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Classification theorems for cohomology rings of finite H-spaces

Dedicated to Prof. John Hubbuck on his 60th birthday

By

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1. Introduction

Let X denote a 1-connected mod 3 finite H-space with $H_*(X;\mathbb{F}_3)$ associative. Hemmi and Lin in [9] show that the even degree algebra generators of $H^*(X;\mathbb{F}_3)$ lie in degrees 8 and 20 and the homology Hopf algebra $H_*(X;\mathbb{F}_3)$ is primitively generated. Then, Lin in [21] shows that the cohomology algebra $H^*(X;\mathbb{F}_3)$ is isomorphic to the mod 3 cohomology algebra of a finite product of E_8's, $X(3)$'s, and odd (≥ 3) dimensional spheres where E_8 is the compact, connected, simple, exceptional Lie group of rank 8 and $X(3)$ is the 1-connected mod 3 finite H-space constructed by Harper [7].

The purpose of this paper is, for X as above, to study $K(n)_*(X)$, $n=2,3$, where $K(n)$ is the n-th periodic Morava K-theory at the prime 3 (see Johnson and Wilson [10]) and then to show the following theorem.

Theorem 1. *Let X be a* 1*-connected mod* 3 *finite H-space such that $H_*(X)$ is associative.* (*The coefficient of the* (*co*)*homology is* $\mathbb{F}_3$*.*)

(i) *If $K(2)_*(X)$ is associative, then we have the following inequalities*:

$$\dim_{\mathbb{F}_3} \mathrm{Q}H^{15}(X) - \dim_{\mathbb{F}_3} \mathrm{P}H^{15}(X) \geq \dim_{\mathbb{F}_3} \mathrm{Q}H^{8}(X),$$
$$\dim_{\mathbb{F}_3} \mathrm{Q}H^{11}(X) - \dim_{\mathbb{F}_3} \mathrm{P}H^{11}(X) \geq \dim_{\mathbb{F}_3} \mathrm{Q}H^{8}(X) - \dim_{\mathbb{F}_3} \mathrm{Q}H^{20}(X).$$

(ii) *If $K(3)_*(X)$ is associative, then we have the following inequalities*:

$$\dim_{\mathbb{F}_3} \mathrm{Q}H^{j}(X) - \dim_{\mathbb{F}_3} \mathrm{P}H^{j}(X) \geq \dim_{\mathbb{F}_3} \mathrm{Q}H^{20}(X), \quad j = 27, 35, 39, 47.$$

Here, the symbols P and Q denote the primitives and the indecomposables respectively. (See Milnor and Moore [23].) In [21], the inequalities in (ii), $j=35,47$, (and the inequalities $\dim_{\mathbb{F}_3} \mathrm{Q}H^{j}(X) \geq \dim_{\mathbb{F}_3} \mathrm{Q}H^{20}(X)$, $j=27,39$,) are shown without the assumption that $K(3)_*(X)$ is associative. In this paper,

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we show all of the above inequalities in one principle. The key of the principle is Lemma 5. Applying it several times to $K(2)_*(X)$ and $K(3)_*(X)$, we can show Theorem 1. The principle can also be applied to the study of the mod p cohomology of a 1-connected mod p finite H-space where p is an odd prime ≥ 5. (However, we need extra assumptions corresponding to the theorem of Hemmi and Lin for $p = 3$, [9].) For example, we can show the following theorem, of which we omit the proof. (See Yagita [36]. Also see Kudou and Yagita [18].)

Theorem 2. *Let X' be a 1-connected mod 5 finite H-space such that $H_*(X')$ and $K(2)_*(X')$ (at the prime 5) are associative and that $\mathrm{Q}H^{2j}(X') = 0$ unless $j = 6$. (The coefficient of the (co)homology is $\mathbb{F}_5$.) Then, we have the following inequalities:*

$$\dim_{\mathbb{F}_5} \mathrm{Q}H^j(X') - \dim_{\mathbb{F}_5} \mathrm{P}H^j(X') \geq \dim_{\mathbb{F}_5} \mathrm{Q}H^{12}(X'),$$
$$j = 15, 23, 27, 35, 39, 47.$$

We know certain characterization of the mod 3 cohomology Hopf algebras over $\mathcal{A}_3$, the mod 3 Steenrod algebra, of F_4 and E_8 where F_4 is the compact, connected, simple, exceptional Lie group of rank 4. (Also we know certain characterization of the mod 5 cohomology Hopf algebra over $\mathcal{A}_5$ of E_8. See Kane [11], [13], [14] and Yagita [36].) The study of this paper is inspired by such studies and is much based on the technique of Rao [26] and Yagita [35], [36]. Observe that Theorems 1 and 2 imply the following theorems.

Theorem 3. *Let X be as in Theorem 1. If $K(2)_*(X)$ is associative (e.g. if X is homotopy associative), then, the cohomology algebra $H^*(X; \mathbb{F}_3)$ is isomorphic to the mod 3 cohomology algebra of a finite product of F_4's, E_8's, and odd (≥ 3) dimensional spheres.*

Theorem 4. *Let X' be as in Theorem 2. Then, the cohomology algebra $H^*(X'; \mathbb{F}_5)$ is isomorphic to the mod 5 cohomology algebra of a finite product of E_8's and odd (≥ 3) dimensional spheres.*

Note that the isomorphism in Theorem 3 (resp. Theorem 4) is as an algebra and in general, is not as a Hopf algebra over $\mathcal{A}_3$ (resp. $\mathcal{A}_5$) under the product multiplication. For example, consider E_6 and E_7, the compact, 1-connected, simple, exceptional Lie groups of rank 6 and 7, at the prime 3. (See Mimura [24]. Also see [16] and [17] for certain characterization of the mod 3 cohomology Hopf algebras over $\mathcal{A}_3$ of E_6, $\mathrm{Ad}E_6$, and E_7 where $\mathrm{Ad}E_6$ is the quotient group of E_6 by its center.) Also note that $X(3)$ satisfies neither the hypothesis nor the conclusion in (i) of Theorem 1. (According to Yagita [35], $K(2)_*(X(3))$ is not associative.) We can easily see that in the preceding two theorems, any factor of the product space satisfies the hypothesis on X or on X' (see Adams [1], Kane [14], and Mimura [24]) and hence, so does the product space with the product multiplication. Thus, the preceding two theorems are classification theorems.

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2. An adjoint algebra

In this section, we work in linear algebra over a fixed field K. We consider an object $\Xi = (m; V_1, V_2, V_3; \varphi, \psi)$ defined as follows.

Let m be an integer ≥ 2. Let V_1, V_2, V_3 be K-vector spaces where V_1 is of finite dimension. Let $\varphi\colon V_1^{\otimes^m} \to V_2$ and $\psi\colon V_1 \otimes V_2 \to V_3$ be K-linear maps. Denote $\psi(a \otimes b)$ by $a * b$ and $\psi(a \otimes B)$ by $a * B$ where $a \in V_1, b \in V_2$, and B is a subspace of V_2. We assume in V_3 that

$$a * \varphi(b \otimes b \otimes \cdots \otimes b \otimes c) = c * \varphi(b \otimes b \otimes \cdots \otimes b \otimes a) \tag{2.1}$$

for any $a, b, c \in V_1$ and that

$$a * \varphi(a \otimes a \otimes \cdots \otimes a) \neq 0 \tag{2.2}$$

for any $0 \neq a \in V_1$.

In this paper, we call such an object Ξ enjoying the properties stated above *an adjoint algebra*.

The purpose of this section is to prove the following lemma. Let Ξ be an adjoint algebra and $\dim V_1 = n$. Given an ordered basis $\alpha = (a_1, a_2, \ldots, a_n)$ of V_1 and $i, j = 1, 2, \ldots, n$, put

$$v_{i,j} = \varphi(a_i \otimes a_i \otimes \cdots \otimes a_i \otimes a_j) \in V_2.$$

(Note that by definition, we have $a_l * v_{i,j} = a_j * v_{i,l}$ and $a_i * v_{i,i} \neq 0$ for any α.) Then, we have

Lemma 5. *We can find an ordered basis α of V_1 which satisfies*

$$\dim\langle v_{i,j} \mid i, j = 1, 2, \ldots, n\rangle \geq n. \tag{2.3}$$

The rest of this section is devoted to the proof of this lemma. For the purpose, we need the following definitions.

Definition. *An n-sequence of length N* is a sequence of non-negative integers $\nu = (n_1, n_2, \ldots, n_N)$ which satisfies $\sum_{h=1}^{N}(n_h + 1) < n$. In the following, let $\nu = (n_1, n_2, \ldots, n_N)$ be an n-sequence of length N. Then, for $h = 1, 2, \ldots, N+1$, set

$$S_h^\nu = \left\{ h, h+1, \ldots, n - \sum_{i=1}^{h-1} n_i \right\}$$

and for $h = 1, 2, \dots, N$, set

$$T_h^\nu = \left\{ h, n - \sum_{i=1}^{h} n_i + 1, n - \sum_{i=1}^{h} n_i + 2, \dots, n - \sum_{i=1}^{h-1} n_i \right\}.$$

(Set $\sum_{i=1}^{0} n_i = 0$. Note that $\{1, 2, \dots, n\} = S_1^\nu \supset S_2^\nu \supset \cdots$, that $S_h^\nu \supset T_h^\nu$, and that $S_h^\nu \setminus T_h^\nu = S_{h+1}^\nu$.) An ordered basis $\alpha = (a_1, a_2, \dots, a_n)$ of V_1 satisfies condition (C_ν) if the following two conditions are satisfied.

(i) The set $\{v_{h,j} \mid h = 1, 2, \dots, N; j \in T_h^\nu\}$ is linearly independent.

(ii) Set $W_h^\nu = \langle v_{h,j} \mid j \in T_h^\nu \rangle$. Then, $v_{h,j} \in \bigoplus_{i \leq h-1} W_i^\nu$ for $h = 1, 2, \dots, N$ and for $j \in S_{h+1}^\nu$. (Set $W_0^\nu = 0$.)

Pick any ordered basis α of V_1. We will prove the following two assertions.

Assertion (a). We can modify α so that it satisfies (C_ν) where ν is an n-sequence of length 1, otherwise α satisfies (2.3).

Assertion (b). If α satisfies (C_ν) where ν is an n-sequence of length N, then we can modify α so that it satisfies (C_{ν_+}) where ν_+ is an n-sequence of length $N+1$ which is an extension of ν, otherwise α satisfies (2.3).

Thus, if we begin with (a) and iterate (b) at most $n-1$ times, then we have α which satisfies (2.3).

Proof of (a). If the set $\{v_{1,j} \mid j = 1, 2, \dots, n\}$ is linearly independent, then α satisfies (2.3). Now, assume that

$$\dim\langle v_{1,j} \mid j = 1, 2, \dots, n \rangle = n_1 + 1 < n.$$

(Recall that $v_{1,1} \neq 0$.) Set $\nu = (n_1)$. Permuting $\{a_j \mid j \in S_1^\nu \setminus \{1\} = \{2, 3, \dots, n\}\}$, we may assume that $\{v_{1,j} \mid j \in T_1^\nu = \{1, n - n_1 + 1, n - n_1 + 2, \dots, n\}\}$ is linearly independent. Then, we have

$$v_{1,l} = \sum_{j \in T_1^\nu} k_{l,j} v_{1,j} \in W_1^\nu = \langle v_{1,j} \mid j \in T_1^\nu \rangle$$

for $l \in \{2, 3, \dots, n - n_1\} = S_1^\nu \setminus T_1^\nu = S_2^\nu$ where $k_{l,j} \in K$. This implies that

$$\varphi \left(a_1 \otimes a_1 \otimes \cdots \otimes a_1 \otimes \left(a_l - \sum_{j \in T_1^\nu} k_{l,j} a_j \right) \right) = 0.$$

Then, $\alpha' = (a_1, a'_2, \dots, a'_{n-n_1}, a_{n-n_1+1}, \dots, a_n)$ satisfies (C_ν) where $a'_l = a_l - \sum_{j \in T_1^\nu} k_{l,j} a_j$ for $l \in S_2^\nu$. Thus, replace a_l by a'_l for $l \in S_2^\nu$.

Proof of (b). Let $\nu = (n_1, n_2, \dots, n_N)$ be an n-sequence of length N and assume that an ordered basis $\alpha = (a_1, a_2, \dots, a_n)$ satisfies (C_ν). If the set $\{v_{h,j} \mid h = 1, 2, \dots, N; j \in T_h^\nu\} \cup \{v_{N+1,j} \mid j \in S_{N+1}^\nu\}$ is linearly independent, then α satisfies (2.3). Now, assume that it is not linearly independent.

Lemma 6. *$a_j * W_i^\nu = 0$ for $j \in T_h^\nu, h = 2,3,\dots,N,$ and $h > i \geq 1$.*

Proof. We prove by induction on i.

Recall that $W_1^\nu = \langle v_{1,l} \mid l \in T_1^\nu \rangle$. For $j \in T_h^\nu, h = 2,3,\dots,N$, and $l \in T_1^\nu$, we have $a_j * v_{1,l} = a_l * v_{1,j} = 0$ since $j \in T_h^\nu \subset S_2^\nu$. It follows that $a_j * W_1^\nu = 0$.

Next, assume that the lemma is true if $i \leq g$. Recall that $W_{g+1}^\nu = \langle v_{g+1,l} \mid l \in T_{g+1}^\nu \rangle$. For $j \in T_h^\nu, h = g+2, g+3,\dots,N$, and $l \in T_{g+1}^\nu$, we have

$$a_j * v_{g+1,l} = a_l * v_{g+1,j} \in a_l * \left(\bigoplus_{f \leq g} W_f^\nu \right) = \sum_{f=1}^{g} (a_l * W_f^\nu) = 0$$

since $j \in T_h^\nu \subset S_{g+2}^\nu$ and by the induction hypothesis. It follows that $a_j * W_{g+1}^\nu = 0$ and thus, the lemma is true also for $i = g+1$. □

Lemma 7. *$a_{N+1} * W_h^\nu = 0$ for $h = 1,2,\dots,N$.*

Proof. Recall that $W_h^\nu = \langle v_{h,j} \mid j \in T_h^\nu \rangle$. We have $v_{h,N+1} \in \bigoplus_{i \leq h-1} W_i^\nu$ since $N+1 \in S_{h+1}^\nu$. Thus, for $j \in T_h^\nu$, we have

$$a_{N+1} * v_{h,j} = a_j * v_{h,N+1} = 0$$

by $v_{1,N+1} = 0$ for $h = 1$ and by the previous lemma for $h \geq 2$. Hence, the lemma follows. □

Lemma 8. *$v_{N+1,N+1} \notin \bigoplus_{i \leq N} W_i^\nu$.*

Proof. The lemma immediately follows from the previous lemma and $a_{N+1} * v_{N+1,N+1} \neq 0$. □

By the previous lemma, there exists a non-negative integer n_{N+1} such that

$$\dim \langle \{ v_{h,j} \mid h = 1,2,\cdots,N; j \in T_h^\nu \} \cup \{ v_{N+1,j} \mid j \in S_{N+1}^\nu \} \rangle = \sum_{h=1}^{N+1} (n_h + 1) < n.$$

Set $\nu_+ = (n_1, n_2, \dots, n_N, n_{N+1})$, which is an n-sequence of length $N+1$. (Note that $S_h^\nu = S_h^{\nu_+}$ for $h = 1,2,\dots,N+1$, and that $T_h^\nu = T_h^{\nu_+}$ for $h = 1,2,\dots,N$.) Permuting $\{ a_j \mid j \in S_{N+1}^{\nu_+} \setminus \{N+1\} \}$, we may assume that $\{ v_{h,j} \mid h = 1,2,\dots,N+1; j \in T_h^{\nu_+} \}$ is linearly independent. Then, we have $v_{N+1,l} \in \bigoplus_{i \leq N+1} W_i^{\nu_+}$ for $l \in S_{N+2}^{\nu_+}$. Let $\sum_{j \in T_{N+1}^{\nu_+}} k_{l,j} v_{N+1,j} \in W_{N+1}^{\nu_+}$ be its component in $W_{N+1}^{\nu_+}$ where $k_{l,j} \in K$. This implies that

$$\varphi \left(a_{N+1} \otimes a_{N+1} \otimes \cdots \otimes a_{N+1} \otimes \left(a_l - \sum_{j \in T_{N+1}^{\nu_+}} k_{l,j} a_j \right) \right) \in \bigoplus_{i \leq N} W_i^{\nu_+}.$$

Further, for $h = 1, 2, \ldots, N$, we have

$$\varphi\left(a_h \otimes a_h \otimes \cdots \otimes a_h \otimes \left(a_l - \sum_{j \in T^{\nu_+}_{N+1}} k_{l,j} a_j\right)\right) = v_{h,l} - \sum_{j \in T^{\nu_+}_{N+1}} k_{l,j} v_{h,j} \in \bigoplus_{i \le h-1} W_i^{\nu_+}$$

since $l \in S^{\nu_+}_{N+2} \subset S^{\nu_+}_{h+1}$ and $j \in T^{\nu_+}_{N+1} \subset S^{\nu_+}_{h+1}$. Replace a_l by $a_l - \sum_{j \in T^{\nu_+}_{N+1}} k_{l,j} a_j$ for $l \in S^{\nu_+}_{N+2}$. Then, α satisfies (C_{ν_+}).

3. Proof of Theorem 1

3.1. Preliminaries

In this section, we use Lemma 5 where $K = \mathbb{F}_3$ to prove Theorem 1. We use the following notations. The subscript of an element of a graded algebra designates the degree. Given a Hopf algebra A, let $\mathrm{P}A$ and $\mathrm{Q}A$ denote the primitives and the indecomposables respectively, and let $\bar{x} \in \mathrm{Q}A$ denote the class of an element x of A. The coefficient for the ordinary (co)homology theory is $\mathbb{F}_3$ and the periodic Morava K-theory $K(n)$ and the connective Morava K-theory $k(n)$ are at the prime 3 unless otherwise stated.

First, let X denote a 1-connected mod 3 finite H-space such that $H_*(X)$ is associative. Let $\bar{\Delta}$ be the reduced coproduct map of $H^*(X)$ induced by the multiplication of X. We recall some facts about $H^*(X)$ and $H_*(X)$.

According to Hemmi-Lin [9], we have $\mathrm{Q}H^{2j}(X) = 0$ unless $j = 4, 10$ and $x^3 = 0$ for any $x \in H^*(X)$. Hence, by Lin [19], we may put

$$\begin{aligned} H^*(X) = &\bigotimes_{\lambda \in \Lambda} \left(\frac{\mathbb{F}_3[x(\lambda)_8, x(\lambda)_{20}]}{(x(\lambda)_8^3, x(\lambda)_{20}^3)} \otimes \wedge(x(\lambda)_3, x(\lambda)_7, x(\lambda)_{15}, x(\lambda)_{19}) \right) \\ &\bigotimes_{\gamma \in \Gamma} \left(\frac{\mathbb{F}_3[x(\gamma)_8]}{(x(\gamma)_8^3)} \otimes \wedge(x(\gamma)_3, x(\gamma)_7) \right) \\ &\otimes \wedge(z(l), l \in L), \end{aligned}$$

where Λ, Γ, and L are finite sets, $\wp^1 x(\lambda)_3 = x(\lambda)_7, \beta x(\lambda)_7 = x(\lambda)_8, -\wp^3 x(\lambda)_7 = \wp^1 x(\lambda)_{15} = x(\lambda)_{19}, -\wp^3 x(\lambda)_8 = \beta x(\lambda)_{19} = x(\lambda)_{20}$ for $\lambda \in \Lambda$, $\wp^1 x(\gamma)_3 = x(\gamma)_7, \beta x(\gamma)_7 = x(\gamma)_8, \wp^3 x(\gamma)_8 = 0$ for $\gamma \in \Gamma$, and $|z(l)|$ is odd for $l \in L$. In particular, for $\theta \in \Theta = \Lambda \amalg \Gamma$ and $\lambda \in \Lambda$, we have

$$Q_1 x(\theta)_3 = Q_0 x(\theta)_7 = x(\theta)_8, \quad Q_2 x(\lambda)_3 = Q_1 x(\lambda)_{15} = Q_0 x(\lambda)_{19} = x(\lambda)_{20},$$

where Q_j is the j-th Milnor operation defined by $Q_0 = \beta$ and $Q_{j+1} = Q_j \wp^{3^j} - \wp^{3^j} Q_j$.

The subalgebra

$$B^* = \bigotimes_{\lambda \in \Lambda} \frac{\mathbb{F}_3[x(\lambda)_8, x(\lambda)_{20}]}{(x(\lambda)_8^3, x(\lambda)_{20}^3)} \bigotimes_{\gamma \in \Gamma} \frac{\mathbb{F}_3[x(\gamma)_8]}{(x(\gamma)_8^3)}$$

is a primitively generated subHopf algebra of $H^*(X)$ invariant under $\mathcal{A}_3$. The submodule $R^* = \langle 1\rangle \oplus \{x \in H^+(X) \mid \bar\Delta(x) \in B^* \otimes H^*(X)\}$ of $H^*(X)$ satisfies $\bar\Delta(R^*) \subset B^* \otimes R^*$ and hence is a subcoalgebra of $H^*(X)$ invariant under $\mathcal{A}_3$. Moreover, $R^{\mathrm{even}} = B^*$ and the composite of the inclusion and the natural projection $R^* \to H^*(X) \to \mathrm{Q}H^*(X)$ is an isomorphism in odd degree. Thus we can choose $x(\lambda)_{15}$ and $z(l)$ to lie in R^*. (Note that $x(\lambda)_j$ for $j \neq 15$ and $x(\gamma)_j$ lie in $\mathrm{P}H^*(X) \subset R^*$.) For B^* and R^*, see Baum-Browder [4], Lin [19], and Kane [13].

Moreover, we can choose $z(l)$ also to satisfy

$$\beta\wp^1 Z^3 = \beta Z^7 = \beta\wp^1 Z^{15} = \beta Z^{19} = 0$$

where $Z^* = \langle z(l) \mid l \in L\rangle$. This fact easily follows from the fact that R^* is invariant under $\mathcal{A}_3$. For example, if $\beta\wp^1 z(l) = \sum_{\theta\in\Theta} k(\theta)x(\theta)_8$ where $|z(l)| = 3$ and $k(\theta) \in \mathbb{F}_3$, then replace $z(l)$ with $z(l) - \sum_{\theta\in\Theta} k(\theta)x(\theta)_3$. Thus, we have

$$Z^j = \begin{cases} (\mathrm{Ker}\beta\wp^1) \cap R^j, & j = 3, 15, \\ (\mathrm{Ker}\beta) \cap R^j, & j = 7, 19, \end{cases}$$

and $\wp^1 Z^j \subset Z^{j+4}$ for $j = 3, 15$. Further, we have $\beta\wp^3 Z^7 = (\wp^3\beta - \wp^1\beta\wp^2)Z^7 = 0$ by the Adem relation and by $\wp^1\beta\wp^2 Z^7 \subset \wp^1 R^{16} = 0$. This implies that $\wp^3 Z^7 \subset Z^{19}$.

Here, recall that $H_*(X)$ has a right $\mathcal{A}_3$-module structure defined by the rule $\langle a\varphi, x\rangle = \langle a, \varphi x\rangle$ for any $a \in H_*(X), x \in H^*(X)$, and $\varphi \in \mathcal{A}_3$, and also has a left $\mathcal{A}_3$-module structure defined by the rule $\chi(\varphi)a = a\varphi$ for any $a \in H_*(X)$ and $\varphi \in \mathcal{A}_3$ where χ is the canonical anti-automorphism of $\mathcal{A}_3$. (In particular, we have $Q_j a = -aQ_j$.) In this paper, we use the left one. Also recall that $\mathrm{Q}H^*(X)$ and $\mathrm{P}H_*(X)$ inherit the left $\mathcal{A}_3$-module structures of $H^*(X)$ and $H_*(X)$ respectively. The relations of Z^* stated above help us to consider the duality between $\mathrm{Q}H^*(X)$ and $\mathrm{P}H_*(X)$ as left $\mathcal{A}_3$-modules.

For $\theta \in \Theta$, let $a(\theta)_j \in \mathrm{P}H_*(X)$ be the dual element of $\bar x(\theta)_j \in \mathrm{Q}H^*(X)$ as to the obvious basis of $\mathrm{Q}H^*(X)$. Then, for $\theta \in \Theta$ and $\lambda \in \Lambda$, we have the following table which describes some actions.

	Q_0	Q_1	Q_2	$\wp^3$
$a(\theta)_8$	$-a(\theta)_7$	$-a(\theta)_3$		
$a(\lambda)_{20}$	$-a(\lambda)_{19}$	$-a(\lambda)_{15}$	$-a(\lambda)_3$	$a(\lambda)_8$

Moreover, in degree ≤ 8, $\mathrm{Ker}[Q_2\colon H^*(X) \to H^*(X)]$ is the subalgebra generated by the generators other than $x(\lambda)_3$ ($\lambda \in \Lambda$), and $\mathrm{Im}[Q_2\colon H_*(X) \to H_*(X)]$ is the ideal generated by $a(\lambda)_3$ ($\lambda \in \Lambda$). Also, in all degrees, $Q_3\colon H^*(X) \to H^*(X)$ vanishes and $\mathrm{Im}[Q_3\colon H_*(X) \to H_*(X)] = 0$. We can easily check these facts by using the properties of R^* stated above and the fact that $\mathrm{P}H^{2j}(X) = \mathrm{P}B^{2j} = 0$ unless $j = 4, 10$. (Also note that Q_j is primitive in $\mathcal{A}_3$.) We omit the detail.

We recall some facts about $K(n)$ and $k(n)$. We use the symbol $T\colon k(n)_*(-) \to H_*(-)$ to denote the Thom map. Recall that T fits into the Sullivan exact sequence

$$\cdots \to k(n)_{j-2(3^n-1)}(-) \xrightarrow{v_n} k(n)_j(-) \xrightarrow{T} H_j(-) \xrightarrow{\rho} k(n)_{j-2(3^n-1)-1}(-) \to \cdots$$

where $T \circ \rho = \pm Q_n$. Recall that $K(n)_*(-)$ is the localization of $k(n)_*(-)$ with respect to the multiplicative set $\{1, v_n, v_n^2, \ldots\}$. Thus, $\operatorname{Ker} i = v_n$-Torsion$(k(n)_*(-))$ where $i\colon k(n)_*(-) \to K(n)_*(-)$ is the canonical map. The theories $k(n)$ and $K(n)$ are multiplicative. (See Morava [25], Shimada-Yagita [28], and Würgler [30].)

3.2. Structure of $K(2)_*(X)$ and the proof of (i)

In this subsection, suppose that X is a 1-connected mod 3 finite H-space such that $H_*(X)$ and $K(2)_*(X)$ are associative. Recall that $K(2)_*(X)$ is a Hopf algebra over $K(2)_*$. For some of the following arguments, refer to Yagita [36].

Note that $T\colon k(2)_*(X) \to H_*(X)$ is a homomorphism of $\mathbb{F}_3$-algebras and that it is isomorphic for $* \leq 18$ if reduced. For $\theta \in \Theta$ and $j = 3, 7, 8$, put $a'(\theta)_j = T^{-1}(a(\theta)_j) \in k(2)_j(X)$ and $a''(\theta)_j = i(a'(\theta)_j) \in K(2)_j(X)$. According to Kane [12], $k(2)_*(X)$ has no higher v_2-torsion and hence $\operatorname{Ker} i = \operatorname{Ker} v_2 \subset k(2)_*(X)$. It follows by the Sullivan exact sequence that $T(\operatorname{Ker} i) = \operatorname{Im} Q_2 \subset H_*(X)$. Thus, in degree ≤ 8, $\operatorname{Ker} i$ is the ideal generated by $a'(\lambda)_3$ ($\lambda \in \Lambda$) and by definition, $a''(\lambda)_3 = 0$ ($\lambda \in \Lambda$).

Lemma 9. *For elements $a, b \in \langle a''(\theta)_8 \mid \theta \in \Theta\rangle$, we have $[a, b] = 0$ in $K(2)_*(X)$.*

Proof. It suffices to show that $[a''(\theta)_8, a''(\eta)_8] = 0$ for $\theta, \eta \in \Theta$. Since $i\colon k(2)_*(X) \to K(2)_*(X)$ is a homomorphism of $k(2)_*$-algebras, it suffices to show that $[a'(\theta)_8, a'(\eta)_8] = 0$ in $k(2)_*(X)$. Note that $k(2)_{16}(X) \cong H_{16}(X) \oplus \langle v_2\rangle$ via T and ε where $\varepsilon\colon k(2)_*(X) \to k(2)_*$ is the augmentation. It is easy to see that $T([a'(\theta)_8, a'(\eta)_8]) = [a(\theta)_8, a(\eta)_8] = 0$ and $\varepsilon([a'(\theta)_8, a'(\eta)_8]) = 0$. Thus, the lemma follows. $\square$

The idea of the proof of the following lemma is due to Yagita [36], but we do not use the Atiyah-Hirzebruch spectral sequence which converges to $K(2)^*(X)$. We treat $K(2)_*(X)$ directly.

Lemma 10. *In $K(2)_*(X)$, we have $a''(\theta)_8^3 = -v_2 a''(\theta)_8$ for $\theta \in \Theta$.*

Proof. Pick $\theta_0 \in \Theta$. By $T((a'(\theta_0)_8)^2 a'(\theta_0)_8) = (a(\theta_0)_8)^3 = 0$ and by the Sullivan exact sequence, we can put $(a'(\theta_0)_8)^2 a'(\theta_0)_8 = v_2 y$ where $y \in k(2)_8(X)$. Applying i, we have $(a''(\theta_0)_8)^3 = v_2 i(y)$. Note that $T\colon k(2)_*(X \times X) \to H_*(X \times X)$ is isomorphic for $* \leq 15$ and that the following diagram is commutative where $d\colon X \to X \times X$ is the diagonal map.

$$\begin{tikzcd}
K(2)_*(X) \arrow[d, "d_*"'] & k(2)_*(X) \arrow[l, "i"'] \arrow[r, "T"] \arrow[d, "d_*"] & H_*(X) \arrow[d, "d_*"] \\
K(2)_*(X \times X) & k(2)_*(X \times X) \arrow[l, "i"'] \arrow[r, "T"] & H_*(X \times X) \\
K(2)_*(X) \underset{K(2)_*}{\otimes} K(2)_*(X) \arrow[u, "\cong"] & & H_*(X) \otimes H_*(X) \arrow[u, "\cong"']
\end{tikzcd}$$

Then, we can easily see that $\mathrm{P}K(2)_8(X) \cap \operatorname{Im} i = \langle a''(\theta)_8 \mid \theta \in \Theta\rangle$. Since $a''(\theta_0)_8$ is primitive, so is $(a''(\theta_0)_8)^3 = v_2 i(y)$. (Note that the multiplication of $K(2)$ (at the prime 3) is commutative.) Thus, we have $i(y) \in \langle a''(\theta)_8 \mid \theta \in \Theta\rangle$. Put $i(y) = \sum_{\theta\in\Theta} k(\theta)a''(\theta)_8$ where $k(\theta) \in \mathbb{F}_3$.

For $\theta \in \Theta$, let $f_\theta\colon X \to K(\mathbb{Z}, 3)$ be a map which represents the integral class of $x(\theta)_3 \in H^3(X)$. Note that f_θ is an H-map. Let $\iota \in H^3(K(\mathbb{Z}, 3))$ be the fundamental class and let $u_8 \in H_8(K(\mathbb{Z}, 3)) \cong \mathbb{F}_3$ be the element satisfying $\langle u_8, Q_1\iota\rangle = 1$. Then, we can easily see that $(f_\theta)_*(a(\theta)_8) = u_8$ and $(f_\theta)_*(a(\eta)_8) = 0$ for $\theta, \eta \in \Theta, \theta \neq \eta$. According to Ravenel-Wilson [27], we can see that $u''_8 = i \circ T^{-1}(u_8) \in K(2)_8(K(\mathbb{Z}, 3))$ satisfies $(u''_8)^3 = -v_2 u''_8 \neq 0$. (Indeed, we have $u''_8 = \pm\delta_*(a_{(0,1)})$ where $\delta\colon K(\mathbb{F}_3, 2) \to K(\mathbb{Z}, 3)$ is the standard map and $a_{(0,1)} \in K(2)_8(K(\mathbb{F}_3, 2))$ is the element defined in [27].) Note that $(f_\theta)_*(a''(\theta)_8) = u''_8$ and $(f_\theta)_*(a''(\eta)_8) = 0$ for $\theta, \eta \in \Theta, \theta \neq \eta$. Applying $(f_{\theta_0})_*$ to $(a''(\theta_0)_8)^3 = v_2 \sum_{\theta\in\Theta} k(\theta)a''(\theta)_8$, we have $-v_2 u''_8 = (u''_8)^3 = v_2 k(\theta_0)u''_8$ and hence $k(\theta_0) = -1$. Similarly, applying $(f_\theta)_*$ for $\theta \neq \theta_0$, we have $0 = v_2 k(\theta)u''_8$ and hence $k(\theta) = 0$. Thus, we have $(a''(\theta_0)_8)^3 = -v_2 a''(\theta_0)_8$. □

Lemma 11. *For an element $a \in \langle a''(\theta)_8 \mid \theta \in \Theta\rangle$, we have $a^3 = -v_2 a$ in $K(2)_*(X)$.*

Proof. This follows from Lemmas 9, 10, and the direct computation.

As an another proof, we can show this lemma as follows. If $a = 0$, the lemma is obvious. Take $0 \neq a \in \langle a''(\theta)_8 \mid \theta \in \Theta\rangle$. We can rearrange the Borel decomposition of $H^*(X)$ which we describe in Subsection 1 of Section 3 so that $a = a''(\theta)_8$ for some $\theta \in \Theta$. Then, applying Lemma 10 for this rearranged decomposition of $H^*(X)$, we have $a^3 = -v_2 a$. □

For $j = 0, 1$, let $Q'_j \in k(2)^*(k(2))$ be the bordism operation which covers $Q_j \in \mathcal{A}_3$, and let $Q''_j \in K(2)^*(K(2))$ be the corresponding operation. (See Würgler [31] and Yagita [32], [33], [34].) Then, we have $-Q''_0 a''(\theta)_8 = a''(\theta)_7$ and $-Q''_1 a''(\theta)_8 = a''(\theta)_3$ for $\theta \in \Theta$. (Recall that $a''(\lambda)_3 = 0$ for $\lambda \in \Lambda$.) Thus, the restrictions $Q''_0\colon \langle a''(\theta)_8 \mid \theta \in \Theta\rangle \to K(2)_7(X)$ and $Q''_1\colon \langle a''(\gamma)_8 \mid \gamma \in \Gamma\rangle \to K(2)_3(X)$ are monomorphisms of $\mathbb{F}_3$-vector spaces. Recall that $Q''_j\colon K(2)_*(X) \to K(2)_*(X)$ is a derivation of the $K(2)_*$-algebra $K(2)_*(X)$.

Now we can prove

Lemma 12. *Set* $V_1 = \langle a''(\theta)_8 \mid \theta \in \Theta\rangle$ *and* $V_j = K(2)_{8j-1}(X)$ *for* $j = 2, 3$. *Define* $\varphi\colon V_1 \otimes V_1 \to V_2$ *by* $\varphi(a \otimes b) = [a, Q_0''b]$ *and* $\psi\colon V_1 \otimes V_2 \to V_3$ *by* $\psi(a \otimes b) = [a, b]$. *Then,* $(2; V_1, V_2, V_3; \varphi, \psi)$ *is an adjoint algebra.*

Proof. We prove (2.1) and (2.2) for $(2; V_1, V_2, V_3; \varphi, \psi)$ above. For elements $a, b \in V_1$, we can show that $[a, Q_0''b] = [b, Q_0''a]$ by applying Q_0'' to $[a, b] = 0$. Further, for elements $a, b, c \in V_1$, we have $[a, [b, Q_0''c]] = [b, [a, Q_0''c]]$ by the Jacobi identity. Thus $[a, [b, Q_0''c]]$ is invariant under any permutation of $\{a, b, c\}$ and hence (2.1) is proved.

For an element $a \in V_1$, $a \neq 0$, we can show that

$$[a, [a, Q_0''a]] = -v_2 Q_0''a \neq 0$$

by applying Q_0'' to $a^3 = -v_2 a$. (See Yagita [36].) Thus (2.2) is proved. □

Corollary 13. $\dim_{\mathbb{F}_3} \mathrm{Q}H^{15}(X) - \dim_{\mathbb{F}_3} \mathrm{P}H^{15}(X) \geq \dim_{\mathbb{F}_3} \mathrm{Q}H^{8}(X)$.

Proof. By Lemma 5, there exists an invertible linear transformation f'' of $\langle a''(\theta)_8 \mid \theta \in \Theta\rangle$ such that

$$\dim_{\mathbb{F}_3}\langle c''(\theta, \eta) \mid \theta, \eta \in \Theta\rangle \geq \dim_{\mathbb{F}_3}\langle a''(\theta)_8 \mid \theta \in \Theta\rangle = \dim_{\mathbb{F}_3} \mathrm{Q}H^8(X)$$

where $c''(\theta, \eta) = [f''(a''(\theta)_8), Q_0''f''(a''(\eta)_8)]$. Let f be the linear transformation of $\langle a(\theta)_8 \mid \theta \in \Theta\rangle$ such that $i \circ T^{-1} \circ f = f'' \circ i \circ T^{-1}$. Then, we have $i \circ T^{-1}(c(\theta, \eta)) = c''(\theta, \eta)$ where $c(\theta, \eta) = [f(a(\theta)_8), Q_0 f(a(\eta)_8)]$. Hence

$$\begin{aligned}
&\dim_{\mathbb{F}_3} \mathrm{Q}H^{15}(X) - \dim_{\mathbb{F}_3} \mathrm{P}H^{15}(X) = \dim_{\mathbb{F}_3} \mathrm{P}H_{15}(X) - \dim_{\mathbb{F}_3} \mathrm{Q}H_{15}(X) \\
&\quad \geq \dim_{\mathbb{F}_3}\langle c(\theta, \eta) \mid \theta, \eta \in \Theta\rangle \geq \dim_{\mathbb{F}_3}\langle c''(\theta, \eta) \mid \theta, \eta \in \Theta\rangle \geq \dim_{\mathbb{F}_3} \mathrm{Q}H^8(X).
\end{aligned}$$

□

Similarly we can prove

Lemma 14. *Set* $V_1 = \langle a''(\theta)_8 \mid \theta \in \Theta\rangle$ *and* $V_j = K(2)_{8j-5}(X)$ *for* $j = 2, 3$. *Define* $\varphi\colon V_1 \otimes V_1 \to V_2$ *by* $\varphi(a \otimes b) = [a, Q_1''b]$ *and* $\psi\colon V_1 \otimes V_2 \to V_3$ *by* $\psi(a \otimes b) = [a, b]$. *Then,* $(2; V_1, V_2, V_3; \varphi, \psi)$ *is an adjoint algebra.*

Corollary 15.

$$\dim_{\mathbb{F}_3} \mathrm{Q}H^{11}(X) - \dim_{\mathbb{F}_3} \mathrm{P}H^{11}(X) \geq \dim_{\mathbb{F}_3} \mathrm{Q}H^{8}(X) - \dim_{\mathbb{F}_3} \mathrm{Q}H^{20}(X).$$

3.3. Structure of $K(3)_*(X)$ and the proof of (ii)

In this subsection, suppose that X is a 1-connected mod 3 finite H-space such that $H_*(X)$ and $K(3)_*(X)$ are associative. Recall that $K(3)_*(X)$ is a Hopf algebra over $K(3)_*$. We apply an argument similar to that of the previous subsection. For some of the following arguments, refer to Yagita [36].

Note that $T\colon k(3)_*(X) \to H_*(X)$ is a homomorphism of $\mathbb{F}_3$-algebras and that it is isomorphic for $* \leq 54$ if reduced. Put $a'(\theta)_j = T^{-1}(a(\theta)_j) \in k(3)_j(X)$ and $a''(\theta)_j = i(a'(\theta)_j) \in K(3)_j(X)$. Also note that $k(3)_*(X)$ is $k(3)_*$-free and hence $i\colon k(3)_*(X) \to K(3)_*(X)$ is monomorphic.

Lemma 16. *For elements $a, b \in \langle a''(\lambda)_{20} \mid \lambda \in \Lambda \rangle$, we have $[a, b] = 0$ in $K(3)_*(X)$.*

Proof. Similar to that of Lemma 9. □

The idea of the proof of the following lemma is also due to Yagita [36].

Lemma 17. *In $K(3)_*(X)$, we have $a''(\lambda)_{20}^3 = v_3 a''(\lambda)_8$ for $\lambda \in \Lambda$.*

Proof. The proof is similar to that of Lemma 10. Pick $\lambda_0 \in \Lambda$. First, by a similar argument, we may put $(a''(\lambda_0)_{20})^3 = v_3 \sum_{\theta \in \Theta} k(\theta) a''(\theta)_8$ where $k(\theta) \in \mathbb{F}_3$. For $\theta \in \Theta$, let $f_\theta \colon X \to K(\mathbb{Z}, 3)$, $\iota \in H^3(K(\mathbb{Z}, 3))$, and $u_8 \in H_8(K(\mathbb{Z}, 3))$ be as in the proof of Lemma 10. Recall that $(f_\theta)_*(a(\theta)_8) = u_8$ and $(f_\theta)_*(a(\eta)_8) = 0$ for $\theta, \eta \in \Theta, \theta \neq \eta$. Let $u_{20} \in H_{20}(K(\mathbb{Z}, 3)) \cong \mathbb{F}_3$ be the element satisfying $\langle u_{20}, Q_2 \iota \rangle = 1$. Then, we can easily see that $(f_\lambda)_*(a(\lambda)_{20}) = u_{20}$ and $(f_\theta)_*(a(\lambda)_{20}) = 0$ for $\lambda \in \Lambda$, $\theta \in \Theta$, $\lambda \neq \theta$. According to Ravenel-Wilson [27], we can see that $u_8'' = i \circ T^{-1}(u_8) \in K(3)_8(K(\mathbb{Z}, 3))$ and $u_{20}'' = i \circ T^{-1}(u_{20}) \in K(3)_{20}(K(\mathbb{Z}, 3))$ satisfy the relation $(u_{20}'')^3 = v_3 u_8'' \neq 0$. (Indeed, we have $u_8'' = \alpha \delta_*(a_{(0,1)})$ and $u_{20}'' = -\alpha \delta_*(a_{(0,2)})$ where $0 \neq \alpha \in \mathbb{F}_3$, $\delta \colon K(\mathbb{F}_3, 2) \to K(\mathbb{Z}, 3)$ is the standard map, and $a_{(0,1)} \in K(3)_8(K(\mathbb{F}_3, 2))$ and $a_{(0,2)} \in K(3)_{20}(K(\mathbb{F}_3, 2))$ are the elements defined in [27], because of the relation $\wp^3 u_{20} = u_8$ in $H_*(K(\mathbb{Z}, 3))$ and of the definitions of $a_{(0,1)}$ and $a_{(0,2)}$.) Now, in a similar manner, we can show that $k(\lambda_0) = 1$ and $k(\theta) = 0$ for $\theta \neq \lambda_0$. Thus, we have the lemma. □

Define a linear map $(\wp^3)'' \colon i \circ T^{-1}(\mathrm{P}H_{20}(X)) = \langle a''(\lambda)_{20} \mid \lambda \in \Lambda \rangle \to \langle a''(\lambda)_8 \mid \lambda \in \Lambda \rangle = i \circ T^{-1} \circ \wp^3(\mathrm{P}H_{20}(X))$ by $(\wp^3)'' a''(\lambda)_{20} = a''(\lambda)_8$. (Note that $(\wp^3)'' = i \circ T^{-1} \circ \wp^3 \circ T \circ i^{-1}$.)

Lemma 18. *For an element $a \in \langle a''(\lambda)_{20} \mid \lambda \in \Lambda \rangle$, we have $a^3 = v_3 (\wp^3)'' a$ in $K(3)_*(X)$.*

Proof. Similar to that of Lemma 11. □

As in the previous subsection, let $Q_j'' \in K(3)^*(K(3))$ be the operation which corresponds to $Q_j \in \mathcal{A}_3$ for $j = 0, 1, 2$. (See Würgler [31] and Yagita [32], [33], [34].) Note that $Q_1''(\wp^3)'' a''(\lambda)_{20} = -a''(\lambda)_3 = Q_2'' a''(\lambda)_{20}$ and thus $Q_1''(\wp^3)'' = Q_2'' \colon \langle a''(\lambda)_{20} \mid \lambda \in \Lambda \rangle \to K(3)_3(X)$ is a monomorphism of $\mathbb{F}_3$-vector spaces.

Now, we have

Lemma 19. *Set $V_1 = \langle a''(\lambda)_{20} \mid \lambda \in \Lambda \rangle$ and $V_j = K(3)_{20j-5}(X)$ for $j = 2, 3$. Define $\varphi \colon V_1 \otimes V_1 \to V_2$ by $\varphi(a \otimes b) = [a, Q_1'' b]$ and $\psi \colon V_1 \otimes V_2 \to V_3$ by $\psi(a \otimes b) = [a, b]$. Then, $(2; V_1, V_2, V_3; \varphi, \psi)$ is an adjoint algebra.*

Proof. Similar to that of Lemma 12. □

Corollary 20. $\dim_{\mathbb{F}_3} \mathrm{Q}H^{35}(X) - \dim_{\mathbb{F}_3} \mathrm{P}H^{35}(X) \geq \dim_{\mathbb{F}_3} \mathrm{Q}H^{20}(X)$.

Proof. Similar to that of Corollary 13. □

Here, we refer to the result of Kane [13]. We can easily see that Theorem 1.1 of [13] can be applied to the elements $a(\lambda)_8, a(\lambda)_{20} \in \mathrm{P}H_{\mathrm{even}}(X)$ ($\lambda \in \Lambda$) with respect to the relation $\wp^3\bar{x}(\lambda)_8 = -\bar{x}(\lambda)_{20}$ in $\mathrm{Q}H^{\mathrm{even}}(X)$. In particular, we have $[a(\lambda)_8, a(\lambda)_7] \neq 0$ for $\lambda \in \Lambda$. By an argument similar to that of the second proof of Lemma 11, we can show that $[a, Q_0 a] \neq 0$ for $0 \neq a \in \langle a(\lambda)_8 \mid \lambda \in \Lambda\rangle$. It follows that $[a, Q_0'' a] \neq 0$ for $0 \neq a \in \langle a''(\lambda)_8 \mid \lambda \in \Lambda\rangle$.

Now we prove

Lemma 21. *Set* $V_1 = \langle a''(\lambda)_{20} \mid \lambda \in \Lambda\rangle$ *and* $V_{j,m} = K(3)_{20(j+m-2)-1}(X)$ *for* $j = 2, 3$ *and* $m = 2, 3, 4, 5$. *Define* $\varphi_m\colon V_1^{\otimes^m} \to V_{2,m}$ *by* $\varphi_m(a(1) \otimes a(2) \otimes \cdots \otimes a(m)) = [a(1), [a(2), [\cdots [a(m-1), Q_0'' a(m)] \cdots]]]$ *and* $\psi_m\colon V_1 \otimes V_{2,m} \to V_{3,m}$ *by* $\psi_m(a \otimes b) = [a, b]$. *Then,* $(m; V_1, V_2, V_3; \varphi_m, \psi_m)$ *is an adjoint algebra for* $m = 2, 3, 4, 5$.

Proof. We can prove (2.1) in a manner similar to that of the proof of Lemma 12. We prove (2.2). It suffices to show it for $m = 5$. For an element $a \in V_1, a \neq 0$, we can show that

$$[a, [a, Q_0'' a]] = v_3 Q_0'' (\wp^3)'' a$$

by applying Q_0'' to $a^3 = v_3 (\wp^3)'' a$. Hence we have

$$\mathrm{ad}^5(a)(Q_0'' a) = v_3 [a^3, Q_0'' (\wp^3)'' a] = v_3^2 [(\wp^3)'' a, Q_0'' (\wp^3)'' a] \neq 0$$

since $0 \neq (\wp^3)'' a \in \langle a''(\lambda)_8 \mid \lambda \in \Lambda\rangle$ where $\mathrm{ad}^1(a)(y) = [a, y]$ and $\mathrm{ad}^{j+1}(a)(y) = [a, \mathrm{ad}^j(a)(y)]$. (See Yagita [36].) Thus (2.2) is proved. □

Corollary 22. $\dim_{\mathbb{F}_3} \mathrm{Q}H^j(X) - \dim_{\mathbb{F}_3} \mathrm{P}H^j(X) \geq \dim_{\mathbb{F}_3} \mathrm{Q}H^{20}(X)$ *for* $j = 27, 39, 47$.

Proof. For $j = 39$, we can show the corollary by Lemma 21 for $m = 2$ and in a manner similar to that of the proof of Corollary 13. By Lemma 21 for $m = 4, 5$, we have linearly independent $\dim_{\mathbb{F}_3} \mathrm{Q}H^{20}(X)$ vectors of the form $\mathrm{ad}^3(a)(Q_0'' b) = [a^3, Q_0'' b] = v_3[(\wp^3)'' a, Q_0'' b]$ and of the form $\mathrm{ad}^4(a)(Q_0'' b) = v_3[a, [(\wp^3)'' a, Q_0'' b]]$, respectively, where $a, b \in \langle a''(\lambda)_{20} \mid \lambda \in \Lambda\rangle$. Multiplying such vectors by v_3^{-1} and applying an argument similar to that of the proof of Corollary 13, we have the corollary for $j = 27, 47$. □

4. Remarks

4.1. Relation to the result of Kane

In this subsection, we compare the result of Kane [11] for a special case with that of this paper. Let p be an odd prime.

On one hand, using secondary operations, Kane showed in [11] that if X is a 1-connected homotopy associative mod p finite H-space such that $\mathrm{Q}H^{2j}(X; \mathbb{F}_p) = 0$ unless $j = p+1$, then we have $0 \neq \mathrm{ad}^{p-2}(a)(Q_j a) \in \mathrm{P}H_*(X; \mathbb{F}_p)$ for any a,

$0 \neq a \in \mathrm{P}H_{2(p+1)}(X;\mathbb{F}_p)$ and for $j=0,1$. Moreover, applying the argument of this paper, we have $0 \neq \mathrm{ad}^{p-1}(a'')(Q''_j a'') \in K(2)_*(X)$ (in the notation similar to that in Section 3) and then we have

(4.1)
$$\dim_{\mathbb{F}_p} \mathrm{Q}H^{d(p,j,m)}(X;\mathbb{F}_p) - \dim_{\mathbb{F}_p} \mathrm{P}H^{d(p,j,m)}(X;\mathbb{F}_p) \geq \dim_{\mathbb{F}_p} \mathrm{Q}H^{2(p+1)}(X;\mathbb{F}_p)$$

for $j=0,1$ and $m=1,2,\ldots,p-2$ where $d(p,j,m)=2p^j+1+2m(p+1)$.

On the other hand, we have the following observation. Let n be a positive integer and k an integer with $n/(p-1) \leq k < n$. Let $M(p,n,k)_*$, a restricted Lie algebra over $\mathcal{A}_p$, be defined as follows. It has an $\mathbb{F}_p$-basis

$$\begin{aligned}\{a(r) \mid 1 \leq r \leq n\} \cup \{b(r,s),\ b'(r,s) \mid 1 \leq r \leq n,\ 0 \leq s \leq p-3\}\\ \cup \{c(t),\ c'(t) \mid 1 \leq t \leq k\},\end{aligned}$$

where $|a(r)| = 2(p+1)$, $b(r,s) = \mathrm{ad}^s(a(r))(Q_0 a(r))$, and $b'(r,s) = \mathrm{ad}^s(a(r))(Q_1 a(r))$ while $c(t) = \mathrm{ad}^{p-2}(a(r))(Q_0 a(r))$ and $c'(t) = \mathrm{ad}^{p-2}(a(r))(Q_1 a(r))$ for r congruent to t modulo k. (As in Lemma 21, put $\mathrm{ad}^0(a)(y)=y$, $\mathrm{ad}^1(a)(y)=[a,y]$, and $\mathrm{ad}^{j+1}(a)(y)=[a,\mathrm{ad}^j(a)(y)]$.) Moreover, it has the relation $[a(r),Q_j a(r')]=0$ for $r \neq r'$ and $j=0,1$. (Thus, a bracket $[a(r_1),[a(r_2),\ldots,[a(r_{l-1}),Q_j a(r_l)]\cdots]]$ of length $l \leq p-1$ is nonzero if and only if $r_1=\cdots=r_l$, because of the Jacobi identity.) The Frobenius map is trivial. The structure over $\mathcal{A}_p$ is given by the description above and by the restriction that the Lie bracket product respects the Cartan formula. (In particular, we have $-\wp^1 b(r,s)=b'(r,s)$ and $-\wp^1 c(t)=c'(t)$.)

Let $U(p,n,k)_*$ be the universal enveloping Hopf algebra over $\mathcal{A}_p$ of $M(p,n,k)_*$. (Note that $U(p,n,k)_*$ is associative and primitively generated, and that $\mathrm{P}U(p,n,k)_* \cong M(p,n,k)_*$.) Then, we have $0 \neq \mathrm{ad}^{p-2}(a)(Q_j a) \in \mathrm{P}U(p,n,k)_*$ for any a, $0 \neq a \in \mathrm{P}U(p,n,k)_{2(p+1)}$ and for $j=0,1$. So far, $U(p,n,k)_*$ might be realizable as the mod p homology of a 1-connected mod p finite H-space. (For example, as that of a product of $X(p)$'s and spheres with some multiplication where $X(p)$ is the 1-connected mod p finite H-space constructed by Harper [7].) However, the multiplication cannot be homotopy associative because of (4.1).

4.2. Adjoint action of a homotopy associative H-space

Let X be a 1-connected homotopy associative mod p finite H-space such that $\mathrm{Q}H^{2j}(X;\mathbb{F}_p)=0$ unless $j=p+1$. We can define the adjoint actions of X on itself and on ΩX (See Kono-Kozima [15].) and we can show (4.1) by using the adjoint actions, instead of the second Morava K-theory, and Lemma 5. For example, at $p=3$, extend the relations $a_8*(a_8*t_2)=\pm t_6^3$ and $a_8*(a_8*\bar{t}_6)=\bar{t}_{22}$ in $H_*(\Omega F_4;\mathbb{F}_3)$ and in $\mathrm{Q}H_*(\Omega F_4;\mathbb{F}_3)$ respectively (See Hamanaka-Hara [5].) to those in the general case and form the adjoint algebras corresponding to these relations. (Also see Hamanaka-Hara-Kono [6] for the case $p=5$.)

However, the author found some difficulties when he attempted to show (ii) of Theorem 1 by using the adjoint actions.

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