper transform of $E(s,t)$ on V_N by the same notation. Let C_0 be an irreducible curve on V_0 such that p_0 is a one-place point of C_0, let d_0 be the local intersection number $i(C_0 \cdot l_0; p_0)$ of C_0 and l_0 at p_0 and let d_1 be the multiplicity $\mathrm{mult}_{p_0}(C_0)$ of C_0 at p_0. Assume that $d_0 > d_1$. The proper transform $C_i := (\sigma_1 \cdots \sigma_i)'(C_0)$ passes through p_i so that $(C_i \cdot E(s,t)) = d_s$ and the intersection number of C_i with the proper transform of $E(s-1, q_{s-1})$ on V_i is $d_{s-1} - td_s$, where $i = q_1 + \cdots + q_{s-1} + t$. The smaller one of d_s and $d_{s-1} - td_s$ is the multiplicity of C_i at p_i for p_i is a one-place point of C_i. Note that the proper transform $\sigma'(C_0)$ on V_N meets the last exceptional curve $E(\alpha, q_\alpha)$ with order d_α and does not meet $E_0 := \sigma'(l_0)$ and other exceptional curves arising in the blowing-up process σ.

We now explain EM-transformations, which is called an (e, i)-transformation in Miyanishi [5] and [6, p. 100]. Let V_0, p_0 and l_0 be the same as above. Let $r > 0$ be a positive integer. An *equi-multiplicity transformation* (or *EM-transformation*, for short) of *length* r with center at p_0 is a composite $\tau = \tau_1 \cdots \tau_r$ of blowing-ups defined as follows. For $1 \le i \le r$, $\tau_i : V_i \to V_{i-1}$ is defined inductively as the blowing-up with center at p_{i-1} and p_i is a point on $\tau_i^{-1}(p_{i-1})$ other than the intersection point $\tau_i'(\tau_{i-1}^{-1}(p_{i-2})) \cap \tau_i^{-1}(p_{i-1})$ ($\tau_1'(l_0) \cap \tau_1^{-1}(p_0)$ if $i = 1$). Let C_0 be an irreducible curve on V_0 such that p_0 is a one-place point of C_0. Suppose $d_0 := i(C_0 \cdot l_0; p_0)$ is equal to $d_1 := \text{mult}_{\,p_0}(C_0)$. Let $\tau_1 : V_1 \to V_0$ be the blowing-up with center p_0, and set $E_1 := \tau_1^{-1}(p_0)$ and $C_1 := \tau_1'(C_0)$. Then the point $p_1 := C_1 \cap E_1$ differs from $\tau_1'(l_0) \cap E_1$. Set $d_0^{(1)} := i(C_1 \cdot E_1; p_1) = d_1$ and $d_1^{(1)} := \text{mult}_{\,p_1}(C_1)$. Suppose $d_0^{(1)} = d_1^{(1)}$. As above, let $\tau_2 : V_2 \to V_1$ be the blowing-up with center p_1, let $E_2 := \tau_2^{-1}(p_1)$ and let $C_2 := \tau_2'(C_1)$. Then $p_2 := C_2 \cap E_2$ differs from the point $\tau_2'(E_1) \cap E_2$. Thus this process can be repeated as long as the intersection number of the proper transform of C_0 with the last exceptional curve is equal to the multiplicity of the proper transform of C_0 at the intersection point. If we perform the blowing-ups r times, the composite of r blowing-ups is an EM-transformation of length r.

3. Proof of Theorem 1.1

In this section, we shall determine all the boundary graphs for m.n.c.'s of $\mathbf{A}^2$. So, let $\mathbf{A}^2 \hookrightarrow V$ be an m.n.c. and $D := V - \mathbf{A}^2$ denote the boundary divisor. Let $C_0 \cong \mathbf{A}^1$ be an affine line in $\mathbf{A}^2$ and let g be an irreducible polynomial of $\Gamma(\mathcal{O}_{\mathbf{A}^2})$ defining the curve C_0. This polynomial g defines a polynomial map:

$$\varphi : \mathbf{A}^2 \to \mathbf{A}^1 = \text{Spec}\,(\mathbf{C}[g]),\ \ P \mapsto g(P).$$

By the Abhyankar-Moh Embedding Theorem (cf. [1]), all the fibers of φ are isomorphic to $\mathbf{A}^1$ scheme-theoretically. Since the base curve $\mathbf{A}^1$ of φ is rational, the closures of fibers of φ in V generate an irreducible linear pencil Λ on V such that $\Phi_\Lambda|_{\mathbf{A}^2}$ coincides with φ, where Φ_Λ is the rational mapping defined by Λ. We consider two cases according as Λ has base points or not.

Case I. Λ has no base points.

Then Λ defines a $\mathbf{P}^1$-fibration $\Phi_\Lambda : V \to \mathbf{P}^1$. The boundary D contains a cross-section S of Φ_Λ and all other irreducible components of D are contained in the fibers of it. We write $D - S$ as a disjoint union:

$$D - S = B_1 + \cdots + B_r,$$

where B_i is a connected component of $D - S$ for $1 \le i \le r$. Then we have the following result:

Lemma 3.1. *$r = 1$ and B_1 is an irreducible component with self-intersection number zero.*

Proof. Since the $\mathbf{A}^1$-fibration φ is parametrized by the affine line $\mathbf{A}^1$, one of $B_1, \ldots, B_r$, say B_1, supports a unique full fiber of Φ_Λ lying outside $\mathbf{A}^2$. Suppose that $r > 1$. Let $Q := \Phi_\Lambda(B_2)$ and C the closure of $\varphi^*(Q)$ in V. Let F be the member of Λ containing B_2. The support of F is a union of B_2 and C. Note that C is contained in F with multiplicity one because all the fibers of φ are irreducible and reduced. Hence there exists a (-1)-curve E in B_2 by Lemma 2.1 (3). But the minimality of D and Lemma 2.1 imply that the (-1)-curve E meets the cross-section S and two distinct components in B_2. Then the contraction of E leads to a contradiction. Hence $r = 1$. If the fiber B_1 is reducible, it contains a (-1)-curve. Then we get a contradiction by the same argumemt as above. Thus B_1 is irreducible and $({B_1}^2) = 0$. □

Thus we have that if $\mathrm{Bs}\,\Lambda = \emptyset$ the dual graph of D is of type (b) in Figure 1.

Case II. Λ has a base point.

Since $G \cap \mathbf{A}^2 \cong \mathbf{A}^1$ for a general member G of Λ, $\mathrm{Bs}\,\Lambda$ consists only of one point, say p_0, which is a one-place point of G and is located on D. Let $\sigma : \widetilde{V} \to V$ be the shortest succession of blowing-ups with centers at p_0 and its infinitely near points such that the proper transform $\widetilde{\Lambda} := \sigma'(\Lambda)$ of Λ has no base points. Then $\widetilde{\Lambda}$ defines a $\mathbf{P}^1$-fibration $\Phi_{\widetilde{\Lambda}} : \widetilde{V} \to \mathbf{P}^1$ on $\widetilde{V}$ such that $\Phi_{\widetilde{\Lambda}}|_{\mathbf{A}^2}$ coincides with φ, where we identify $\sigma^{-1}(\mathbf{A}^2)$ with $\mathbf{A}^2$. Among the components of $\widetilde{V} - \mathbf{A}^2 = \mathrm{Supp}\,(\sigma^{-1}(D))$, the last exceptional curve of σ is a cross-section and all the others are contained in some members of $\widetilde{\Lambda}$. Since φ is parametrized by $\mathbf{A}^1$, exactly one member of $\widetilde{\Lambda}$, say F_∞, lies outside $\mathbf{A}^2$. Note that the process σ is written as a composite of Euclidean transformations and EM-transformations (cf. Section 2 for the definitions) since a general member of Λ has the point p_0 as a one-place point. According to the location of the point p_0, we need to consider two cases:

Case II-(1). p_0 lies on only one component, say A, of D.

We then prove the following result with respect to the process σ:

Lemma 3.2. *The process σ ends with an EM-transformation.*

Proof. Otherwise σ ends with an Euclidean transformation. Let S be the last exceptional component of σ. Then $\mathrm{Supp}\,(\sigma^{-1}(D)) - S$ consists of two connected components, say B_1 and B_2. One of B_1 and B_2, say B_1, contains the proper transform of D and the other B_2 consists of a part of the exceptional components arising from the last Euclidean transformation. Hence the self-intersection number of each component of B_2 is less than or equal to -2. Note that one of B_1 and B_2 supports the member F_∞ of $\widetilde{\Lambda}$ lying outside $\mathbf{A}^2$ and the other plus one irreducible component C such that $C \cap \mathbf{A}^2$ is a fiber of φ supports a member F of $\widetilde{\Lambda}$ different from F_∞. It is clear that B_2 cannot support F_∞ because there are no (-1) components contained in it. Meanwhile, it is also impossible that B_2+C supports F. Indeed, otherwise, by Lemma 2.1 (3), there exists a (-1)-curve in B_2 since the multiplicity of C in F is one (see the argument at the beginning of this section). This is a contradiction. $\square$

We write σ as:

$$\sigma = \tau_1 \cdot \sigma_1 \cdots \tau_{n-1} \cdot \sigma_{n-1} \cdot \tau_n \quad \text{with} \quad n \geq 1,$$

where σ_j (resp. τ_j) is the j-th Euclidean transformation (resp. EM-transformation) in σ. Although τ_j for $1 \leq j < n$ might be the identity morphism, τ_n is not so by Lemma 3.2. For $1 \leq j < n$, let $\mathcal{D}_j = \{p_0^{(j)}, l_0^{(j)}, d_0^{(j)}, d_1^{(j)}\}$ be the datum of σ_j, let $d_2^{(j)}, \ldots, d_{\alpha_j}^{(j)}$ and $q_1^{(j)}, \ldots, q_{\alpha_j}^{(j)}$ be positive integers obtained by the Euclidean algorithm with respect to $d_0^{(j)} > d_1^{(j)}$ and let $E^{(j)}(s,t)$ denote the proper transform on $\widetilde{V}$ of the exceptional component arising from the $(q_1^{(j)} + \cdots + q_{s-1}^{(j)} + t)$-th blowing-up in σ_j for $1 \leq s \leq \alpha_j$ and $1 \leq t \leq q_s^{(j)}$. To simplify the notations we put $C_j := E^{(j)}(\alpha_j, q_{\alpha_j}^{(j)})$ for $1 \leq j \leq n$, that is, C_j is the proper transform on $\widetilde{V}$ of the last exceptional component arising from σ_j. Let l_j be the length of τ_j and $E^{(j)}(k)$ the proper transform on $\widetilde{V}$ of the exceptional component from the k-th blowing-up in τ_j for $1 \leq k \leq l_j$ and $1 \leq j \leq n$. Note that the last exceptional component $E^{(n)}(l_n)$ is a cross-section of $\widetilde{\Lambda}$ and $\mathrm{Supp}\,(\sigma^{-1}(D)) - E^{(n)}(l_n)$ supports F_∞. We can specify σ_j and τ_j as follows:

Lemma 3.3. *With the notations as above, the following hold*:

(1) *For $2 \leq j \leq n$, we have $l_j > 0$, i.e., τ_j is not the identity. But τ_1 might be the identity.*

(2) *For $1 \leq j < n$, τ_{j+1} determines the foregoing σ_j as follows*:

$$\begin{cases} \alpha_j = 1 \text{ and } q_1^{(j)} = 2 & \text{if} \quad l_{j+1} = 1 \\ \alpha_j = 2, q_1^{(j)} = 1 \text{ and } q_2^{(j)} = l_{j+1} & \text{if} \quad l_{j+1} > 1 \end{cases} \quad (j \neq n-1),$$

$$\begin{cases} \alpha_{n-1} = 1 \text{ and } q_1^{(n-1)} = 2 & \text{if} \quad l_n = 2 \\ \alpha_{n-1} = 2, q_1^{(n-1)} = 1 \text{ and } q_2^{(n-1)} = l_n - 1 & \text{if} \quad l_n > 2. \end{cases}$$

Proof. Near the cross-section $E^{(n)}(l_n)$, the fiber F_∞ lying outside $\mathbf{A}^2$ has the configuration as in Figure 3, where $S_{n-1}(+)$ (resp. $S_{n-1}(-)$) consists of $E^{(n-1)}(s,t)$ with s even (resp. odd) except for the last component C_{n-1} from σ_{n-1}, T_n is a linear chain supported by the (-2)-curves $E^{(n)}(1), \ldots, E^{(n)}(l_n - 1)$. Note that $S_{n-1}(-)$ and T_n contain no (-1)-curves, so the leftside of C_{n-1} in Figure 3 is contractible by Remark 2.2.

After the contraction of it, the image of F_∞ has the configuration as described in Figure 4, where the (-1)-curve is the image of C_{n-1}. Hence $S_{n-1}(-)$ consists only of one component $E^{(n-1)}(1,1)$ with self-intersection number $-l_n$.

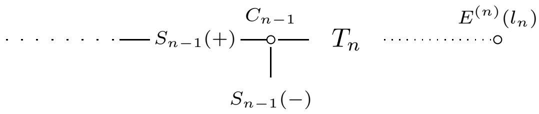

Figure 3

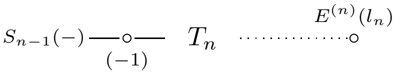

Figure 4

Since $(E^{(n-1)}(1,1)^2) \leq -2$, we have $l_n \geq 2$. Since $S_{n-1}(-)$ consists only of $E^{(n-1)}(1,1)$ it follows that $\alpha_{n-1} \leq 2$. If $l_n = 2$ we have $\alpha_{n-1} = 1$ and $q_1^{(n-1)} = 2$. Indeed, if $\alpha_{n-1} = 2$ the self-intersection number of $E^{(n-1)}(1,1)$ is less than -2. Meanwhile, if $l_n > 2$ then it is not hard to show that $\alpha_{n-1} = 2$. If $q_1^{(n-1)} > 1$ then $S_{n-1}(-)$ contains at least two components, which is a contradiction. Hence $q_1^{(n-1)} = 1$. Since $(E^{(n-1)}(1,1)^2) = -(1 + q_2^{(n-1)}) = -l_n$, we have that $q_2^{(n-1)} = l_n - 1$. Note that $S_{n-1}(+)$ is void if $\alpha_{n-1} = 1$ or a linear chain of (-2)-curves if $\alpha_{n-1} = 2$. We prove the following claim:

Claim. *τ_{n-1} is not the identity if $n \geq 3$.*

Proof. Assume the contrary that $\tau_{n-1} = id$. Then the fiber F_∞ has the configuration as in Figure 5, where $S_{n-2}(+)$ (resp. $S_{n-2}(-)$) consists of $E^{(n-2)}(s,t)$ with s even (resp. odd) except for the last component C_{n-2} from σ_{n-2}. Note that $S_{n-2}(-)$ and the rightside of C_{n-2} contain no (-1)-curves. Hence the leftside of C_{n-2} is contractible (Remark 2.2). After the contraction of it, we contract C_{n-2} and the components in $S_{n-1}(+)$. Then the image of C_{n-1} is a (-1)-curve meeting three other fiber components. This is a contradiction to Lemma 2.1 (4). Hence we obtain that τ_{n-1} is not the identity.

By repeating the above arguments downward $n, n-1, \ldots$, we can prove the assertions of Lemma 3.3. □

Note that the process σ to resolve $\mathrm{Bs}\,\Lambda = \{p_0\}$ does not affect on the components of the boundary D other than A, and that the proper transform of D is contained in a member F_∞ of $\widetilde{\Lambda}$. Every component in F_∞ other than the proper transform $A' := \sigma'(A)$ of A has self-intersection number less than

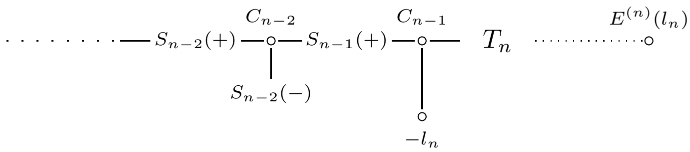

Figure 5

$$L \;\overset{A'}{-\!\circ\!-}\; T(l_1) - S(l_2) - T(l_2) - \cdots\cdots - S(l_{n-1}) - T(l_{n-1}) - S(l_n-1) - \; T_n \;\cdots\cdots\cdots \overset{E^{(n)}(l_n)}{\circ}$$

Figure 6

or equal to -2 by the minimality of D and the constructions of Euclidean and EM-transformations. Hence, unless A' itself is the fiber F_∞, A' is a unique (-1)-curve in F_∞. Note that the case $A' = F_\infty$ occurs only if $D = A$ and σ is a single blowing-up. In this case the dual graph of the boundary D is (a) in Figure 1. We exclude the case $A' = F_\infty$ in the subsequent arguments. If $D - A$ consists of two or more other connected parts, the (-1)-curve A' meets three or more other fiber components of F_∞ including an exceptional component of σ. This is a contradiction to Lemma 2.1 (4). Thus D is written as

$$D = A + L,$$

where L is connected (L might be void). By Lemma 3.3, the dual graph of the fiber F_∞ is given as in Figure 6, where the notations $S(l)$ and $T(l)$ are those defined in Figure 7, T_n is a linear chain supported by the (-2)-curves $E^{(n)}(1), \ldots, E^{(n)}(l_n - 1)$ and where the part $T(l_1)$ may be empty.

We then prove the following result:

Lemma 3.4.

(1) *The self-intersection number* $({A'}^2)$ *of* A' *is written in terms of* (A^2) *as follows*:

$$({A'}^2) = \begin{cases} (A^2) - 2 & \text{if} \quad T(l_1) = \emptyset, \\ (A^2) - 1 & \text{if} \quad T(l_1) \neq \emptyset. \end{cases}$$

(2) *The part* L *is determined by* $T(l_1)$ *as follows*:

(i) *If* $T(l_1) = \emptyset$, L *is empty.*

(ii) *If* $T(l_1) \neq \emptyset$, L *consists only of one component with self-inter section number* $-(1 + l_1)$ *if* $n \geq 2$ (*resp.* $-l_1$ *if* $n = 1$).

Proof. (1) If $T(l_1) = \emptyset$, the process σ starts with an Euclidean transformation σ_1. The datum of σ_1 is $\{p_0, A, d_0^{(1)}, d_1^{(1)}\}$, where $d_0^{(1)} := i(G \cdot A; p_0), d_1^{(1)} := \text{mult}_{\,p_0} G$ for a general member G of Λ. By Lemma 3.3, we have either $d_0^{(1)} = 2d_1^{(1)}$ or $d_0^{(1)} = d_1^{(1)} + d_2^{(1)}$ with $d_2^{(1)} < d_1^{(1)}$. Hence, we blow up in σ two points p_0 and its infinitely near point on A. Thus $({A'}^2) = (A^2) - 2$. On the other hand, if $T(l_1) \neq \emptyset$ then we blow up only one point p_0 on A in σ. Hence $({A'}^2) = (A^2) - 1$.

(2) The dual graph of F_∞ (cf. Figure 6) determines the part L as given in the statement in such a way that F_∞ is brought to a smooth fiber by contractions of successive (-1)-curves starting with A'. $\square$

In view of Lemma 3.4 and $(A'^2) = -1$, the dual graph of the boundary D is of type either (a) or (b) with $m \leq -2$ in Figure 1.

Case II-(2). p_0 is an intersection point of two components of D.

Let A and B be the components of D such that $A \cap B = \{p_0\}$. We write D as

$$D = A + B + L_1 + L_2,$$

where L_1 (resp. L_2) is the union of all connected components of $D - (A + B)$ which are linked to A (resp. B). For a general member G of Λ, we put $d_1 := \operatorname{mult}_{p_0} G$. Since p_0 is a one-place point of G, at least one of $i(G \cdot A; p_0)$ and $i(G \cdot B; p_0)$ is equal to d_1. We need to consider three subcases according as how G intersects A and B at p_0.

Case II-(2)-(i). $i(G \cdot A; p_0) > i(G \cdot B; p_0) = d_1$.

We put $d_0 := i(G \cdot A; p_0)$ and obtain positive integers $d_2, \ldots, d_\alpha$ and $q_1, \ldots, q_\alpha$ by the Euclidean algorithm with respect to $d_0 > d_1$ as performed in Section 2. The shortest process σ to resolve the base points of Λ starts with the Euclidean transformation σ_1 associated with the datum $\mathcal{D}_1 = \{p_0, A, d_0, d_1\}$. Let $E(s, t)$ denote the proper transform on $\widetilde{V}$ of the exceptional components arising from the $(q_1 + \cdots + q_{s-1} + t)$-th blowing-up in σ_1 for $1 \leq s \leq \alpha$ and $1 \leq t \leq q_s$. With the indexing slightly changed, we write σ as

$$\sigma = \sigma_1 \cdot \tau_1 \cdots \sigma_n \cdot \tau_n \quad \text{with} \quad n \geq 1,$$

where σ_j (resp. τ_j) is the j-th Euclidean transformation (resp. EM-transformation) in σ and τ_j might be the identity. The same argument as in the proof of Lemma 3.2 shows that the following result holds:

Lemma 3.5. *If $n \geq 2$, the process σ ends with an EM-transformation.*

We suppose, for the moment, that $n \geq 2$. Let $\mathcal{D}_j = \{p_0^{(j)}, l_0^{(j)}, d_0^{(j)}, d_1^{(j)}\}$ be the datum of σ_j and let $d_2^{(j)}, \ldots, d_{\alpha_j}^{(j)}$ and $q_1^{(j)}, \ldots, q_{\alpha_j}^{(j)}$ be positive integers obtained by the Euclidean algorithm with respect to $d_0^{(j)} > d_1^{(j)}$ for $2 \leq j \leq n$. Let l_j be the length of τ_j for $1 \leq j \leq n$. Let $E^{(j)}(s, t)$ for $1 \leq s \leq \alpha_j, 1 \leq t \leq q_s^{(j)}$ and $E^{(j)}(k)$ for $1 \leq k \leq l_j$ have the same meaning as defined after the proof of Lemma 3.2. We have the following result concerning σ_j and τ_j for $2 \leq j \leq n$:

Lemma 3.6. *Let the notations be the same as above. If $n \geq 2$, we have:*

(1) *For $2 \leq j \leq n$, $l_j > 0$, i.e., τ_j is not the identity. But τ_1 might be the identity.*

(2) *For $2 \leq j \leq n$, τ_j determines the foregoing σ_j as follows:*

$$\left\{ \begin{array}{lll} \alpha_j = 1 \text{ and } q_1^{(j)} = 2 & \text{if} \quad l_j = 1 \\ \alpha_j = 2, q_1^{(j)} = 1 \text{ and } q_2^{(j)} = l_j & \text{if} \quad l_j > 1 \end{array} \right. \quad \text{for } 2 \leq j < n,$$

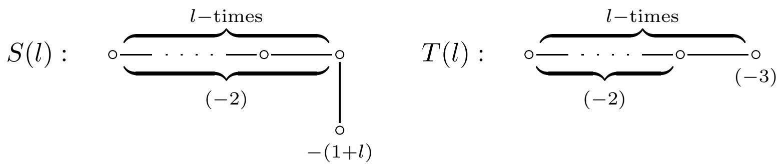

Figure 7

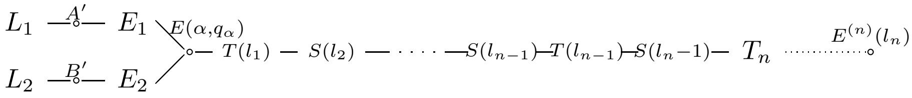

Figure 8

$$\begin{cases} \alpha_n = 1 \text{ and } q_1^{(n)} = 2 & \text{if} \quad l_n = 2, \\ \alpha_n = 2, q_1^{(n)} = 1 \text{ and } q_2^{(n)} = l_n - 1 & \text{if} \quad l_n > 2. \end{cases}$$

Proof. The proof of the present lemma is the same as the one for Lemma 3.3 except for a slight difference of indices. □

Let F_∞ be the member of $\widetilde{\Lambda}$ lying outside $\mathbf{A}^2$. With the notations preceding Lemma 3.6, the last exceptional curve $E^{(n)}(l_n)$ is a cross-section of $\widetilde{\Lambda}$ and the support of F_∞ is Supp $(\sigma^{-1}(D)) - E^{(n)}(l_n)$. By Lemma 3.6, the dual graph of F_∞ is given as in Figure 8, where A' (resp. B') is the proper transform of A (resp. B), the notations $S(l), T(l)$ are those defined in Figure 7, T_n is a linear chain supported by the (-2)-curves $E^{(n)}(1), \ldots, E^{(n)}(l_n - 1)$ and where $T(l_1)$ may be empty. The dual graph of $E_1 + E(\alpha, q_\alpha) + E_2$ is the same as given in Figure 2 with replaced the self-intersection number of $E(\alpha, q_\alpha)$ by -3 if $T(l_1) = \emptyset$ (resp. -2 if $T(l_1) \neq \emptyset$). Note that the process σ does not affect on the components of $L_1 + L_2$.

We prove the following result:

Lemma 3.7. *With the notations as above, let $M_1 := L_1 + A' + E_1$ and $M_2 := L_2 + B' + E_2$ (see Figure 8). If $n \geq 2$ we have the following:*

(1) *Let C_2 be the last exceptional component of σ_2. (The component C_2 corresponds to the rightmost vertex of $S(l_2)$ in Figure 8.) Then the part lying on the leftside of C_2 in Figure 8 is contractible.*

(2) *Suppose $T(l_1) = \emptyset$. Then $E(\alpha, q_\alpha)$ becomes a (-1)-curve after the parts M_1 and M_2 are contracted to smooth points. Moreover, the contractions of the parts M_1 and M_2 start with the contractions of A' and B' and end with the contractions of the components meeting $E(\alpha, q_\alpha)$.*

(3) *Suppose $T(l_1) \neq \emptyset$. If α is odd (resp. if α is even) then M_2 (resp. M_1) is contractible and the contraction of M_2 (resp. M_1) starts with B' (resp. A') and ends with $E(\alpha, q_\alpha - 1)$ (cf. Figure 2).*

Proof. (1) $F_\infty - C_2$ consists of three connected parts. Among them $E^{(2)}(1,1)$, which corresponds to a vertex with weight $-(1 + l_2)$ of $S(l_2)$ in

Figure 8, and the rightside of C_2 contains no (-1)-curves. Hence, the leftside of C_2 is contractible by Remark 2.2.

(2) Suppose that $E(\alpha, q_\alpha)$ becomes a (-1)-curve before M_1 and M_2 are contracted. Then we contract all components between $E(\alpha, q_\alpha)$ and C_2. After this contraction the image of C_2 is a (-1)-curve meeting three or more other irreducible fiber components. This is a contradiction to Lemma 2.1 (4). Thus $E(\alpha, q_\alpha)$ becomes a (-1)-curve after the contractions of M_1 and M_2. Since each component in M_1 and M_2 other than A' and B' has self-intersection number less than or equal to -2 and $E(\alpha, q_\alpha)$ is a (-3)-curve, the contractions of M_1 and M_2 start with A' and B' and end with the components meeting $E(\alpha, q_\alpha)$.

(3) Since $E(\alpha, q_\alpha)$ meets three other fiber components and the rightside of $E(\alpha, q_\alpha)$ in Figure 8 contains no (-1)-curve, at least one of M_1 and M_2 is contractible (see Remark 2.2). But since $E(\alpha, q_\alpha)$ is a (-2)-curve, exactly one of M_1 and M_2 is contractible, and this contraction ends with the contraction of the component meeting $E(\alpha, q_\alpha)$. Suppose that α is odd and M_1 is contractible. After the contraction of M_1, the image of F_∞ contains at least four components and one of them, which is the image of $E(\alpha, q_\alpha)$, is a (-1)-curve. If $l_1 > 1$ then the image of $E(\alpha, q_\alpha)$ meets two (-2)-curves, one in M_2 and the other in $T(l_1)$. This is a contradiction. If $l_1 = 1$ then $T(l_1)$ consists only of a (-3)-curve $E^{(1)}(1)$. After the contraction of M_1, we contract $E(\alpha, q_\alpha), E(\alpha, q_\alpha - 1), E^{(1)}(1)$ and the components in the linear chain between $E^{(1)}(1)$ and C_2 (cf. Figure 8). But it contradicts the assertion (1) of the present lemma. Thus M_2 is contractible if α is odd. Similarly M_1 is contractible if α is even. $\square$

Lemma 3.8. *With the notations as above, suppose that $n \geq 2$. Then we have the following*:

(1) *The self-intersection number of A is determined as follows*:

$$(A^2) = \begin{cases} q_1 & \text{if} \quad \alpha > 1, \\ q_1 - l_1 - 1 & \text{if} \quad \alpha = 1, \end{cases}$$

where we put $l_1 = 0$ when $\tau_1 = id$.

(2) $(B^2) = 0$.

(3) *If $\alpha > 1$ and $\tau_1 = id$ (resp. if $\alpha > 1$ and $\tau_1 \neq id$, resp. if $\alpha = 1$) then the dual graphs of L_1 and L_2 are given as in Figure 9 (resp. as in Figure 9 with (-2) components counted $(q_\alpha - l_1 - 2)$-times instead of $(q_\alpha - 2)$-times, resp. $L_1 = L_2 = \emptyset$), where the $-(1+q_2)$-curve in L_1 meets A and the $-(1+q_1)$-curve in L_2 meets B.*

Proof. Suppose that $\tau_1 = id$, i.e., $T(l_1) = \emptyset$. Then by Lemma 3.7, M_1 and M_2 are contractible and these contractions start with A' and B' and end with the contractions of the components meeting $E(\alpha, q_\alpha)$. Hence A' and B' are (-1)-curves and L_1 and L_2 are uniquely determined by the parts $A' + E_1$ and $B' + E_2$. Noting that $({A'}^2) = (A^2) - (1 + q_1)$ if $\alpha > 1$ (resp. $({A'}^2) = (A^2) - q_1$ if $\alpha = 1$) and $({B'}^2) = (B^2) - 1$, we then easily prove the assertions. Next suppose that $\tau_1 \neq id$, i.e., $T(l_1) \neq \emptyset$. We consider only the case α is odd because the

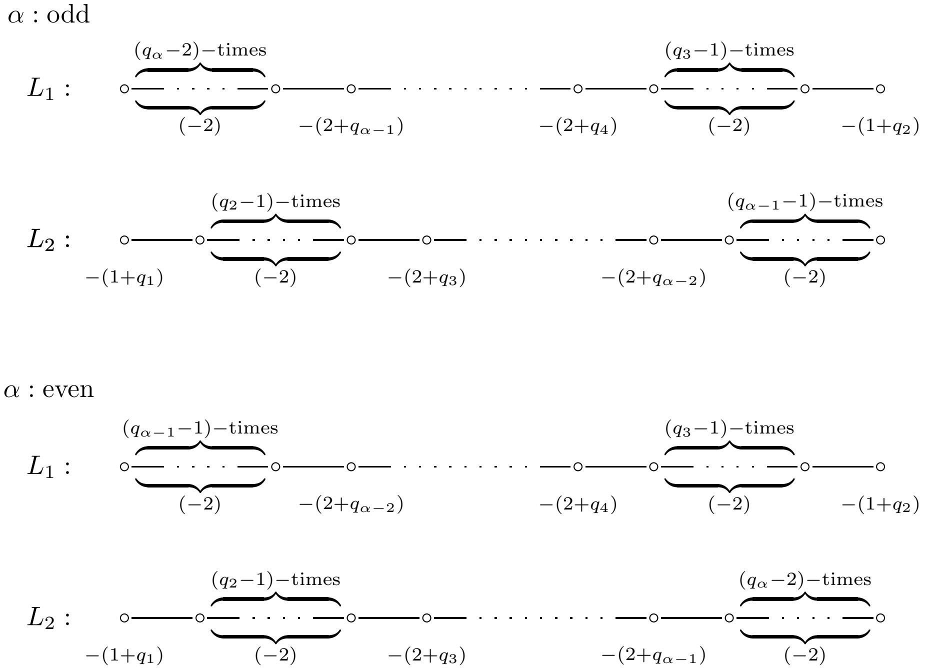

Figure 9

case α is even can be treated similarly. By Lemma 3.7 (3), M_2 is contractible and this contraction starts with B' and ends with $E(\alpha, q_\alpha - 1)$. Hence, B' is a (-1)-curve and L_2 is determined by $B' + E_2$ as described in Figure 9. We contract $E(\alpha, q_\alpha), E^{(1)}(1), \ldots, E^{(1)}(l_1 - 1)$ in this order after the contraction of M_2. Let M_1' be the image of M_1 after the above contraction and let C be the image of the component meeting $E(\alpha, q_\alpha)$ in M_1. The self-intersection number of C increases by l_1. Lemma 3.7 (1) then says that M_1' is contractible and that the contraction of M_1' starts with A' and ends with C. These observations imply the assertions. Note that if $\alpha = 1$, the part E_1 is empty and C is the image of A' satisfying $-1 = (C^2) = ({A'}^2) + l_1 = (A^2) - q_1 + l_1$. $\square$

By Lemma 3.8 we can determine the dual graph of the boundary D as follows. At first, let us suppose $\alpha = 1$. Then $L_1 = L_2 = \emptyset$ and the graph of D is of type (b) in Figure 1. Secondly, suppose $\alpha = 2$. Then $(A^2) = q_1, (B^2) = 0, L_1 = \emptyset$ and L_2 consists of $-(1 + q_1)$-curve plus several (-2) components, so the graph of D is of type (c) in Figure 1. Finally, suppose $\alpha > 2$. Then the graph of D is of type either (e) or (f) in Figure 1 according as α is odd or even by Lemma 3.8.

From now on we consider the case where σ is written as $\sigma = \sigma_1 \cdot \tau_1$. Let l_1 be the length of τ_1 and let $E(k)$ be the proper transform on $\widetilde{V}$ of the exceptional component arising from the k-th blowing-up in τ_1 for $1 \le k \le l_1$ (if $\tau_1 \neq id$). If τ_1 is not the identity, the dual graph of the fiber F_∞ is given

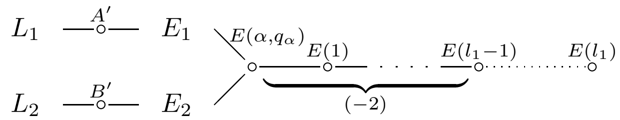

Figure 10

as in Figure 10, where the component $E(l_1)$ is a cross-section of $\widetilde{\Lambda}$, the dual graph of $E_1 + E(\alpha, q_\alpha) + E_2$ is the same as given in Figure 2 with the self-intersection number of $E(\alpha, q_\alpha)$ replaced by -2 and A', B' are respectively the proper transforms of A, B on $\widetilde{V}$.

We prove then the following result:

Lemma 3.9. *Suppose that the shortest process σ to resolve the base points of Λ is written as $\sigma = \sigma_1 \cdot \tau_1$ with $\tau_1 \neq id$. Let $M_1 := L_1 + A' + E_1$ and let $M_2 := L_2 + B' + E_2$ (see Figure 10). Then we have:*

(1) *If $l_1 = 1$ then both of M_1 and M_2 are contractible to smooth points.*

(2) *If $l_1 > 1$ and α is odd (resp. if $l_1 > 1$ and α is even) then M_2 (resp. M_1) is contractible to a smooth point.*

(3) *The contractions of M_1, M_2 stated in (1) (resp. (2)) start with A', B' (resp. one of A', B') and end with the components meeting $E(\alpha, q_\alpha)$.*

Proof. (1) If $l_1 = 1$ then $E(\alpha, q_\alpha)$ meets a cross-section $E(1)$ of $\widetilde{\Lambda}$. So, the multiplicity of $E(\alpha, q_\alpha)$ in F_∞ is one, and we can obtain from F_∞ a smooth fiber which is the image of $E(\alpha, q_\alpha)$. Therefore M_1 and M_2 are contractible to smooth points. Note that all possible (-1)-curves in F_∞ are only A' and B', and that $E(\alpha, q_\alpha)$ is a (-2)-curve. Thus the contraction of M_1 (resp. M_2) starts with A' (resp. B') and ends with the components meeting $E(\alpha, q_\alpha)$.

(2) In the case $l_1 > 1$ we can show the assertion (2) by the same argument as in Lemma 3.7 (3). Moreover, by the same argument there, the contraction of M_2 if α is odd (resp. M_1 if α is even) starts with B' (resp. A') and ends with the contraction of $E(\alpha, q_\alpha - 1)$. The last assertion (3) is thus proved. $\square$

Lemma 3.10. *With the notations and assumptions as in Lemma 3.9, the following assertions hold:*

(1)

$$(A^2) = \begin{cases} q_1 & \text{if} \quad \alpha > 1, \\ q_1 - l_1 & \text{if} \quad \alpha = 1. \end{cases}$$

(2) $(B^2) = 0$.

(3) *If $\alpha > 1$ (resp. if $\alpha = 1$) then the dual graphs of L_1 and L_2 are given as in Figure 9 with (-2) components counted $(q_\alpha - l_1 - 1)$-times instead of $(q_\alpha - 2)$-times (resp. $L_1 = L_2 = \emptyset$).*

Proof. We can prove the assertions from Lemma 3.9 by the same fashion as we proved Lemma 3.8 from Lemma 3.7. $\square$

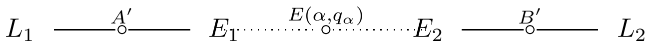

Figure 11

As we determined the dual graph of the boundary D from Lemma 3.8 in the previous case, we can also do the same in the present case from Lemma 3.10. Namely, the graph of D is of type (b) if $\alpha = 1$, of type (c) if $\alpha = 2$ and of type (e) or (f) if $\alpha > 2$ in Figure 1.

We next consider the case where the process σ to resolve $\mathrm{Bs}\,\Lambda$ consists of a single Euclidean transformation σ_1. The configuration of $\mathrm{Supp}\,(\sigma^{-1}(D))$ is then given as in Figure 11, where $E(\alpha, q_\alpha)$ is a cross-section of $\widetilde{\Lambda}$ and the dual graph of $E_1 + E(\alpha, q_\alpha) + E_2$ is the same as given in Figure 2. The component A' (resp. B') is the proper transform of A (resp. B). We then prove the following result:

Lemma 3.11. *Let the notations and assumptions be the same as above. Then one of $M_1 := L_1 + A' + E_1$ and $M_2 := L_2 + B' + E_2$ (see Figure* 11*) supports F_∞ and the other is contractible to a smooth point, starting with the contraction of A' or B'.*

Proof. $\mathrm{Supp}\,(\sigma^{-1}(D)) - E(\alpha, q_\alpha)$ consists of two connected components M_1 and M_2. Since the fibration φ is parametrized by the affine line $\mathbf{A}^1$, one of M_1 and M_2 supports the member F_∞ of $\widetilde{\Lambda}$ lying outside $\mathbf{A}^2$ and the other is contained in a member F of $\widetilde{\Lambda}$ different from F_∞. Since all the fibers of φ are isomorphic to $\mathbf{A}^1$ scheme-theoretically (see the argument at the beginning of this section), it follows that $C_0 := F \cap \mathbf{A}^2 \cong \mathbf{A}^1$ scheme-theoretically. Hence the closure C of C_0 in $\widetilde{V}$ is contained in F with multiplicity one. So, the fiber F can be reduced to a smooth fiber which is the image of C. Thus one of M_1 and M_2 supports F_∞ and the other is contracted to a smooth point. Note that by the minimality of D and the construction of an Euclidean transformation, every component of M_1 and M_2 other than A' and B' has self-intersection number less than or equal to -2. Hence the contraction of M_1 or M_2 starts with A' or B'. $\square$

For each case stated in Lemma 3.11 we can easily determine the dual graph of the boundary D since we know the dual graphs of $A' + E_1$ and $B' + E_2$ concretely (see Figure 11). Namely, we have the following:

Lemma 3.12. *With the notations and assumptions as in Lemma* 3.11*, the following assertions hold*:

(1)

$$(A^2) = \begin{cases} q_1 - 1 & \text{if} \quad \alpha = 1 \text{ and } M_1 \text{ is contractible,} \\ q_1 & \text{otherwise.} \end{cases}$$

(2) $(B^2) = 0$.

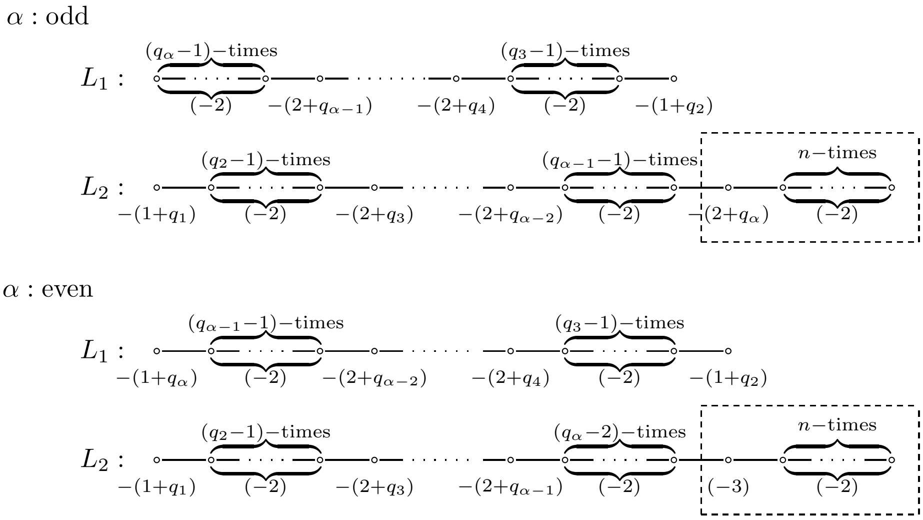

Figure 12

(3) *Suppose that* $\alpha > 1$. *If* M_1 (*resp.* M_2) *supports* F_∞ *and* M_2 (*resp.* M_1) *is contractible, the dual graphs of* L_1 *and* L_2 *are given as in Figure* 12 (*resp. Figure* 13), *where the linear chains in the dotted frames might be empty, n is an arbitrary non-negative integer and where the* $-(1+q_2)$*-curve in* L_1 *and the* $-(1+q_1)$*-curve in* L_2 *meet* A *and* B, *respectively.*

(4) *Suppose that* $\alpha = 1$. *If* M_1 (*resp.* M_2) *supports* F_∞ *and* M_2 (*resp.* M_1) *is contractible, the dual graph of* D *is given by* (i) (*resp.* (ii)) *in Figure* 14, *where the linear chains in the dotted frames might be empty and n is an arbitrary non-negative integer.*

Case II-(2)-(i′). $i(G \cdot B; p_0) > i(G \cdot A; p_0) = d_1$; see the argument after the proof of Lemma 3.4 for the notations.

This case can be treated in the same fashion as in Case II-(2)-(i) and consequently all possible weighted dual graphs of the boundary D are the same as those obtained in Case II-(2)-(i).

Case II-(2)-(ii). $i(G \cdot A; p_0) = i(G \cdot B; p_0) = d_1$.

Then the process σ to resolve the base point $\mathrm{Bs}\,\Lambda = \{p_0\}$ starts with an EM-transformation. Suppose that σ consists only of a single blowing-up. This case can be treated in the same fashion as in Lemmas 3.11 and 3.12 with E_1 and E_2 empty. As a consequence, the dual graph of D is given by 1) in Figure 14 with $q_1 = 1$, that is, of type (b) or (c) in Figure 1 with $m = 1$. In the subsequent we exclude this case. Then the same argument as in Lemma 3.2 shows that the following result holds:

Lemma 3.13. *The process σ ends with an EM-transformation.*

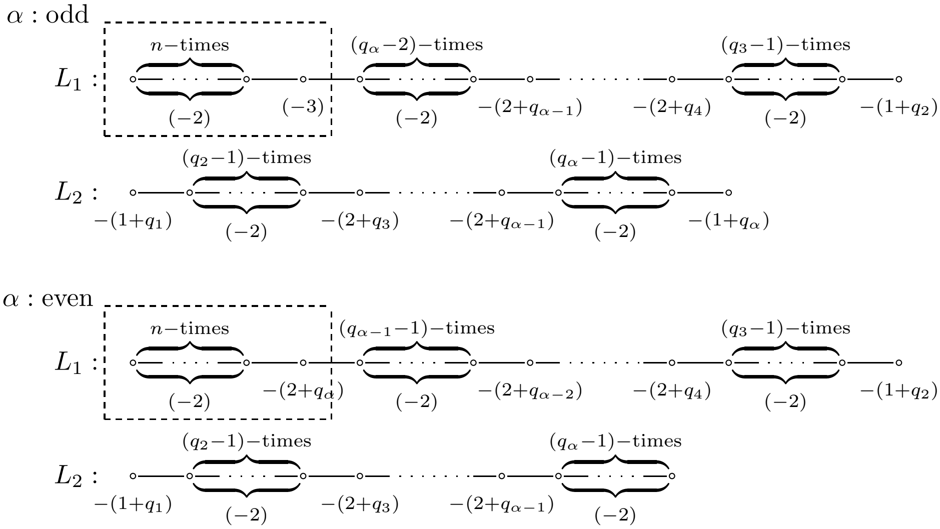

Figure 13

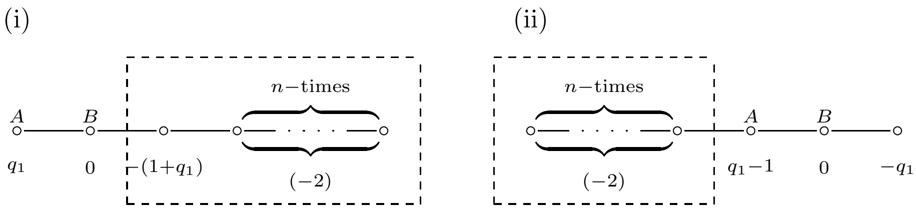

Figure 14

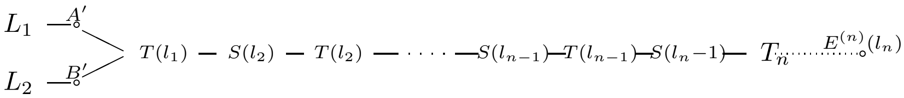

Figure 15

We write σ as

$$\sigma = \tau_1 \cdot \sigma_1 \cdots \tau_{n-1} \cdot \sigma_{n-1} \cdot \tau_n \quad \text{with} \quad n \geq 1,$$

where σ_j (resp. τ_j) is the j-th Euclidean transformation (resp. EM-transformation) in σ. Let the notations $\mathcal{D}_j = \{p_0^{(j)}, l_0^{(j)}, d_0^{(j)}, d_1^{(j)}\}, d_2^{(j)}, \ldots, d_{\alpha_j}^{(j)}, q_1^{(j)}, \ldots, q_{\alpha_j}^{(j)}, l_j, E^{(j)}(s,t)$ and $E^{(j)}(k)$ be the same as those defined after the proof of Lemma 3.2. With these notations, the same assertions as in Lemma 3.3 hold by the same argument there. The last component $E^{(n)}(l_n)$ is a cross-section of $\widetilde{\Lambda}$ and Supp $(\sigma^{-1}(D)) - E^{(n)}(l_n)$ supports the member F_∞ of $\widetilde{\Lambda}$ lying outside $\mathbf{A}^2$. The dual graph of F_∞ is given as in Figure 15, where the notations $S(l)$ and $T(l)$ are those defined in Figure 7 and A', B' are the proper transforms on $\widetilde{V}$ of A, B, respectively.

The part T_n is a linear chain supported by the (-2)-curves $E^{(n)}(1), \ldots, E^{(n)}(l_n - 1)$. We assume for the moment that $n \geq 2$. Then we have the following result:

Lemma 3.14. *With the notations as above, suppose that $n \geq 2$. Then $E^{(1)}(l_1)$, which is the rightmost component of $T(l_1)$ in Figure 15, becomes a (-1)-curve after the leftside of $E^{(1)}(l_1)$ in Figure 15 is contracted to a smooth point.*

Proof. Assume the contrary that $E^{(1)}(l_1)$ becomes a (-1)-curve before the leftside of it is contracted. Then we contract the components in the linear chain between $E^{(1)}(l_1)$ and the last exceptional component C_1 from σ_1, which is the rightmost component of $S(l_2)$. Then the image of C_1 is a (-1)-curve meeting three or more other fiber components. This is a contradiction to Lemma 2.1 (4). □

By Lemma 3.14 we can determine the dual graph of D. Namely, we have the following:

Lemma 3.15. *With the notations and assumptions as above, the dual graph of D is given by (b) in Figure 1 with $m = -l_1 + 1$.*

Proof. Since $E^{(1)}(1)$, which is the leftmost component of $T(l_1)$, meets three other fiber components and the rightside of $E^{(1)}(1)$ in Figure 15 contains no (-1)-curve, at least one of $A' + L_1$ and $B' + L_2$ is contractible (see Remark 2.2). Let us suppose that $A' + L_1$ is contractible. Then A' is a (-1)-curve and

L_1 is empty. Assume the contrary that L_1 is not empty. In the case $l_1 = 1$ (resp. $l_1 > 1$), when we contract $A' + L_1$ the image of $E^{(1)}(1)$ has self-intersection number greater than or equal to -1 (resp. 0). This contradicts Lemma 3.14, so L_1 is empty. We contract the components $A', E^{(1)}(1), \ldots, E^{(1)}(l_1 - 1)$ in this order. Then the images of B' and $E^{(1)}(l_1)$ have self-intersection number $({B'}^2)+l_1-1$ and -2, respectively. Lemma 3.14 then implies that $({B'}^2)+l_1-1 = -1$ and L_2 is empty. Noting that $({A'}^2) = (A^2) - 1$ and $({B'}^2) = (B^2) - 1$, we proved the assertion. In the case where $B'+L_2$ is contracted instead of $A'+L_1$, we can also prove the assertion by the same fashion. $\square$

Finally we consider the case where the process σ consists of a single EM-transformation τ_1. By assumption before Lemma 3.13, the length l_1 of τ_1 is greater than 1. The last component $E^{(1)}(l_1)$ is a cross-section of $\widetilde{\Lambda}$ and the fiber F_∞ is supported by

$$J := A' + L_1 + B' + L_2 + E^{(1)}(1) + \cdots + E^{(1)}(l_1 - 1).$$

The configuration of F_∞ is described as in Figure 15 with the rightside of $T(l_1)$ excluded.

We prove the following result:

Lemma 3.16. *With the notations and assumptions as above, the dual graph of the boundary D is* (*b*) *in Figure* 1 *with $m = -l_1 + 2$.*

Proof. The proof is somewhat similar to the one of Lemma 3.15. Note that $E^{(1)}(1)$ is a (-2)-curve meeting three other fiber components or two fiber components and a cross-section. In order to obtain a smooth fiber from F_∞, one of A' and B', say A', is a (-1)-curve and L_1 is accordingly empty. We contract $A', E^{(1)}(1), \ldots, E^{(1)}(l_1 - 1)$ in this order. Let B'' be the image of B' after this contraction. Since B'' meets a cross-section, we can reduce $B''+L_2$ to a smooth fiber retaining B'' as a final component. So, if L_2 is not empty it contains a (-1)-curve. But this is impossible by the minimality of D. Thus we know that L_2 is empty and B'' itself is a smooth fiber. Noting that $({A'}^2) = (A^2) - 1$ and $({B''}^2) = (B^2) + l_1 - 2$, we proved the assertion. $\square$

We thus obtain that for any m.n.c. of $\mathbf{A}^2$ the dual graph of the boundary is given by the one of (a)–(f) in Figure 1.

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. So, suppose that a smooth affine surface X has a normal compactification $X \hookrightarrow V$ such that each irreducible component of the boundary divisor $D := V - X$ is rational and the weighted dual graph of D is the same as that of the boundary of a suitable normal compactification of the affine plane $\mathbf{A}^2$. Then D can be brought to a smooth rational curve $\overline{D}$ with the self-intersection number 1 on a smooth projective surface $\overline{V}$ after suitable blowing-ups and blowing-downs with centers outside X, because any boundary divisor with respect to the normal compactification of an affine plane $\mathbf{A}^2$ can be brought to a line on a projective plane $\mathbf{P}^2$ via suitable blowing-ups and blowing-downs with centers outside $\mathbf{A}^2$. Now we prepare the following lemma by which we know that $\overline{V}$ is a rational surface:

Lemma 4.1. *Let W be a smooth projective surface. Suppose that there exists a smooth rational curve C on W with positive self-intersection number. Then W is a rational surface.*

Proof. Since $(C \cdot K_W) = -(C^2) - 2 < 0$ and $(C^2) > 0$, it is easy to see that W has negative Kodaira dimension, i.e., W is ruled. Suppose that W has the positive irregularity $q(W) > 0$. Then there is a $\mathbf{P}^1$-fibration

$$f : W \to B,$$

where B is a smooth projective curve with genus $q(W)$. Since $(C^2) > 0$, the curve C is not contained in a fiber of f. Hence C is a quasi-section of f. But this is impossible by Lüroth theorem because C is rational. Thus we have that $q(W) = 0$ and consequently W is rational. □

We blow-up $\overline{V}$ at a point of $\overline{D}$. Let V' be the resulting surface, let D' be the proper transform of $\overline{D}$ on V' and let E' be the exceptional curve. Since $D' \cong \mathbf{P}^1$ and $({D'}^2) = 0$, we have $\mathcal{O}_{D'}(D') \cong \mathcal{O}_{\mathbf{P}^1}$. Consider an exact sequence:

$$0 \to \mathcal{O}_{V'} \to \mathcal{O}_{V'}(D') \to \mathcal{O}_{\mathbf{P}^1} \to 0.$$

Noting that $H^1(V', \mathcal{O}_{V'}) = 0$ because of the rationality of V' (see Lemma 4.1), the induced cohomology exact sequence implies that $h^0(V', \mathcal{O}_{V'}(D')) = 2$. The linear pencil $|D'|$ defines a $\mathbf{P}^1$-fibration $h : V' \to \mathbf{P}^1$ and E' is a cross-section of h. All the fibers of h are irreducible since $X = V' - (D' \cup E')$ is affine and E' is a cross-section of h. Thus we know that $h|_X$ is an $\mathbf{A}^1$-bundle over $\mathbf{A}^1$. Hence $X \cong \mathbf{A}^1 \times \mathbf{A}^1 \cong \mathbf{A}^2$. This completes the proof of Theorem 1.2.

Remark 4.2. In the statement of Theorem 1.2, if we replace the affine plane $\mathbf{A}^2$ by another $\mathbf{Q}$-homology plane then a similar result does not hold in general. For instance, H. Flenner-M. Zaidenberg constructed a family $(V_t, D_t); t \in \mathbf{A}^1$, of smooth projective surfaces V_t and divisors D_t on V_t such that, for $t \neq t'$, the affine surfaces $X_t := V_t - D_t$ and $X_{t'} := V_{t'} - D_{t'}$ are non-isomorphic $\mathbf{Q}$-homology planes, whereas the boundaries $D_t, D_{t'}$ are isomorphic with weight (cf. [3, Example 4.16]).

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Added in proof: Even if we replace, in the statement of Theorem 1.2, X; a *smooth* affine surface by X: a *normal* affine surface, we can show that X is isomorphic to the affine plane $\mathbf{A}^2$. For the details see T. Kishiomto, Abhyankar-Sathaye Embedding problem in dimension Three, preprint.