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Small deviations in p-variation for multidimensional Lévy processes

By

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Abstract

Let Z be an $\mathbb{R}^d$-valued Lévy process with strong finite p-variation for some $p < 2$. We prove that the "decompensated" process $\tilde{Z}$ obtained from Z by annihilating its generalized drift has a small deviations property in p-variation. This property means that the null function belongs to the support of the law of $\tilde{Z}$ with respect to the p-variation distance. Thanks to the continuity results of T. J. Lyons/D. R. E. Williams [19], [35], this allows us to prove a support theorem with respect to the p-Skorohod distance for canonical SDE's driven by Z without any assumption on Z, improving the results of H. Kunita [15]. We also give a criterion ensuring the small deviation property for Z itself, noticing that the characterization under the uniform distance, which we had obtained in [26], no more holds under the p-variation distance.

1. Introduction

In a series of celebrated papers [19] and [20], T. J. Lyons has built a general theory of rough differential equations. One of the main interests of this theory is the possibility to solve path-wise multidimensional stochastic equations whose driving paths have finite p-variation only for some $p > 1$. In contrast to Itô's theory, the equations are solved through convergence of the Picard iteration scheme, so that their solutions can be viewed as continuous functionals of the driving signal, with respect to some p-variation distance. Lyons' papers dealt only with the continuous case, and recently D. R. E. Williams ([35], [36]) has extended this theory to discontinuous stochastic equations, in particular when the driving noise is a Lévy process. Williams considered equations with jumps of Itô and Marcus type, but got continuity results only in the latter case. The continuity problem for equations of Itô type seems namely quite difficult, even in Dimension 1.

The purpose of this paper is to apply Lyons/Williams' results to the proof of a support theorem for S.D.E.'s of Marcus type driven by a multidimensional

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Lévy process. Namely, this continuity property allows us to reduce the difficult part of this kind of theorem (the approximation of an element of the support by the S.D.E. with positive probability) to a control on the driving path itself. In [25] we had already used this idea when the underlying Lévy process is one-dimensional. In this situation the continuity property holds with respect to the local uniform norm, so that we could appeal to the small deviation results for Lévy processes in uniform topology which we had obtained in [26]. Here our driving Lévy process Z is multidimensional, and we restrict ourselves to the case where it has finite p-variation for some $p < 2$. In particular we only make use of the "little theorem" of Lyons/Williams, and no control on the area process is involved.

However we make no other assumption on Z, so that our support theorem covers a wide class of driving Lévy processes without Gaussian part, and improves significantly the results of Kunita [15] on the subject. Our description of the support is also simpler, and more naturally related to the geometry of the underlying Lévy measure ν, as in Tortrat's famous article [34]: up to finitely many jumps and some fixed drift, this support is made out of a family of O.D.E.'s driven by functions with regular p-variation which are valued in the subspace of $\mathbb{R}^d$ consisting in the completely asymptotic directions of ν—see (2) below for details. Notice finally that thanks to the continuity property, we obtain the support theorem in a stronger topology than the local Skorohod one, taking into account the p-variation, and which we call the *p-Skorohod topology.*

The core of this article consists in the proof of the small deviations property in p-variation norm for the "decompensated process" $\tilde{Z}$ obtained from Z after annihilating its generalized drift. This generalized drift is just the sum of the usual drift and of the projection of finite 1-variations of the compensator in the Lévy-Khintchine formula—in particular the projection of $\tilde{Z}$ with finite 1-variations is the sum of its jumps. The small deviation property simply means that the null function belongs to the support of $\tilde{Z}$ with respect to the p-variation norm. In [26] we had already obtained this property for $\tilde{Z}$ under the uniform norm. Here the proof is somewhat analogous, but much more delicate. Roughly, denoting by L the above completely asymptotic subspace and by Π_L the orthogonal projection operator onto L, we need to approximate $\tilde{Z}$ by some "saw-function" whose slope is

$$v_L^\eta = \Pi_L \left(\int_{\eta \leq |z| \leq 1} z \, \nu(dz) \right)$$

for every small η along some subsequence. In other words, we must approximate v_L^η by a sum of the type

$$\sum_{i=1}^{r} \alpha_i^\eta x_i^\eta,$$

where r is a fixed integer, $x_1^\eta, \ldots, x_r^\eta \in \operatorname{Supp} \nu \cap \{|z| \leq \eta\}$, and $\alpha_1^\eta, \ldots, \alpha_r^\eta$ are minimizing integers verifying

$$\alpha_i^\eta |x_i^\eta|^p \to 0$$

for every $i = 1, \dots, r$, as η tends to 0 along the subsequence. The latter convergence is crucial because of the p-variation norm, but it is quite hard to obtain in full generality on the Lévy measure. We overcome the difficulties with the help of elementary analysis and geometry which require a lot of care, and where strict positivity (or strict convexity) plays a central rôle. A useful tool is also Skorohod's absolute continuity theorem for Lévy processes, which comes rather unexpectedly since the involved transformations are far less tractable than in the Cameron-Martin theorem.

In [26] a *characterization* of the small deviation property under the uniform norm for general multidimensional Lévy processes was obtained, in terms of interactions between the drift and the projection of finite 1-variations of the Lévy measure. Simple examples show that this characterization no more holds under the p-variation norm: there are Lévy processes with finite p-variation which have small deviations under the uniform norm but not under the p-variation norm. The characterization in the case $p = 1$ is easily proved to be $Z = \tilde{Z}$, that is Z itself is the sum of its jumps. In the case $p > 1$ and when the Lévy measure has infinite variations in every direction, we also prove that the small deviation property in p-variation always holds for Z. In the general case when $p > 1$ and the projection of Z with finite 1-variation is non trivial, we give a criterion involving the strict convexity of the asymptotic cône generated by Supp ν, a criterion which in some sense is optimal.

The organization of this article is as follows: in Section 2 we present the framework and state the main result of this paper—the small deviation property for $\tilde{Z}$, as well as its two corollaries—the criterion mentioned above and the support theorem for Marcus equations driven by Z. In this section we also give some examples, which might be helpful for the understanding of the proof of the main theorem. The latter, which is unfortunately quite technical, is given in Section 4. Before that, we give in Section 3 a few lemmas concerning some deterministic functions—in particular the saw-functions which are of central use in the proof of the main result—and their p-variation. In Section 5 we prove the corollaries.

2. Notations and results

2.1. Lévy processes and their p-variation

We work on $\mathbb{R}^d$ endowed with $|\cdot|$ any Euclidean norm. Let $p \geq 1$ and I be an interval of $\mathbb{R}^+$. A function $f : \mathbb{R}^+ \to \mathbb{R}^d$ is said to have finite (strong) p-variation over I if

$$\|f\|_{I,p} = \left(\sup_{t_0 < \cdots < t_k \in I} \sum_{j=1}^{k} |f(t_j) - f(t_{j-1})|^p \right)^{1/p} < \infty.$$

If $I = [0, T]$ for some $T \geq 0$ we will use the simpler notation $\|f\|_{T,p}$ for $\|f\|_{[0,T],p}$. Notice that for every $q \geq p$,

$$\|f_0\|_{I,\infty} \leq \|f\|_{I,q} \leq \|f\|_{I,p},$$

where we wrote $f_0(t) = f(t) - f(0)$ for every $t \geq 0$ and $\|\cdot\|_{I,\infty}$ stands for the uniform norm over I. Besides $\|\cdot\|_{I,p}$ is a Banach semi-norm which satisfies in particular the triangle inequality. We denote by $\mathcal{W}_p$ the space of functions having finite p-variation over every compact interval, factored by the set of constant functions. The family of semi-norms

$$\{\|\cdot\|_{n,p},\ n \geq 1\}$$

makes $\mathcal{W}_p$ into a Banach space with a norm $\|\cdot\|_p$ defined in the usual way:

$$\|f\|_p = \sum_{n\geq 1} 2^{-n}(1 \wedge \|f\|_{n,p})$$

for every $f \in \mathcal{W}_p$. A function $f : \mathbb{R}^+ \to \mathbb{R}^d$ is said to have *regular* finite p-variation over an interval I if

$$\lim_{\varepsilon\to 0}\left(\sup_{\substack{t_0<\cdots<t_k\in I\\ |t_j-t_{j-1}|\leq\varepsilon}} \sum_{j=1}^{k} |f(t_j) - f(t_{j-1})|^p\right) = 0.$$

Notice that this notion is only of interest for $p > 1$, and that if f is continuous with finite p-variation over I, then it has regular finite q-variation over I for every $q > p$. Notice also that every function with regular finite p-variation is necessarily continuous.

We now fix $x \in \mathbb{R}^d$ and denote by $\mathcal{W}_p(x)$ the space of functions having finite p-variation over every compact interval and starting from x. In the following we shall also work on $\mathbb{R}^m$ for some $m \neq d$, and we will still denote by $\mathcal{W}_p(x)$ the space of functions having finite p-variation over every compact interval and starting from $x \in \mathbb{R}^m$.

It is well-known and easy to see that every member of $\mathcal{W}_p(x)$ has left and right limits at every point of $\mathbb{R}^+$. We denote by $\mathcal{D}_p(x)$ the subspace of $\mathcal{W}_p(x)$ made out of *càd-làg* functions. We endow it with the following distance: if $f, g \in \mathcal{D}_p(x)$

$$\mathbf{d}_p(f,g) = \sum_{n\geq 1} 2^{-n}(1 \wedge \mathbf{d}_p^n(f,g)),$$

where for every $n \in \mathbb{N}^*$ $\mathbf{d}_p^n$ is defined by

$$\mathbf{d}_p^n(f,g) = \inf_{\lambda\in\Lambda}\left\{\sup_{s\leq t}\left|\log\frac{\lambda_t - \lambda_s}{t-s}\right| + \|k_n f(\lambda_.) - k_n g(\cdot)\|_{n+1,p}\right\},$$

Λ designing the set of all continuous strictly increasing functions $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lambda_0 = 0$ and $\lambda_t \uparrow +\infty$ as $t \uparrow +\infty$, and k_n being given by

$$k_n(t) = \begin{cases} 1 & \text{if} \quad t \leq n, \\ n+1-t & \text{if} \quad n < t \leq n+1, \\ 0 & \text{if} \quad t \geq n+1. \end{cases}$$

Such a λ will be called a *change of time* in the sequel. Making the same considerations as in [12, pp. 293–294] and using the fact that $\|\cdot\|_{n+1,p}$ is a semi-norm entails that $\mathbf{d}_p$ is actually a distance on $\mathcal{W}_p(x)$, which dominates the usual local Skorohod distance $\mathbf{d}$: for every $f, g \in \mathcal{D}_p(x)$,

$$\mathbf{d}(f,g) \leq \mathbf{d}_p(f,g).$$

In the sequel $\mathbf{d}_p$ will be called the *p-Skorohod distance* and the topology induced by $\mathbf{d}_p$ on $\mathcal{D}_p(x)$ the *p-Skorohod topology.*

Let $\{Z_t, t \geq 0\}$ be an $\mathbb{R}^d$-valued Lévy process starting from 0, without Gaussian part. Its Lévy-Itô decomposition writes

$$Z_t = \alpha t + \int_0^t \int_{|z|\leq 1} z\, \widetilde{\mu}(ds,dz) + \int_0^t \int_{|z|>1} z\, \mu(ds,dz),$$

where $\alpha \in \mathbb{R}^d$, ν is a positive Borel measure on $\mathbb{R}^d - \{0\}$ satisfying

$$\int_{\mathbb{R}^d} \frac{|z|^2}{|z|^2+1} \nu(dz) < \infty,$$

μ is the Poisson measure over $\mathbb{R}^+ \times \mathbb{R}^d$ with intensity $ds \otimes \nu(dz)$, and $\widetilde{\mu} = \mu - ds \otimes \nu$ is the compensated measure. Bretagnolle [4] obtained the following characterization: for every $1 \leq p < 2$

$$Z \in \mathcal{W}_p(0) \ \text{ a.s. } \iff \int_{|z|\leq 1} |z|^p \nu(dz) < \infty$$

(notice that the equivalence is trivial for $p = 1$). In particular every stable process has finite p-variation for some $p < 2$. Recall that on the contrary Brownian Motion has infinite 2-variation, so that the above characterization only makes sense for Lévy processes without Gaussian part.

In the case $p > 1$, Bretagnolle got actually a sharper result: the existence of two universal constants c_p and C_p depending only on p such that

$$c_p \int_{|z|\leq 1} |z|^p \nu(dz) \leq \mathbb{E}[\|Z\|_{1,p}^p] \leq C_p \int_{|z|\leq 1} |z|^p \nu(dz)$$

if ν is concentrated on $\{|z| \leq 1\}$ and $\alpha = 0$. Using the inequality $|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$ for any $a, b \in \mathbb{R}^d$ entails easily that for every $T > 0$

$$2^{1-p} c_p T \int_{|z|\leq 1} |z|^p \nu(dz) \leq \mathbb{E}[\|Z\|_{T,p}^p] \leq 2^{p-1} C_p T \int_{|z|\leq 1} |z|^p \nu(dz)$$

if ν is concentrated on $\{|z| \leq 1\}$ and $\alpha = 0$. This latter estimate yields readily the following approximation lemma—notice that the case $p = 1$ is trivial:

Lemma 1. *Let $[\mathcal{E}_\eta, \eta > 0]$ be a family of subsets of $\mathbb{R}^d$ included, for every $\eta > 0$, in the ball of radius η centered at the origin. Let $Z \in \mathcal{W}_p(0)$ be* *a Lévy process with drift α and Lévy measure ν. Let Z^η be the Lévy process with same drift and Lévy measure $\mathbf{1}_{\mathcal{E}_\eta^c}\nu$ and set $\tilde{Z}^\eta = Z - Z^\eta$. Then for every $T > 0$,*

$$\mathbb{E}[\|\tilde{Z}^\eta\|_{T,p}] \longrightarrow 0$$

as $\eta \downarrow 0$.

The following u.c.p. lemma (also trivial in the case $p = 1$) is another immediate application of Bretagnolle's estimate, thanks to the Markov inequality:

Lemma 2. *Let ν_0 be a Lévy measure on $\mathbb{R}^d$ concentrated on $\{|z| \leq 1\}$, and integrating $|z|^p$. For every $\varepsilon > 0$, $0 < c < 1$ and $T > 0$, there exists $\eta_0 > 0$ such that for every $\eta < \eta_0$ and every Lévy process $Z \in \mathcal{W}_p(0)$ with drift $\alpha = 0$ and Lévy measure $\nu \leq \nu_0$,*

$$\mathbb{P}[\|\tilde{Z}^\eta\|_{T,p} < \varepsilon] > c$$

with the notations of Lemma 1, and where the notation $\nu \leq \nu_0$ means that $\nu(A) \leq \nu_0(A)$ for every measurable $A \subset \mathbb{R}^d$.

2.2. Small deviations in p-variation norm for Lévy processes

As in [34] and [23] we introduce the following vector space

$$K = \left\{ x \in \mathbb{R}^d \,\middle|\, \int_{|z|\leq 1} |x * z| \, \nu(dz) < \infty \right\},$$

where $*$ is the scalar product defining the chosen Euclidean norm on $\mathbb{R}^d$. Notice that the vector space $L = K^\perp$, which can be viewed as the completely asymptotic direction of $\mathbf{1}_{|z|\geq\eta}\nu(dz)$ as $\eta \downarrow 0$, depends *only* on ν and not on the choice of this Euclidean structure. In the following, every Lévy process with the above Lévy-Itô decomposition will be said to have characteristics (α, ν), and K will always implicitly stand for the orthogonal space of L with respect to the chosen scalar product. We define the *generalized drift* of a Lévy process Z with characteristics (α, ν) by

$$\alpha_\nu = \alpha - \int_{|z|\leq 1} z_K \, \nu(dz),$$

where z_K is the orthogonal projection of z onto K, so that the integral makes sense. We finally introduce the *decompensated process* $\tilde{Z}$ associated with Z, which is the Lévy process with characteristics $(\alpha - \alpha_\nu, \nu)$. Equivalently

$$\tilde{Z}_t = \int_0^t \int_{|z|\leq 1} z_K \, \mu(ds, dz) + \int_0^t \int_{|z|\leq 1} z_L \, \widetilde{\mu}(ds, dz) + \int_0^t \int_{|z|>1} z \, \mu(ds, dz)$$

for every $t > 0$, where z_L denotes the orthogonal projection of z onto L. The main result of this paper is the following

Theorem. *Let $Z \in \mathcal{W}_p(0)$ be a Lévy process with characteristics (α, ν). Its decompensated process $\tilde{Z}$ has the following small deviation property*:

$$\mathbb{P}[\|\tilde{Z}\|_{T,p} < \varepsilon] > 0$$

for every $\varepsilon > 0$ and $T > 0$.

We now recall a few notations from [34], [23], and [26]: for every $\eta > 0$, set $\mathcal{C}^\eta$ for the closed convex cône with vertex 0 generated by $\mathcal{S}^\eta = \operatorname{Supp}\nu \cap \{|z| \le \eta\}$, and

$$\mathcal{C} = \bigcap_{\eta > 0} \mathcal{C}^\eta.$$

Let Π_K be the operator of orthogonal projection onto K,

$$\mathcal{A}_K = \left(\int_{|z|\le 1} z_K\,\nu(dz)\right) - \overline{\Pi_K(\mathcal{C})} \text{ and } \mathcal{B}_K = \left(\int_{|z|\le 1} z_K\,\nu(dz)\right) - \bigcap_{\eta>0} \overline{\Pi_K\left(\mathcal{C}^\eta\right)}.$$

It follows from the main result of [26] that if $Z \in \mathcal{W}_p(0)$ is a Lévy process with characteristics (α, ν), then the following equivalence holds:

$$\alpha \in \Pi_K^{-1}(\mathcal{B}_K) \Longleftrightarrow \mathbb{P}[\|Z\|_{T,\infty} < \varepsilon] > 0 \qquad \text{for} \quad \text{every } \varepsilon > 0 \text{ and } T > 0.$$

One could wonder if the same characterization holds under the p-variation norm, i.e. if one could replace ∞ by p in the above right-hand side. However this is not true, as shows the following example.

Example 3. Consider on $\mathbb{R}^2 = \{(z_1, z_2)\}$ endowed with the canonical basis (e_1, e_2), the following measure

$$\nu(dz) = \mathbf{1}_{\{0<z_1<|z_2|^r<c_r\}}|z_2|^{-2-q}\,dz,$$

where q and r are such that $1 < (1+q)/2 < r < q < r+1$ (notice that q can take any value strictly greater than 1), and c_r is the unique positive solution to $x^2 + x^{2/r} = 1$. Then ν is a jumping measure concentrated on $\{|z| \le 1\}$, whose asymptotic subspaces are $K = \operatorname{Vect}\{e_1\}$ and $L = \operatorname{Vect}\{e_2\}$. Besides, with the above notations, $\Pi_K^{-1}(\mathcal{A}_K) = \Pi_K^{-1}(\mathcal{B}_K) = \{z_1 \le c\}$ where we set

$$c = \frac{1}{2r - q - 1}.$$

Let Z be the Lévy process given by

$$Z_t = \int_0^t \int_{|z|\le 1} z\,\widetilde{\mu}(ds, dz)$$

for every $t > 0$. With our notations, $\alpha = 0 \in \Pi_K^{-1}\left(\mathcal{B}_K\right)$. On the other hand, $Z \in \mathcal{W}_p(0)$ a.s. for every $p > 1 + q - r$ (notice that $0 < q - r < 1$). Set now $p \in]1+q-r, r[$ (notice that $r > 1+q-r$) and let Z^1 (resp. Z^2) be the projection of Z onto K (resp. onto L). For every $0 < \varepsilon < c/2$,

$$\begin{aligned}\{\|Z\|_{1,p} < \varepsilon\} &\subset \left\{\sup_{t\leq 1}|\Delta Z_t| < \varepsilon,\ \sum_{t\leq 1}\Delta Z_t^1 > c/2\right\}\\ &\subset \left\{\sup_{t\leq 1}|\Delta Z_t| < \varepsilon,\ \sum_{t\leq 1}|\Delta Z_t^2|^r > c/2\right\}\\ &\subset \left\{\sum_{t\leq 1}|\Delta Z_t|^p > \varepsilon^{p-r}c/2\right\},\end{aligned}$$

which proves that

$$\mathbb{P}[\|Z\|_{1,p} < \varepsilon] = 0$$

as soon as $\varepsilon < (c/2)^{1/r}$, since obviously

$$\|Z\|_{1,p} \geq \left(\sum_{t\leq 1}|\Delta Z_t|^p\right)^{1/p} \quad \text{a.s.}$$

Nevertheless, our main result makes it possible to obtain the following criterion. We say that a closed convex cône with vertex 0 in $\mathbb{R}^d$ is *strictly convex* if it contains no half-plane.

Corollary A. *Let $Z \in \mathcal{W}_p(0)$ be a Lévy process on $\mathbb{R}^d$ with characteristics (α, ν). With the above notations,*

(a) *If $K = \mathbb{R}^d$, then Z has small deviations in 1-variation if and only if $\alpha_\nu = 0$.*

(b) *If $L = \mathbb{R}^d$, then Z has small deviations in p-variation $(p > 1)$.*

(c) *If $L \neq \mathbb{R}^d, \alpha \in \Pi_K^{-1}(\mathcal{A}_K)$ and $\mathcal{C}$ is strictly convex, then Z has small deviations in p-variation $(p > 1)$.*

(d) *If $L \neq \mathbb{R}^d$ and $\alpha \notin \Pi_K^{-1}(\mathcal{B}_K)$, then Z does not have small deviations in p-variation $(p > 1)$.*

Remarks. (a) Even if $\mathcal{C}$ is strictly convex, the condition $\alpha \in \Pi_K^{-1}(\mathcal{B}_K)$ is not sufficient, as the following example easily shows. Consider

$$\nu(dz) = \mathbf{1}_{\{0<z_1<z_2^r<c_r\}} z_2^{-(2+q)}\, dz$$

on $\mathbb{R}^+ \times \mathbb{R}^+$, with the notations of Example 3. Here,

$$\Pi_K^{-1}(\mathcal{A}_K) = \left\{z_1 = \frac{1}{2(2r-q-1)}\right\} \text{ and } \Pi_K^{-1}(\mathcal{B}_K) = \left\{z_1 \leq \frac{1}{2(2r-q-1)}\right\}.$$

The process Z defined as in Example 3 verifies of course $\alpha \in \Pi_K^{-1}(\mathcal{B}_K)$, but does not have small deviations in p-variation either.

(b) If $\mathcal{C}$ is strictly convex, the condition $\alpha \in \Pi_K^{-1}(\mathcal{A}_K)$ is sufficient but *not* necessary, as shows the following example. Consider on $\mathbb{R}^+ \times \mathbb{R}^+$

$$\nu(dz) = \nu_1(dz) + \nu_2(dz),$$

where

$$\nu_1(dz) = \mathbf{1}_{\{0<z_1<z_2^r<c_r\}}\, dz \quad \text{and} \quad \nu_2(dz) = \sum_{n\geq 1} n^q \delta_{(0,n^{-1})}(dz).$$

Here $0 < q < 1 < r$ and c_r is the unique positive solution to $x^2 + x^{2/r} = 1$. Then ν is a jumping measure concentrated on $\{|z| \leq 1\}$, whose asymptotic subspaces are $K = \text{Vect}\{e_1\}$ and $L = \text{Vect}\{e_2\}$. Besides

$$\Pi_K^{-1}(\mathcal{A}_K) = \left\{ z_1 = \frac{1}{2(2r+1)} \right\} \quad \text{and} \quad \Pi_K^{-1}(\mathcal{B}_K) = \left\{ z_1 \leq \frac{1}{2(2r+1)} \right\}.$$

Let $\alpha \in \mathbb{R}^2$ with $\alpha_1 < 1/(2(2r+1))$ and consider Z the Lévy process with characteristics (α, ν). Clearly $Z \in \mathcal{W}_p(0)$ a.s. if $p > q + 1$. We briefly show that Z has small deviations in p-variation norm, even though $\alpha \notin \Pi_K^{-1}(\mathcal{A}_K)$. Set μ_i for the Poisson measure on $\mathbb{R}^+ \times \mathbb{R}^d$ with compensator $ds \otimes \nu_i(dz)$, $i = 1, 2$. Introduce the compound Poisson process

$$Z_t^1 = \int_0^t \int_{|z|\leq 1} z\, \mu_1(ds, dz) = \sum_{s\leq t} \Delta Z_s^1$$

for every $t > 0$, and let $\{T_n, Z_n\}_{n\geq 1}$ be the sequence of its successive jumping times and sizes. Let $T_0 = 0$ and $S_n = T_n - T_{n-1}$ for every $n \geq 1$. Set finally

$$\beta = \frac{1}{2(2r+1)} - \alpha_1.$$

Fix $p > q + 1$, $\varepsilon, T > 0$. We will show that

$$\mathbb{P}[\|Z\|_{T,p} < \varepsilon] > 0$$

in using the independence of Z^1 and μ_2. Take $\eta > 0$ such that $6cT\eta^{p-r} < \varepsilon$. Consider $P_\eta = (\eta^r, \eta) \in \text{Supp}\, \nu_1$ and $t_\eta = \eta^r/cT$. For every $\lambda > 0$ the event

$$\Omega_\lambda = \{|S_n - t_\eta| < \lambda,\ |Z_n - P_\eta| < \lambda,\ \forall n = 1, \ldots, (\text{Ent}[cT/\eta^r] + 1)\}$$

has positive probability. On the other hand, it follows from Lemma 11 below that

$$\Omega_\lambda \subset \{\|Z_\eta^1\|_{T,p} < \varepsilon/2\}$$

if λ is small enough, where we set

$$Z_\eta^1(t) = \sum_{s\leq t} \Delta Z_s^1 - tc(e_1 + \eta^{1-r} e_2)$$

for every $t > 0$. But by Corollary A,

$$\mathbb{P}[\|Z_\eta^2\|_{T,p} < \varepsilon/2] > 0,$$

where we introduced the process

$$Z_\eta^2(t) = \int_0^t \int_{|z|\le 1} z\, \widetilde{\mu}_2(ds, dz) + t(\eta^{1-r}c + \alpha_2)e_2$$

for every $t > 0$. Since $Z = Z_\eta^1 + Z_\eta^2$ for every $\eta > 0$, with Z_η^1 and Z_η^2 independent, we finally get

$$\mathbb{P}[\|Z\|_{T,p} < \varepsilon] > 0$$

by the triangle inequality. We stress finally that the above "approximation event" Ω_λ will be introduced repeatedly during the proof of our main result, in various forms.

We next give two more classical examples which fall into the scope of our Theorem and Corollary A. Every concerned Lévy process shall be written in its canonical form

$$Z_t = \alpha t + \int_0^t \int_{|z|\le 1} z\, \widetilde{\mu}(ds, dz) + \int_0^t \int_{|z|>1} z\, \mu(ds, dz),$$

and we shall discuss the shape of the jumping measure ν. We refer to Chapter 3 in [22] for an extensive account on these two examples.

Example 4 (Stable processes)**.** The measure ν is given in the integral form

$$\nu(B) = \int_{\mathcal{S}^{d-1}} \lambda(d\xi) \int_0^{+\infty} \mathbf{1}_B(r\xi) \frac{dr}{r^{1+\beta}}$$

for every measurable set $B \subset \mathbb{R}^d$, where $0 < \beta < 2$ and λ is some finite positive measure on $\mathcal{S}^{d-1}$. We suppose that ν is non-degenerated, i.e. $\operatorname{Supp}\lambda$ is not included in any hyper-plane of $\mathbb{R}^d$. Hence, with the above notations, either $\beta < 1$ and $K = \mathbb{R}^d$, or $1 \le \beta < 2$ and $L = \mathbb{R}^d$. It is clear that $Z \in \mathcal{W}_p(0)$ if and only if $p > \beta$. Corollary A reads

(a) If $\beta < 1$, then Z has small deviations in 1-variation norm if and only if

$$\alpha = \frac{1}{1-\beta}\left(\int_{\mathcal{S}^{d-1}} \xi\, \lambda(d\xi)\right),$$

i.e. if and only if Z is strictly stable (or the sum of its jumps).

(b) If $\beta \ge 1$, then Z has small deviations in p-variation norm for every $p > \beta$.

Besides, since here $\mathcal{C} = \mathcal{C}_\lambda$ where $\mathcal{C}_\lambda$ is the convex cône generated by $\operatorname{Supp}\lambda$, one can improve (c) and (d) in Corollary A and show that if $\beta < 1$, then Z has small deviations in p-variation norm ($p > 1$) if and only if

$$\frac{1}{1-\beta}\left(\int_{\mathcal{S}^{d-1}} \xi\, \lambda(d\xi)\right) - \alpha \in \mathcal{C}_\lambda.$$

Of course, much more can be said about stable processes. If $\alpha = 0$ and λ is a symmetric measure (the so-called symmetric β-stable case) then for every $\gamma > \beta$, the γ-variation of Z over $[0,1]$ is given by

$$\|Z\|_{1,\gamma} = \left(\sum_{t \le 1} |\Delta Z_t|^\gamma \right)^{1/\gamma}$$

if $\gamma \le 1$ (notice that this expression makes sense even if $\beta < \gamma < 1$), and satisfies

$$\|Z\|_{1,\gamma} \ge \left(\sum_{t \le 1} |\Delta Z_t|^\gamma \right)^{1/\gamma}$$

if $\gamma > 1$. Notice that the process

$$S : t \longmapsto \sum_{s \le t} |\Delta Z_s|^\gamma$$

is a (β/γ)-stable subordinator, whose Laplace transform is given by

$$\mathbb{E}[\exp -uS_1] = \exp - \left[\frac{c_\lambda \Gamma(1-\delta)}{\beta} u^\delta \right],$$

where we set $\delta = \beta/\gamma$ and

$$c_\lambda = \int_{\mathcal{S}^{d-1}} |\xi|^\gamma \lambda(d\xi)$$

(see e.g. Example 24.12. in [22]). Hence, by De Bruijn's Tauberian theorem (see Theorem 4.12.9 in [2]), we get

$$-\log \mathbb{P}[S_1 < \varepsilon] \sim (1-\delta) \left(\frac{c_\lambda \Gamma(1-\delta)}{\gamma \varepsilon} \right)^{\delta/(1-\delta)}$$

as $\varepsilon \to 0$. This leads to

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{\gamma\beta}{\gamma-\beta}} \log \mathbb{P}[\|Z\|_{1,\gamma} < \varepsilon] = - \left(\frac{(\gamma-\beta)(c_\lambda \Gamma(1-\beta/\gamma))^{\frac{\beta}{\gamma-\beta}}}{\gamma^{\frac{\gamma}{\gamma-\beta}}} \right) = -C_{\beta,\gamma}$$

if $\beta < \gamma \le 1$, and to

$$\limsup_{\varepsilon \to 0} \varepsilon^{\frac{\gamma\beta}{\gamma-\beta}} \log \mathbb{P}[\|Z\|_{1,\gamma} < \varepsilon] \le - \left(\frac{(\gamma-\beta)(c_\lambda \Gamma(1-\beta/\gamma))^{\frac{\beta}{\gamma-\beta}}}{\gamma^{\frac{\gamma}{\gamma-\beta}}} \right)$$

if $\gamma > 1$. This can be viewed as small ball probability estimates for symmetric β-stable processes under the γ-variation norm, and can probably be extended to more general symmetric Lévy processes without Gaussian part. In [28], we prove that

$$\lim_{\varepsilon \to 0} -\varepsilon^{\frac{\gamma\beta}{\gamma-\beta}} \log \mathbb{P}[\|Z\|_{1,\gamma} < \varepsilon]$$

exists and is finite for *every* $\gamma > \beta$. However we could not identify this limit as yet, when $\gamma > 1$.

Remarks. (a) When γ tends to $+\infty$, the constant $C_{\beta,\gamma}$ tends to 1 whereas heuristically, the above limit tends to

$$\lim_{\varepsilon\to 0} -\varepsilon^{\beta} \log \mathbb{P}[\|Z\|_{1,\omega} < \varepsilon],$$

where $||Z||_{1,\omega}$ stands for the oscillation of Z over $[0,1]$ (see [5] Proposition 2.3. p. 27). It follows from the classical result of Taylor [32] under the uniform norm (see also [3] Proposition VIII.3) and standard sublinearity arguments that the latter limit actually exists and belongs to $]0,+\infty[$ (but nothing is known about its explicit value).

(b) For a linear Brownian motion W, it follows from the general result of Stolz [29] that

$$0 < \liminf_{\varepsilon\to 0} -\varepsilon^{\frac{2p}{p-2}} \log \mathbb{P}[\|W\|_{1,p} < \varepsilon] \leq \limsup_{\varepsilon\to 0} -\varepsilon^{\frac{2p}{p-2}} \log \mathbb{P}[\|W\|_{1,p} < \varepsilon] < +\infty$$

for every $p > 2$. This speed of convergence is in accordance with the results of Baldi and Roynette under the Hölder norms [1], and with our previous computation for non-Gaussian symmetric stable processes. It follows from the general results of [18] that the limit actually exists, but we were not able to identify it as yet. As far as we know, the value of the small ball constant for linear Brownian motion is still unknown under the Hölder norms (see the introduction in [1]).

(c) We notice finally that in two celebrated papers [33] and [8], the *exact* variation functions of Brownian motion and stable processes had been computed.

Example 5 (Self-decomposable processes)**.** The measure ν is given in the integral form

$$\nu(B) = \int_{\mathcal{S}^{d-1}} \lambda(d\xi) \int_0^{+\infty} \mathbf{1}_B(r\xi) k_\xi(r) \frac{dr}{r}$$

for every measurable set $B \subset \mathbb{R}^d$, where λ is some finite positive measure on $\mathcal{S}^{d-1}$, and $k_\xi(r)$ is a non-negative function measurable in ξ and decreasing in r.

These class of processes includes the above stable ones, but its range is much wider. In particular it is possible that K and L together are non-trivial: consider for example over $\mathbb{R}^3 = \{(z_1, z_2, z_3)\}$ endowed with its canonical Euclidean strcture,

$$\lambda(d\theta, d\phi) = \mathbf{1}_{\{0\leq\theta,\phi\leq\pi/2\}} d\theta\, d\phi \quad \text{and} \quad k_{\theta,\phi}(r) = \frac{\sin\phi\, \cos\theta}{r^{\cos\theta}} \mathbf{1}_{\{r\leq 1\}},$$

where we used the spherical coordinates given by $z_1 = r\sin\phi, z_2 = r\cos\phi\sin\theta$ and $z_3 = r\cos\phi\cos\theta$. Then, with the above notations, we get $K = \mathrm{Vect}\{e_1\}$, $L = \mathrm{Vect}\{e_2, e_3\}$, and

$$\Pi_K^{-1}(\mathcal{A}_K) = \Pi_K^{-1}(\mathcal{B}_K) = \{z_1 \leq 1 + \pi/2\}.$$

The associated process Z has finite p-variation for every $p > 1$ and since $\mathcal{C}$ is clearly strictly convex, Corollary A entails that Z has small deviations in p-variation norm if

$$\alpha_1 \leq 1 + \pi/2$$

(notice that here the reverse inclusion is actually true, since $\alpha_1 > 1 + \pi/2$ entails, with the above notation for Z^1, that $Z_1^1 > \alpha_1 - (1 + \pi/2)$ a.s.).

In this example, we stress that even though $\operatorname{Dim} L = 2$, there is actually just *one* asymptotic direction $u \in L$ as far as our problem is concerned. Indeed, if we set

$$v_L^\eta = \Pi_L \left(\int_{\eta \leq |z| \leq 1} z \, \nu(dz) \right),$$

then we see that $v_L^\eta = \rho_\eta u$ where u is the fixed unit vector

$$u = \frac{2e_2 + \pi e_3}{\sqrt{\pi^2 + 4}}$$

and where

$$0 < \rho_\eta \leq \int_{\eta \leq |z| \leq 1} |z| \, \nu(dz),$$

for every $0 < \eta < 1$. This property, which is not at all a feature of self-decomposable processes (think of $k_{\theta,\phi}(r) = r^{-1}\mathbf{1}_{\{\theta=0,\phi=0\}} + r^{-3/2}\mathbf{1}_{\{\theta=0,\phi=\pi/2\}}$ on $\mathbb{R}^3$), makes the proof of the Theorem somewhat simpler (see Subsection 4.3.1 below).

In any case, we notice that the self-decomposable case is not really relevant to the generality of our result. The monotonicity condition on $k_\xi(r)$ entails namely that

$$\mathcal{C} = \mathcal{C}^1 = \mathcal{C}^\eta$$

for every $\eta > 0$, a feature which also simplifies significantly the proof of the Theorem (see the first paragraph in Subsection 4.3.2 below). Actually we have even more: for every $x \in \mathcal{C}$, there exists $x_1, \ldots, x_d \in \operatorname{Supp} \nu$ and $\lambda_1, \ldots, \lambda_d > 0$ such that

$$x = \sum_{i=1}^{d} \lambda_i x_i$$

together with $\mu x_i \in \operatorname{Supp} \nu$ for *every* $\mu > 0$ and $i = 1, \ldots, d$. This latter property makes the proof really easy. The following example, which revisits Example 3, depicts a typical situation where our main result is more difficult to obtain.

Example 6 (A pathological measure)**.** Consider on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ endowed with the canonical Euclidean structure, the following measure:

$$\nu(dz) = \mathbf{1}_{\{0 < z_1 < z_2^r < z_3^{rs} < 1\}} z_3^{-2-q} dz,$$

where $r, s > 1$ and $s(r+1) + s < q + 1 < s(r+1) + 2 \wedge rs$ (notice that q can take any value strictly greater than 2). Then ν is a jumping measure whose asymptotic subspaces are $K = \mathrm{Vect}\{e_1\}$ and $L = \mathrm{Vect}\{e_2, e_3\}$. In this example, the first difficulty comes from the fact that

$$\frac{|v_L^\eta|}{|v_3^\eta|} \to 1, \ \frac{|v_3^\eta|}{|v_2^\eta|} \to +\infty \quad \text{and} \quad |v_2^\eta| \to +\infty$$

as $\eta \downarrow 0$ (with the obvious notations for v_2^η and v_3^η), so that here we must cope with more than one asymptotic direction. The second difficulty comes from the degenerescence of $\mathcal{C} = \{z_1 = z_2 = 0, \ z_3 \geq 0\}$. In particular

$$\Pi_3^\perp(\mathcal{C}) = \{0\} \neq \{z_1 \geq 0, z_2 \geq 0\} = \bigcap_{\eta > 0} \overline{\Pi_3^\perp(\mathcal{C}^\eta)},$$

where $\Pi_3^\perp$ stands for the operator of orthogonal projection onto $\mathrm{Vect}\{e_1, e_2\}$. This very pathological situation is the matter of Subsection 4.3.2, more particularly of its second paragraph.

2.3. Support theorem in p-Skorohod topology for Marcus equations

The principal motivation for our above small deviation result is to prove a support theorem [30] for a class of stochastic integral equations driven by Z, without any assumption on Z but the finiteness of its p-variation for some $1 \leq p < 2$. In this subsection, every index p will be implicitly supposed to belong to $[1, 2[$. We consider on $\mathbb{R}^m$

$$X_t = x + \int_0^t f(X_{s-}) \diamond dZ_s, \tag{1}$$

where $x \in \mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}^m \otimes \mathbb{R}^d$ is a function which is α-Lipschitz for some $p < \alpha < 2$: f is bounded with bounded derivatives $\partial_j f$ verifying

$$\sup_{x \neq y} \frac{|\partial_j f(x) - \partial_j f(y)|}{|x - y|}^{\alpha - 1} < +\infty.$$

for $j = 1, \ldots, m$. In (1), the integral is defined followingly:

$$\int_0^t f(X_{s-}) \diamond dZ_s = \int_0^t f(X_{s-}) dZ_s + \sum_{s \leq t} g(X_{s-}, \Delta Z_s),$$

where the first integral is a standard Itô integral and $g : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^m$ is a local Lipschitz function such that when (x, z) stays in a fixed compact set of $\mathbb{R}^m \times \mathbb{R}^d$

$$|g\,(x, z)| \leq K|z|^2$$

for some constant K, and such that in (1), each time t when Z jumps, X_t is given by the integral in time 1 of the vector field $x \mapsto f(x)\Delta Z_t$, starting from X_{t-}.

Introduced by Marcus [25], these stochastic equations are fairly often studied in the literature (see e.g. [9], [16], [14], [10], [11], [36], [7]), even though they concern a specific class of integrand. Their main interest is that they share nice flow properties, and this is not always the case for classical Itô equations with jumps. Quoting Theorem 7.3.1 in [35] (which mostly follows from the main result of [19]), we get the following result, which will be the central tool in proving our support theorem:

Theorem 7 (T. J. Lyons and D. R. E. Williams)**.** *Equation* (1) *can be solved path-wise, and has a unique solution. Besides, the map*

$$\Phi : \begin{cases} \mathbb{R}^m \times \mathcal{W}_p \longrightarrow \mathcal{W}_p, \\ (x, Z) \longmapsto X, \end{cases}$$

is local Lipschitz, where X is the equivalence class of the unique solution to (1) *starting from x, and $\mathcal{W}_p$ is endowed with the p-variation norm.*

Remarks. (a) The local Lipschitz property of Φ means: for every $T > 0$, for every compact set of $\mathbb{R}^m \times \mathcal{W}_p\left([0,T], \mathbb{R}^d\right)$ with respect to the norm $|\cdot| + \|\cdot\|_{T,p}$, there exists a constant K such that for every $(x,u),(y,v)$ in this compact set,

$$\|\Phi(x,u) - \Phi(y,v)\|_{T,p} \leq K(|x-y| + \|u-v\|_{T,p}).$$

(b) If Z is one-dimensional, then $\Phi : \mathbb{R}^m \times \mathcal{D} \longrightarrow \mathcal{D}$ is actually continuous with respect to the local uniform norm [7]. Of course, this is no more true when Z is multidimensional and the vector fields defining f do not commute, as it is easily seen by transferring Sussmann's well-known counterexample (see p. 40 in [31]) to pure jump processes.

We want to find the support of X solution of (1) in $(\mathcal{D}_p(x), \mathbf{d}_p)$. Recall that by definition this set is made out of functions $\phi \in \mathcal{D}_p(x)$ such that for every $n \in \mathbb{N}^*$ and $\varepsilon > 0$

$$\mathbb{P}[\mathbf{d}_p^n(X, \phi) < \varepsilon] > 0.$$

As in [25] we set $\mathbf{U}$ for the set of sequences $u = \{u_p\} = \{t_p, z_p\}$, where $\{t_p\}$ is an increasing sequence in $(0, +\infty)$ tending to $+\infty$ and $\{z_p\}$ a sequence in Supp $\nu - \{0\}$. For every $u \in \mathbf{U}$ and every function $\phi_L : \mathbb{R}^+ \to L$ with regular p-variation, we set

$$\phi_t^L = \phi_L(t) + t\alpha_\nu$$

for every $t \geq 0$, and introduce the following piecewise differential equation:

$$\psi_t = x + \int_0^t f(\psi_s) d\phi_s^L + \sum_{t_p \leq t} g_f(\psi_{t_p-}, z_p), \tag{2}$$

where we wrote $g_f(x,z) = f(x)z + g(x,z)$ for every $(x,z) \in \mathbb{R}^m \times \mathbb{R}^d$. Notice that since (2) has finitely many jumps on every compact time interval, since ϕ^L has regular p-variation, and since f is α-Lipschitz with $\alpha > p$, the main result of [19] states precisely that there exists a unique solution to (2), which belongs to $\mathcal{D}_p(x)$. Let $\mathcal{S}$ be the set of solutions to (2), u varying in $\mathbf{U}$ and ϕ_L in the set of functions from $\mathbb{R}^+$ to L with regular p-variation. Set $\overline{\mathcal{S}}$ for the closure of $\mathcal{S}$ in $(\mathcal{D}_p(x), \mathbf{d}_p)$. Using together our Theorem and Theorem 7 entails the following

Corollary B. *Let f be α-Lipschitz and $X \in \mathcal{D}_p(x)$ be the unique solution to* (1). *Then*

$$\operatorname{Supp} X = \overline{\mathcal{S}}.$$

Remarks. (a) The condition that f is α-Lipschitz entails in particular that f is bounded, which is a bit annoying if one wishes to consider e.g. linear equations. In this Corollary we can actually get rid of the boundedness assumption through a standard approximation argument, which we did not include here for the sake of brevity.

(b) In [25] a support theorem was proved in the local Skorohod topology for equation (1) without any assumption on Z and under weaker assumptions on f, provided the stochastic part of Z is one-dimensional. H. Kunita [15] had treated the multidimensional case, but his description of the support is complicated, and his results holds under stringent conditions on the Lévy measure.

(c) In both papers [15] and [25] the driving process was allowed to have a Gaussian part. Here we cannot cope with this situation, since Theorem 7 no more holds when the driving process has only finite p-variation for some $p \geq 2$. Notice that Lévy processes without Gaussian part may also have infinite p-variation for every $p < 2$ —see Example 2.1 in [36]. In the Brownian case, Ledoux, Qian and Zhang ([17]) got recently a new proof of Stroock-Varadhan's theorem with the help of Lyons' continuity theorem ([20]) and a suitable control in p-variation ($2 < p < 3$) on the driving Brownian path together with its Lévy area process. The same kind of arguments combined with Williams' adaptation of Lyons' theory to jump processes ([35], [36]) could actually be a successful approach to prove the support theorem for Marcus equations in full generality on the Lévy driving path. However this method promises to be highly technical.

(d) The unique solution to equation (2) is invariant under $(1+\alpha)$-Lipschitz changes of coordinates—see the final Remarks in [19]. On the other hand, since the integral $\diamond$ is defined through exponentiation of vector fields, (1) is also coordinate-free and may be studied on a nice manifold ([9], [14]). It is essentially trivial that with an intrinsic definition for the function g_f, Corollary B also holds in this more general framework.

(e) Viewing $\mathbb{R}^m$ not as a manifold but as a vector space, classical Itô equations with jumps are more natural (and more general) objects than Marcus equations. But even in dimension 1 continuity results are not known for such equations, and seem actually quite difficult to prove. We refer however to a recent survey of Dudley and Norvaiša [6] for results in this direction in the case of the Doléans-Dade equation. In [27] a support theorem is obtained in full generality on Z for 1-dimensional Itô equations, viewing the latter as perturbations of Marcus equations and using a comparison's lemma. This method no more holds in the multidimensional case. See also [24] for partial results, under heavy assumptions on the Lévy measure.

(f) In the literature, there does not seem to exist controllability results in p-variation for equation (2). However, considering supports for the local Skorohod topology and using the classical results of [13] we can prove, as in [25], that if $L = \mathbb{R}^d$

$$\{\mathrm{Lie}_f(y) = \mathbb{R}^m \ \forall y \in \mathbb{R}^m\} \Longrightarrow \{\mathcal{C}_x \subset \mathrm{Supp}\, X\},$$

where $\mathcal{C}_x$ stands for the set of continuous functions $\mathbb{R}^+ \to \mathbb{R}^m$ starting from x and with the obvious notation for $\mathrm{Lie}_f(y)$, and that

$$\{\mathrm{Lie}_f(y) = \mathbb{R}^m \ \forall y \in \mathbb{R}^m \text{ and } \mathrm{Supp}\, \nu = \mathbb{R}^d\} \Longrightarrow \{\mathrm{Supp}\, X = \mathcal{D}_x\},$$

where $\mathcal{D}_x$ stands for the set of càd-làg functions $\mathbb{R}^+ \to \mathbb{R}^m$ starting from x.

3. Some deterministic lemmas

In this section we gather some easy lemmas about p-variation which we will use in proving the Theorem and the Corollaries. We begin with two fairly trivial results:

Lemma 8. *Let $n \in \mathbb{N}^*$, $T > 0$ and $v_0, \ldots, v_n \in \mathbb{R}^d$. Let $f : [0, T] \to \mathbb{R}^d$ be the following step-function*:

$$f(t) = v_i \qquad \text{if} \quad \frac{Ti}{n} \le t < \frac{T(i+1)}{n}.$$

Then

$$\|f\|_{T,p}^p \le n \max_{0 \le i,j \le n} |v_i - v_j|^p.$$

Proof. Straightforward. □

Lemma 9. *Let $f : \mathbb{R}^+ \to \mathbb{R}^d$ be a linear function*: *$f(t) = ta$ for some $a \in \mathbb{R}^d$. Then for every $T > 0$,*

$$\|f\|_{T,p} = T|a| = \|f\|_{T,\infty}.$$

Proof. Let $0 = t_0 < t_1 < \cdots < t_k = T$ be a partition of $[0, T]$. Writing $s_j = t_j - t_{j-1} > 0$ for $j = 1, \ldots, k$, we get

$$\sum_{j=1}^k |f(t_j) - f(t_{j-1})|^p = |a|^p \left(\sum_{j=1}^k s_j^p \right) \le |a|^p \left(\sum_{j=1}^k s_j \right)^p = T^p |a|^p$$

since $p \ge 1$. The above inequality is of course an equality when $k = 1$. □

The following definition and lemma will be of constant use in proving the Theorem. The lemma itself is a direct consequence of Lemma 9.

Definition 10. Let $n \in \mathbb{N}^*$, $T > 0$ and $v \in \mathbb{R}^d$. The following càd-làg function from $[0,T]$ to $\mathbb{R}^d$:

$$t \longmapsto \left(\frac{nt}{T} - k\right) v \qquad \text{if} \quad \frac{kT}{n} \leq t < \frac{(k+1)T}{n}$$

is called a *saw-function* with parameters (n, T, v) and is denoted by $\mathrm{Saw}_v^{n,T}$.

Lemma 11. *For every $n \in \mathbb{N}^*$, $T > 0$, $v \in \mathbb{R}^d$ and $p \geq 1$ we have*

$$\|\mathrm{Saw}_v^{n,T}\|_{T,p}^p = 2n|v|^p.$$

Proof. Let $0 = t_0 < t_1 < \cdots < t_k = T$ be a partition of $[0,T]$. Writing $q_0 = 0$ and

$$q_j = \sup\left\{q \in \{0, \ldots, k\}/t_q < \frac{jT}{n}\right\}$$

for every $j = 1, \ldots, n+1$, we get

$$\begin{aligned}
&\sum_{i=1}^{k} |\mathrm{Saw}_v^{n,T}(t_i) - \mathrm{Saw}_v^{n,T}(t_{i-1})|^p \\
&\quad = \sum_{j=0}^{n} \sum_{q_j+1 \leq q \leq q_{j+1}} |\mathrm{Saw}_v^{n,T}(t_q) - \mathrm{Saw}_v^{n,T}(t_{q-1})|^p \\
&\quad \leq \sum_{j=1}^{n} |\mathrm{Saw}_v^{n,T}(t_{q_j+1}) - \mathrm{Saw}_v^{n,T}(t_{q_j})|^p \\
&\qquad + \sum_{j=0}^{n-1} \sum_{k_j < q \leq q_{j+1}} |\mathrm{Saw}_v^{n,T}(t_q) - \mathrm{Saw}_v^{n,T}(t_{q-1})|^p \\
&\quad \leq n|v|^p + \sum_{j=0}^{n-1} \sum_{k_j < q \leq q_{j+1}} |\mathrm{Saw}_v^{n,T}(t_q) - \mathrm{Saw}_v^{n,T}(t_{q-1})|^p \\
&\quad \leq n|v|^p + n|v|^p = 2n|v|^p,
\end{aligned}$$

where in the third line we wrote $k_0 = 0$, $k_j = q_j + 1$ for $1 \leq j \leq n+1$, and used the fact that $q_n + 1 = q_{n+1} = k$, and where in the last inequality we used Lemma 9.

Considering now the following partition of $[0,T]$:

$$0 < \frac{T-\rho}{n} < \frac{T}{n} < \frac{2T-\rho}{n} < \frac{2T}{n} < \cdots < T - \frac{\rho}{n} < T,$$

and letting ρ tend to 0, it is easy to see that the above upper bound is actually the lowest possible. □

The next elementary lemma, whose statement is trivial if $|a| = 0$, will be used in proving Corollary A.

Lemma 12. *Let ϕ be a càd-làg function $\mathbb{R}^+ \to \mathbb{R}^d$ with finite 1-variations such that for every $t > 0$*

$$\phi_t = ta + \sum_{s \le t} \Delta\phi_s,$$

for some fixed vector a. Then for every $T > 0$

$$\|\phi\|_{T,1} = T\,|a| + \sum_{t \le T} |\Delta\phi_t|\,.$$

Proof. Fix $T > 0$. Introduce

$$\phi^n : t \mapsto ta + \sum_{s \le t} \Delta\phi_s \mathbf{1}_{\{|\Delta\phi_s| \ge 1/n\}},$$

for every $n \ge 1$. Since ϕ^n has finitely many discontinuities and reasoning as in Lemma 11, it is clear that

$$\|\phi^n\|_{T,1} = T\,|a| + \sum_{t \le T} |\Delta\phi^n_t| \longrightarrow T|a| + \sum_{t \le T} |\Delta\phi_t|$$

as $n \uparrow +\infty$. On the other hand,

$$\|\phi - \phi^n\|_{T,1} = \sum_{t \le T} |\Delta\phi_t| \mathbf{1}_{\{|\Delta\phi_t| \le 1/n\}} \longrightarrow 0$$

as $n \uparrow +\infty$, which completes the proof of the lemma. □

The next approximation lemma will be used in proving Corollary B. It is probably well-known in the literature on p-variation. Nevertheless we give a proof in order to be more complete, even though this is quite tedious.

Lemma 13. *Let $\phi : \mathbb{R}^+ \to \mathbb{R}^d$ have regular finite p-variation. For every $\varepsilon > 0$, $T > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$*

$$\|\phi - \phi^n\|_{T,p} < \varepsilon,$$

where ϕ^n is the polygonal approximation of ϕ over $[0, T]$ with step T/n.

Proof. Fix $\varepsilon > 0$ and $T > 0$. Since ϕ has regular paths, we can find $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$,

$$\sup_{\substack{0 = t_0 < \cdots < t_k = T \\ |t_j - t_{j-1}| \le T/n}} \sum_{j=1}^{k} |\phi(t_j) - \phi(t_{j-1})|^p < \frac{\varepsilon}{2^{3p+2}}.$$

Let $n \ge n_0$. We first show that

$$\sup_{\substack{0 = t_0 < \cdots < t_k = T \\ |t_j - t_{j-1}| < T/n}} \sum_{j=1}^{k} |\phi^n(t_j) - \phi^n(t_{j-1})|^p < \frac{\varepsilon}{2^{2p+1}}.$$

Let $0 = t_0 < t_1 < \cdots < t_k = T$ be a partition of $[0, T]$ such that $|t_j - t_{j-1}| < T/n$ for every $j \geq 0$. Write $q_0^- = q_0^+ = 0$ and set

$$q_j^- = \sup\{q \in \{0, \ldots, k\}/t_q < s_j\}, \;\; q_j^+ = \inf\{q \in \{0, \ldots, k\}/t_q \geq s_j \wedge T\}$$

with the notation $s_j = jT/n$, for every $j \geq 0$. Notice that $q_n^+ = k = q_r^- = q_r^+$ for every $r > n$, and that

$$s_j \leq t_{q_j^+} \leq t_{q_{j+1}^-} < s_{j+1}$$

for every $j \geq 0$. We get, reasoning as in Lemma 1,

$$\begin{aligned}
\sum_{i=1}^{k} |\phi^n(t_i) - \phi^n(t_{i-1})|^p &\leq \sum_{j=0}^{n-1} |\phi^n(t_{q_{j+1}^-}) - \phi^n(t_{q_j^+})|^p + \sum_{j=1}^{n} |\phi^n(t_{q_j^+}) - \phi^n(t_{q_j^-})|^p \\
&\leq \sum_{j=0}^{n-1} |\phi(s_{j+1}) - \phi(s_j)|^p + \sum_{j=1}^{n} |\phi^n(t_{q_j^+}) - \phi^n(t_{q_j^-})|^p \\
&\leq \frac{\varepsilon}{2^{3p+2}} + \sum_{j=1}^{n} |\phi^n(t_{q_j^+}) - \phi^n(t_{q_j^-})|^p.
\end{aligned}$$

Writing $s_j^T = s_j \wedge T$ for every $j \geq 0$ and reasoning again as in Lemma 1, we can control the second term of the right-hand side:

$$\begin{aligned}
\sum_{j=1}^{n} |\phi^n(t_{q_j^+}) - \phi^n(t_{q_j^-})|^p &\leq \sum_{j=1}^{n} 2^{p-1}(|\phi^n(t_{q_j^+}) - \phi^n(s_j)|^p + |\phi^n(s_j) - \phi^n(t_{q_j^-})|^p) \\
&\leq \sum_{j=1}^{n} 2^{p-1}(|\phi(s_{j+1}^T) - \phi(s_j)|^p + |\phi(s_j) - \phi(s_{j-1})|^p) \\
&\leq \frac{\varepsilon}{2^{2p+2}}.
\end{aligned}$$

This yields finally

$$\sum_{i=1}^{k} |\phi^n(t_i) - \phi^n(t_{i-1})|^p \leq \frac{\varepsilon}{2^{3p+2}}(1 + 2^p) \leq \frac{\varepsilon}{2^{2p+1}}$$

which is the desired result.

Let now $0 = t_0 < t_1 < \cdots < t_k = T$ be *any* partition of $[0, T]$. Define q_j^- and q_j^+ as above, and

$$J = \{j \geq 0 \text{ such that } t_{q_{j+1}^+} > t_{q_j^+} \text{ or } j = n\}.$$

Notice that again

$$s_j \leq t_{q_j^+} \leq t_{q_{j+1}^-} < s_{j+1}$$

if $j \in J$. Writing $\psi^n = \phi - \phi^n$ for simplicity, we get

$$\sum_{i=1}^{k} |\psi^n(t_i) - \psi^n(t_{i-1})|^p = \sum_{j \in J} |\psi^n(t_{q_j^+}) - \psi^n(t_{q_j^-})|^p + \sum_{j \in J} \left(\sum_{t_{q_j^+} \le t_q < t_{q+1} \le t_{q_{j+1}^-}} |\psi^n(t_{q+1}) - \psi^n(t_q)|^p \right).$$

On the one hand, for every $j \in J$ and $t_{q_j^+} \le t_q < t_{q+1} \le t_{q_{j+1}^-}$,

$$|\psi^n(t_{q+1}) - \psi^n(t_q)|^p \le 2^{p-1}(|\phi^n(t_{q+1}) - \phi^n(t_q)|^p + |\phi(t_{q+1}) - \phi(t_q)|^p),$$

so that after summation,

$$\sum_{j \in J} \left(\sum_{t_{q_j^+} \le t_q < t_{q+1} \le t_{q_{j+1}^-}} |\psi^n(t_{q+1}) - \psi^n(t_q)|^p \right) \le 2^{p-1} \left(\frac{\varepsilon}{2^{3p+2}} + \frac{\varepsilon}{2^{2p+1}} \right) \le \frac{\varepsilon}{2}.$$

On the other hand, after summation,

$$\begin{aligned}
&\sum_{j \in J} |\psi^n(t_{q_j^+}) - \psi^n(t_{q_j^-})|^p \\
&\qquad \le 2^{p-1} \sum_{j \in J} (|\psi^n(t_{q_j^+}) - \psi^n(s_j)|^p + |\psi^n(s_{j+1}^T) - \psi^n(t_{q_{j+1}^-})|^p) \\
&\qquad \le 4^{p-1} \sum_{j \in J} (|\phi(t_{q_j^+}) - \phi(s_j)|^p + |\phi^n(t_{q_j^+}) - \phi^n(s_j)|^p \\
&\qquad\quad + |\phi(s_{j+1}^T) - \phi(t_{q_{j+1}^-})|^p + |\phi^n(s_{j+1}^T) - \phi^n(t_{q_{j+1}^-})|^p) \\
&\qquad \le 4^{p-1} \left(\frac{2\varepsilon}{2^{3p+2}} + \frac{2\varepsilon}{2^{2p+1}} \right) \le \frac{\varepsilon}{2}.
\end{aligned}$$

Finally, we get

$$\sum_{i=1}^{k} |\psi^n(t_i) - \psi^n(t_{i-1})|^p \le \varepsilon.$$

$\square$

Remark. Francis Hirsch gave me the following counterexample when ϕ is continuous and has finite p-variation but not regular paths, in the case $p = 1$: let μ be a probability measure over $[0, 1]$, singular with respect to Lebesgue measure, and such that

$$\phi(x) = \int_0^x \mu(dy)$$

is continuous (ϕ is a so-called Lebesgue function). ϕ has finite (but not regular finite) 1-variation. For every $n \geq 1$ we can write

$$\phi^n(x) = \int_0^x \mu^n(dy),$$

where μ^n is absolutely continuous with respect to Lebesgue measure. Hence we get

$$\|\phi - \phi^n\|_{1,1} = \|\phi\|_{1,1} + \|\phi^n\|_{1,1} = 2.$$

4. Proof of the Theorem

We first make the general remark that, obviously, it suffices to consider the situation where the jumps of Z are bounded by 1, so that in particular

$$\tilde{Z}_t = \int_0^t \int_{|z|\leq 1} z\, \widetilde{\mu}(ds, dz) + t \left(\int_{|z|\leq 1} z_K\, \nu(dz) \right)$$

for every $t > 0$. We will separate the proof according to Dim L with increasing order of difficulty. The arguments are somewhat similar to those of the Proposition in [26], but here the situation is significantly more complicated because of the p-variation norm.

4.1. $\operatorname{Dim} L = 0$

This case is obvious since we can take $p = 1$. In particular,

$$\|\tilde{Z}\|_{T,1} = \sum_{t \leq T} |\Delta Z_t|,$$

and the Theorem follows readily from the fact that

$$\int_{|z|\leq 1} |z| \nu(dz) < \infty.$$

4.2. $\operatorname{Dim} L = 1$

Here the situation is more complicated since we must take $p > 1$. We fix T and $\varepsilon > 0$ once and for all. We first set

$$v^\eta = \int_{\eta \leq |z| \leq 1} z\, \nu(dz) \quad \text{and} \quad v_L^\eta = \int_{\eta \leq |z| \leq 1} z_L\, \nu(dz).$$

The asymptotic study of v_L^η and its suitable approximation by elements of $\operatorname{Supp} \nu$ will be actually the central point in the whole proof.

Since $\operatorname{Dim} L = 1$, it clear that for every η and $\rho > 0$ there exists $x_\rho^\eta \in \operatorname{Supp} \nu$ such that $\left| x_\rho^\eta \right| < \eta$ and

$$\sphericalangle(v_L^\eta, x_\rho^\eta) \leq \rho,$$

where $\sphericalangle(\cdot,\cdot)$ stands for the Euclidean angle between two vectors. Choosing $\eta < \varepsilon/12T$ and $\rho < \eta$ such that $\rho\,|v_L^\eta| < \varepsilon/12T$, we get

$$|v_L^\eta - \alpha_\rho^\eta x_\rho^\eta| < \varepsilon/6T$$

for some minimizing integer α_ρ^η. Besides, we can choose a neighborhood $\mathcal{V}_\rho^\eta$ of x_ρ^η, included in $\{|z| < \eta\}$ and small enough, such that

$$\left| \int_{\mathcal{V}_\rho^\eta} z\,\nu(dz) - \beta_\rho^\eta x_\rho^\eta \right| < \varepsilon/6T$$

for another minimizing integer β_ρ^η. Setting

$$v_\rho^\eta = v_L^\eta + \int_{\mathcal{V}_\rho^\eta} z\,\nu(dz) \quad \text{and} \quad \gamma_\rho^\eta = \alpha_\rho^\eta + \beta_\rho^\eta$$

yields

$$|v_\rho^\eta - \gamma_\rho^\eta x_\rho^\eta| < \varepsilon/3T. \tag{3}$$

We now introduce the saw-function with parameters $\left(\gamma_\rho^\eta, T, -x_\rho^\eta\right)$, which we will write Saw_ρ^η for the sake of simplicity. By Lemma 6,

$$\|\mathrm{Saw}_\rho^\eta\|_{T,p}^p = 2\gamma_\rho^\eta |x_\rho^\eta|^p.$$

Hence, letting η tend to 0,

$$\begin{aligned}\|\mathrm{Saw}_\rho^\eta\|_{T,p}^p &\sim 2|x_\rho^\eta|^{p-1}|v_\rho^\eta| \\ &\le 2|x_\rho^\eta|^{p-1}\int_{\mathcal{A}_\rho^\eta} |z|\nu(dz),\end{aligned}$$

where we wrote $\mathcal{A}_\rho^\eta = \{z,\ 1 \ge |z| \ge |x_\rho^\eta|/2\}$. But since

$$\int_{|z|\le 1} |z|^p \nu(dz) < \infty,$$

in the above inequality the right-hand side tends to 0 as η tends to 0, and we get

$$\lim_{\eta\downarrow 0} \|\mathrm{Saw}_\rho^\eta\|_{T,p} = 0. \tag{4}$$

We now come back to the proof of the Theorem. Set

$$\mathcal{B}_\rho^\eta = \{z,\ |z| \le \eta\} \cap (\mathcal{V}_\rho^\eta)^c \quad \text{and} \quad \tilde{Z}_\rho^\eta(t) = \int_0^t \int_{\mathcal{B}_\rho^\eta} z\,\widetilde{\mu}(ds,dz) + t\left(\int_{\mathcal{B}_\rho^\eta} z_K\,\nu(dz)\right)$$

for every $t \ge 0$. We obviously have

$$\lim_{\eta\downarrow 0}\left(\int_{\mathcal{B}_\rho^\eta} z_K\,\nu(dz)\right) = 0.$$

Hence by Lemmas 1 and 9, and the triangle inequality for $\|\cdot\|_{T,p}$, we see that

$$\lim_{\eta\downarrow 0} \|\tilde{Z}^\eta_\rho\|_{T,p} = 0 \quad \text{a.s.} \tag{5}$$

We next consider the process $Z^\eta_\rho = \tilde{Z} - \tilde{Z}^\eta_\rho$, which can be written

$$Z^\eta_\rho(t) = \int_0^t \int_{(\mathcal{B}^\eta_\rho)^c} z\,\mu(ds, dz) - t v^\eta_\rho$$

for every $t > 0$. Writing, for each $k = 0, \ldots, \gamma^\eta_\rho$, $s^\eta_\rho(k) = kT/\gamma^\eta_\rho$, we see that for every $t \in [0, T]$,

$$Z^\eta_\rho(t) - \mathrm{Saw}^\eta_\rho(t) = \psi^\eta_\rho(t) + \phi^\eta_\rho(t),$$

where we introduced

$$\psi^\eta_\rho(t) = \sum_{s\le t} \Delta Z^\eta_\rho(s) - \sum_{s^\eta_\rho(k)\le t} x^\eta_\rho \quad \text{and} \quad \phi^\eta_\rho(t) = t(\gamma^\eta_\rho x^\eta_\rho - v^\eta_\rho).$$

By (3) and Lemma 9, we have

$$\|\phi^\eta_\rho\|_{T,p} < \varepsilon/3$$

if η was chosen small enough. Introduce $\{T^\eta_\rho(k), U^\eta_\rho(k)\}_{k\ge 1}$, the successive jumping times and sizes of Z^η_ρ. Since $x^\eta_\rho \in \operatorname{Supp}\nu$ and since $\mathcal{V}^\eta_\rho \subset (\mathcal{B}^\eta_\rho)^c$, we see that the event

$$\{|T^\eta_\rho(k) - s^\eta_\rho(k)| < \lambda,\ |U^\eta_\rho(k) - x^\eta_\rho| < \lambda,\ k = 1, \ldots, \gamma^\eta_\rho\}$$

has positive probability for every $\lambda > 0$. But if λ is small enough, then on the latter event ψ^η_ρ is a step function on $[0, T]$ with $2\gamma^\eta_\rho$ jumps and such that

$$|\psi^\eta_\rho(t) - \psi^\eta_\rho(s)| < (1+\lambda)|x^\eta_\rho|$$

for every $s \neq t \in [0, T]$, so that according to Lemma 8

$$\|\psi^\eta_\rho\|_{T,p} < \varepsilon/3$$

if η was chosen small enough. Putting everything together leads to

$$\mathbb{P}[\|Z^\eta_\rho - \mathrm{Saw}^\eta_\rho\|_{T,p} < 2\varepsilon/3] > 0 \tag{6}$$

if η was chosen small enough. Using (4), (5), (6), the independence of Z^η_ρ and $\tilde{Z}^\eta_\rho$ and the triangle inequality for $\|\cdot\|_{T,p}$, we finally get

$$\mathbb{P}[\|\tilde{Z}\|_{T,p} < \varepsilon] > 0,$$

which finishes the proof in the case $\operatorname{Dim} L = 1$.

4.3. $\mathrm{Dim}\, L = 2$

We consider this particular situation in order to clarify the exposition - the arguments are analogous in the case $\mathrm{Dim}\, L > 2$, but involve heavier notations. The outline of the proof will be roughly the same as in the preceding subsection, except that here the estimate (3) does not hold in general, so that we will need more elements of Supp ν to approximate v_L^η.

For each vector $z \in \mathbb{R}^d$, we will write $z = (x, y)$ according to the unique decomposition $z = x + y$ with $x \in K$ and $y \in L$. Fix an orthonormal basis of L. Thanks to the spatial homogeneity of Poisson measures, we first remark that it suffices to consider the case where

$$\mathrm{Supp}\,\nu \subset \{(x, y),\ y_i \geq 0\ \forall i = 1, 2\}. \tag{7}$$

This choice of (strict) positivity will play a crucial rôle in the following. Notice, first, that it entails

$$|v_L^\eta| \to +\infty$$

as $\eta \downarrow 0$. Choose a subsequence $\{\eta\}$ along which

$$\lim_{\eta\downarrow 0} \frac{v^\eta}{|v^\eta|} = \lim_{\eta\downarrow 0} \frac{v_L^\eta}{|v_L^\eta|} = u_1 \in \mathcal{S}^{d-1}.$$

Set L_1 for the line generated by u_1 and consider the orthogonal sum $L = L_1 \oplus L_2$. Let v_i^η be the projection of v_L^η onto L_i for $i = 1, 2$. Clearly we have

$$|v_1^\eta| \to +\infty \quad \text{and} \quad \frac{|v_1^\eta|}{|v_2^\eta|} \to +\infty$$

as η tends to 0 along the subsequence. We will consider two disjoint cases:

Case A. There exists a sub-subsequence $\{\eta\}$ along which $v_2^\eta \to 0$.

Case B. For every sub-subsequence $\{\eta\}$, $\liminf_{\eta\downarrow 0} |v_2^\eta| > 0$.

4.3.1. Case A

The situation is quite analogous to $\mathrm{Dim}\, L = 1$ though a bit more complicated since here, as we said before, one cannot rely on inequality (3). One should keep in mind the two-dimensional example where $\mathrm{Supp}\,\nu \subset \{z = (y_1, y_2),$ $y_2 = |y_1|\}$ and where the restriction of ν on each half-line is the same measure. In the following, each η will be implicitly chosen in the sub-subsequence.

Recall that $\mathcal{C}^\eta$ stands for the closed convex cone generated by $\mathcal{S}^\eta = \mathrm{Supp}\,\nu$ $\cap \{z,\ |z| < \eta\}$, and

$$\mathcal{C} = \bigcap_{\eta > 0} \mathcal{C}^\eta.$$

Notice that clearly $u_1 \in \mathcal{C}$. Besides, since

$$\lim_{\eta\downarrow 0} \frac{v_1^\eta}{|v_1^\eta|} = \lim_{\eta\downarrow 0} \frac{v_L^\eta}{|v_L^\eta|} = u_1,$$

we see that $v_1^\eta \in \mathcal{C}$ for η small enough. In particular there exist some integer $r \leq d$ and distinct $x_1^\eta, \ldots, x_r^\eta \in \mathcal{S}^\eta$ such that

$$\left| v_1^\eta - \sum_{i=1}^r \alpha_i^\eta x_i^\eta \right| \leq \varepsilon/4T \tag{8}$$

for minimizing integers $\alpha_1^\eta, \ldots, \alpha_r^\eta$. Notice that by positivity, (7) yields obviously

$$|\alpha_i^\eta x_i^\eta| \leq |v_1^\eta| \tag{9}$$

for every $i = 1, \ldots, r$. As before we can choose some disjoint neighborhoods $\mathcal{V}_i^\eta$ of the x_i^η, included in $\{|z| < \eta\}$ and small enough, such that

$$\left| \int_{\mathcal{V}_i^\eta} z\, \nu(dz) - \beta_i^\eta x_i^\eta \right| < \varepsilon/8(r+1)T \tag{10}$$

for minimizing integers β_i^η. We consider again

$$w_i^\eta = \alpha_i^\eta x_i^\eta + \int_{\mathcal{V}_i^\eta} z\, \nu(dz), \qquad \gamma_i^\eta = \alpha_i^\eta + \beta_i^\eta,$$

and set Saw_i^η for the saw-function with parameters $(\gamma_i^\eta, T, -x_i^\eta)$. Using (9) and reasoning exactly as in the case $\mathrm{Dim}\, L = 1$ entail

$$\lim_{\eta \downarrow 0} \|\mathrm{Saw}_i^\eta\|_{T,p} = 0 \tag{11}$$

for every $i = 1, \ldots, r$. Writing

$$\mathcal{B}_r^\eta = \{z,\ |z| \leq \eta\} \cap (\mathcal{V}_1^\eta \cup \cdots \cup \mathcal{V}_r^\eta)^c \quad \text{and}$$
$$\tilde{Z}_r^\eta(t) = \int_0^t \int_{\mathcal{B}_r^\eta} z\, \widetilde{\mu}(ds, dz) + t \left(\int_{|z| \leq \eta} z_K\, \nu(dz) \right)$$

for every $t \geq 0$, we get again, by Lemma 1,

$$\lim_{\eta \downarrow 0} \|\tilde{Z}_r^\eta\|_{T,p} = 0 \quad \text{a.s.} \tag{12}$$

We finally consider the processes

$$Z_i^\eta(t) = \int_0^t \int_{\mathcal{V}_i^\eta} z\, \mu(ds, dz) - t w_i^\eta \quad \text{and} \quad Z_{r+1}^\eta(t) = \int_0^t \int_{\eta \leq |z| \leq 1} z\, \mu(ds, dz)$$

for every $t > 0$ and $i = 1, \ldots, r$. Notice that the Z_i^η's are mutually independent and that one can rewrite

$$\tilde{Z}(t) = \tilde{Z}_r^\eta(t) + \sum_{i=1}^{r+1} Z_i^\eta(t) + t \left(\sum_{i=1}^r \alpha_i^\eta x_i^\eta - v_1^\eta \right)$$

for every $t > 0$. Using the inequality (10) and reasoning exactly as in the case $\operatorname{Dim} L = 1$ yield

$$\mathbb{P}[\|Z_i^\eta - \mathrm{Saw}_i^\eta\|_{T,p} < \varepsilon/4(r+1)] > 0 \tag{13}$$

for every $i = 1, \dots, r$. Moreover it is clear that

$$\mathbb{P}[\|Z_{r+1}^\eta\|_{T,p} < \varepsilon/4(r+1)] > 0. \tag{14}$$

Using (12), (14), (13), (11), (8), independence arguments and the triangle inequality, we finally get

$$\mathbb{P}[\|\tilde{Z}\|_{T,p} < \varepsilon] > 0,$$

which completes the proof of the theorem.

4.3.2. Case B

This case is the most complicated: here we need to cope with v_2^η, a vector which does not belong to $\mathcal{C}$ in general. One should keep in mind Example 6.

From Case A it is clear that it suffices to prove the following: there exists a fixed integer r and distinct $x_1^\eta, \dots, x_r^\eta \in \mathcal{S}^\eta$ such that if η is chosen small enough along the subsequence

$$\left| v_L^\eta - \sum_{i=1}^r \alpha_i^\eta x_i^\eta \right| \le \varepsilon/2T, \tag{15}$$

where $\alpha_1^\eta, \dots, \alpha_r^\eta$ are minimizing integers verifying

$$\alpha_i^\eta |x_i^\eta|^p \to 0 \tag{16}$$

for every $i = 1, \dots, s$, as η tends to 0. The estimate (15) would be enough to obtain a small deviation property in uniform norm, as in [26]. But the latter convergences (16) are crucial to obtain this property in p-variation norm, because of Lemma 11. In general (16) is quite difficult to obtain together with (15), since the length of the approximating vectors $\alpha_i^\eta x_i^\eta$ is not controlled a priori by that of v_L^η, as $\eta \downarrow 0$. Notice, however, that (16) follows readily as soon as

$$|\alpha_i^\eta x_i^\eta| \le c|v_L^\eta| \tag{17}$$

for every $i = 1, \dots, s$ and a constant c independent of η. The remainder of this article will be devoted to the proof of (15) and (16), a proof which does not require probability theory anymore, but some amount of elementary analysis and geometry.

Take a sub-subsequence $\{\eta\}$ along which

$$\lim_{\eta \downarrow 0} \frac{v_2^\eta}{|v_2^\eta|} = u_2 \in \mathcal{S}^{d-1}$$

with $v_2^\eta * u_2 > 0$ for every η. Set $z_2 = z * u_2$ and $z_2^+ = \sup(0, z_2)$ for every $z \in \mathbb{R}^d$. Clearly,

$$\int_{|z|\le 1} z_2^+ \mathbf{1}_{\{|z_K|<z_2\}} \nu(dz) = +\infty. \tag{18}$$

Let $\mathcal{D}^\eta$ be the closed convex cône generated by $\mathcal{S}^\eta \cap \{0 < |z_K| < z_2\}$ and

$$\mathcal{D} = \bigcap_{\eta>0} \mathcal{D}^\eta.$$

Set Π_1 (resp. $\Pi_1^\perp$, Π_2) for the operator of orthogonal projection onto L_1 (resp. $L_1^\perp$, L_2). Because of (18) we see that for every $\eta > 0$,

$$\frac{\Pi_1^\perp \left(\int_{\rho\le|z|\le\eta} z\, \mathbf{1}_{\{0<|z_K|<z_2\}} \nu(dz) \right)}{\left| \Pi_1^\perp \left(\int_{\rho\le|z|\le\eta} z\, \mathbf{1}_{\{0<|z_K|<z_2\}} \nu(dz) \right) \right|} \longrightarrow u_2$$

as ρ tends to 0 along the sub-subsequence. Hence, $u_2 \in \overline{\Pi_1^\perp(\mathcal{D}^\eta)}$ for every $\eta > 0$, and

$$v_2^\eta \in \bigcap_{\rho>0} \overline{\Pi_1^\perp(\mathcal{D}^\rho)}$$

for η small enough along the sub-subsequence. In particular there exist some integer $s \le d$ and distinct $x_1^\eta, \dots, x_s^\eta \in \mathcal{S}^\eta \cap \{0 < |z_K| < z_2\}$ such that

$$\left| v_2^\eta - \Pi_1^\perp \left(\sum_{i=1}^s \beta_i^\eta x_i^\eta \right) \right| \le \varepsilon/4T \tag{19}$$

for minimizing integers $\beta_1^\eta, \dots, \beta_s^\eta$. Besides, by positivity, it is clear that

$$|\Pi_2(\beta_i^\eta x_i^\eta)| \le |v_2^\eta|$$

for every $i = 1, \dots, s$. Hence, by the triangle inequality,

$$|\Pi_1^\perp(\beta_i^\eta x_i^\eta) = |\Pi_K(\beta_i^\eta x_i^\eta)| + |\Pi_2(\beta_i^\eta x_i^\eta)| \le 2|\Pi_2(\beta_i^\eta x_i^\eta)| \le 2|v_2^\eta| \tag{20}$$

for every $i = 1, \dots, s$. We now separate the proof according to the degenerescence of $\mathcal{D}$ with respect to L_2.

The case when $\mathcal{D}$ is non-degenerated. We mean the case where

$$u_2 \in \overline{\Pi_1^\perp(\mathcal{D})}.$$

Then there exists c independent of η such that $u_2 + cu_1 \in \mathcal{D}$, and in (19) $x_1^\eta, \dots, x_s^\eta$ can be chosen such that

$$\left| \Pi_1 \left(\sum_{i=1}^s \beta_i^\eta x_i^\eta \right) \right| \le c|v_2^\eta|. \tag{21}$$

Since, by positivity,

$$|\Pi_1(\beta_i^\eta x_i^\eta)| \leq \left|\Pi_1\left(\sum_{i=1}^{s}\beta_i^\eta x_i^\eta\right)\right|$$

for every $i = 1, \ldots, s$, we get, by (20) and the triangle inequality,

$$|\beta_i^\eta x_i^\eta| \leq (2+c)|v_2^\eta| \leq (2+c)|v_L^\eta| \tag{22}$$

for every $i = 1, \ldots, s$. We now write

$$v_L^\eta = \left(\sum_{i=1}^{s}\beta_i^\eta x_i^\eta\right) + \left(v_1^\eta - \Pi_1\left(\sum_{i=1}^{s}\beta_i^\eta x_i^\eta\right)\right) + \left(v_2^\eta - \Pi_1^\perp\left(\sum_{i=1}^{s}\beta_i^\eta x_i^\eta\right)\right) \tag{23}$$

and set

$$w_1^\eta = v_1^\eta - \Pi_1\left(\sum_{i=1}^{s}\beta_i^\eta x_i^\eta\right).$$

Using (21) and the fact that

$$\lim_{\eta\downarrow 0}\frac{|v_1^\eta|}{|v_2^\eta|} = +\infty,$$

we see that there exists η_0 such that for every $\eta < \eta_0$ along the sub-subsequence,

$$w_1^\eta * u_1 > 0 \quad \text{and} \quad |w_1^\eta| < |v_1^\eta|.$$

Hence, by Case A, for every $\eta < \eta_0$ along the sub-subsequence, there exists an integer $r \leq d$ and distinct $x_{s+1}^\eta, \ldots, x_{s+r}^\eta \in \mathcal{S}^\eta$ such that

$$\left|w_1^\eta - \sum_{i=1}^{r}\beta_{s+i}^\eta x_{s+i}^\eta\right| \leq \varepsilon/4T, \tag{24}$$

where $\beta_{s+1}^\eta, \ldots, \alpha_{s+r}^\eta$ are minimizing integers verifying

$$|\beta_{s+i}^\eta x_{s+i}^\eta| \leq |w_1^\eta| \leq |v_L^\eta|. \tag{25}$$

Clearly together (19), (22), (23), (24), and (25) yield (15) and (17), which completes the proof of the Theorem.

The case when $\mathcal{D}$ is degenerated. We mean the delicate situation where

$$u_2 \notin \overline{\Pi_1^\perp(\mathcal{D})} \tag{26}$$

and where in particular (21) no more holds a priori. We denote by $\tilde{\mathcal{D}}$ (resp. $\tilde{\mathcal{D}}^\eta$) the intersection of $\mathcal{D}$ (resp. $\mathcal{D}^\eta$) with $L = L_1 \oplus L_2$. Set (z_1, z_2) for the coordinates on $L_1 \oplus L_2$ with respect to (u_1, u_2). Because of (26), $\tilde{\mathcal{D}}$ is actually reduced to the half-line $\{z_1 \geq 0,\ z_2 = 0,\ z_K = 0\}$. However, since

$$u_2 \in \bigcap_{\eta > 0}\overline{\Pi_1^\perp(\mathcal{D}^\eta)},$$

for every $\eta > 0$ the intersection of $\tilde{\mathcal{D}}^\eta$ with the open quadrant $\{z_1 > 0,\ z_2 > 0,\ z_K = 0\}$ is non void. Set Δ^η for the frontier of $\tilde{\mathcal{D}}^\eta$ in $\{z_1 > 0,\ z_2 > 0,\ z_K = 0\}$. Δ^η is another half-line whose slope with respect to L_1 decreases to 0 as η decreases to 0. Set $(z_1^\eta, z_2^\eta) = (\eta, z_2^\eta)$ for the point of Δ^η with u_1-coordinate η and consider the increasing convex function

$$h : \begin{cases}]0,1] \to \mathbb{R}^+, \\ \eta \mapsto z_2^\eta. \end{cases}$$

The graph of h is located under a smooth curve with slope 0 at $\eta = 0$, but we will see that h cannot grow too slowly in the neighborhood of 0:

Lemma 14. *The function $h :]0,1] \to \mathbb{R}^+$ defined above satisfies*

$$\int_{|z|\le 1} h(|z|)\ \nu(dz) = +\infty.$$

Proof. Since h is positive increasing and since, because of (7), $\operatorname{Supp}\nu \subset \{z_1 \ge 0\}$, it suffices to prove that

$$\int_{|z|\le 1} h(z_1)\,\nu(dz) = +\infty.$$

Suppose first that $d = 2$, i.e. $K = \{0\}$. Then clearly, by definition of h,

$$\operatorname{Supp}\nu \cap \{|z| \le 1\} \subset \{z_2^+ \le h(z_1)\}.$$

Hence

$$\int_{|z|\le 1} h(z_1)\nu(dz) \ge \int_{|z|\le 1} z_2^+\,\nu(dz) = +\infty$$

and this completes the proof of the lemma.

The case $d > 2$, i.e. $K \ne \{0\}$ is more subtle. First, by the Hahn-Banach theorem, there exists a hyper-plane H containing the half-line $\{z_1 \ge 0,\ z_2 = 0,\ z_K = 0\}$ and separating $\mathcal{D}$ from $\{z_1 \ge 0,\ z_2 > 0,\ z_K = 0\}$. Its unitary normal vector n oriented in the direction of $\mathcal{D}$ verifies $n_1 = 0$ and $n_2 < 0$. Besides, we can choose H such that n_2 is the lowest possible, in the sense that if $m \in \mathcal{S}^{d-1} \cap \{z_1 = 0\}$ and if $m_2 < n_2$, then

$$\mathcal{D} \cap \{m * z < 0\} \ne \emptyset.$$

Analogously, for every $\eta > 0$ we set H^η for the hyper-plane containing Δ^η, separating $\mathcal{D}^\eta$ from $\{z_1 > 0,\ z_2 > \alpha_\eta z_1,\ z_K = 0\}$ (where $\alpha_\eta = h(\eta)/\eta$), and such that if n^η is its unitary normal vector oriented in the direction of $\mathcal{D}^\eta$, then $n_2^\eta < 0$ is the lowest possible. Clearly, we have $n^\eta \to n$ and in particular $n_2^\eta \to n_2$ as $\eta \downarrow 0$. Hence there exists $\lambda > 0$ and $\eta_0 > 0$ such that $n_2^\eta < -\lambda$ for every $\eta < \eta_0$. A little Euclidean geometry shows then that if $\eta < \eta_0$, then

$$\mathcal{D}^\eta \cap \{z * n_K \ge 0\} \subset \{\lambda(\alpha_\eta z_1 - z_2) + |z_K| \ge 0\},$$

whereas obviously

$$\mathcal{D}^\eta \cap \{z * n_K \leq 0\} \subset \{z_2 \leq \alpha_\eta z_1\}.$$

Hence, for every $\eta < \eta_0$,

$$\mathcal{D}^\eta \cap \{|z_K| \leq \lambda z_2/2\} \subset \{z_2 \leq 2\alpha_\eta z_1\} \subset \{z_2 \leq 2h(z_1)\}$$

since $\eta \mapsto \alpha_\eta$ is decreasing. In particular

$$\int_{|z|\leq\eta_0} z_2\, \mathbf{1}_{\{0<|z_K|<\lambda z_2/2\}} \nu(dz) \leq 2 \int_{|z|\leq\eta_0} h(z_1)\nu(dz).$$

But the left-hand side equals $+\infty$ and we get

$$\int_{|z|\leq 1} h(z_1)\nu(dz) = +\infty,$$

which completes the proof of the lemma. □

Set now

$$L(\rho) = \frac{h(\rho)}{\rho^p}$$

for every $\rho \in]0,1]$. Since

$$\int_{|z|\leq 1} |z|^p \nu(dz) < +\infty,$$

Lemma 14 obviously entails that

$$\limsup_{\rho\downarrow 0} L(\rho) = +\infty.$$

In the following we will consider $\{\rho_\eta\}$ a sequence in $]0,1]$ with $\rho_\eta \leq \eta$, where $\{\eta\}$ is the original sub-subsequence, and such that

$$L(\rho_\eta) = \sup_{\rho_\eta\leq\rho\leq 1} L(\rho) \uparrow +\infty.$$

By construction of h, we see that for every $\eta > 0$ there exist some integer $s \leq d$, distinct $x_1^\eta, \ldots, x_s^\eta \in \mathcal{S}^{\rho_\eta} \cap \{0 < |z_K| < z_2\}$ and minimizing integers $\beta_1^\eta, \ldots, \beta_s^\eta$ such that

$$\left| v_2^\eta - \Pi_1^\perp \left(\sum_{i=1}^s \beta_i^\eta x_i^\eta \right) \right| \leq \varepsilon/8T \tag{27}$$

and

$$\rho_\eta^{p-1} L(\rho_\eta) \frac{\left| \Pi_1 \left(\sum_{i=1}^s \beta_i^\eta x_i^\eta \right) \right|}{\left| \Pi_1^\perp \left(\sum_{i=1}^s \beta_i^\eta x_i^\eta \right) \right|} \longrightarrow 1$$

as $\eta \downarrow 0$. In particular, for every $i = 1, \ldots, s$

$$|\Pi_1(\beta_i^\eta x_i^\eta)| \leq \left| \Pi_1 \left(\sum_{i=1}^s \beta_i^\eta x_i^\eta \right) \right| \leq \frac{2\rho_\eta^{1-p} |v_2^\eta|}{L(\rho_\eta)}$$

and remembering (20),

$$\beta_i^\eta |x_i^\eta|^p \leq \rho_\eta^{p-1} (|\Pi_1(\beta_i^\eta x_i^\eta)| + |\Pi_1^\perp(\beta_i^\eta x_i^\eta)|) \leq 2 \left(\rho_\eta^{p-1} |v_2^\eta| + \frac{|v_2^\eta|}{L(\rho_\eta)} \right)$$

as $\eta \downarrow 0$. On the one hand, since $\rho_\eta \leq \eta$,

$$\lim_{\eta \downarrow 0} \rho_\eta^{p-1} |v_2^\eta| = 0.$$

On the other hand

$$|v_2^\eta| \leq \int_{\eta \leq |z| \leq 1} z_2^+ \, \nu(dz) \leq \int_{\eta \leq |z| \leq 1} h(z_1) \, \nu(dz) \leq \int_{\rho_\eta \leq |z| \leq 1} |z|^p L(|z|) \nu(dz).$$

But since $L(\rho_\eta) = \sup_{\rho_\eta \leq \rho \leq 1} L(\rho) \uparrow +\infty$, we have

$$\int_{\rho_\eta \leq |z| \leq 1} |z|^p \left(\frac{L(|z|)}{L(\rho_\eta)} \right) \nu(dz) \to 0$$

as $\eta \downarrow 0$. This yields

$$\frac{|v_2^\eta|}{L(\rho_\eta)} \to 0$$

and, putting everything together,

$$\beta_i^\eta |x_i^\eta|^p \to 0 \tag{28}$$

as $\eta \downarrow 0$ along the sub-subsequence, for every $i = 1, \ldots, s$.

The proof draws now to its final step. Suppose first that

$$\left| \Pi_1 \left(\sum_{i=1}^s \beta_i^\eta x_i^\eta \right) \right| \leq |v_1^\eta|$$

along a subsequence $\{\eta\}$ of the original sub-subsequence. Then if we set

$$w_1^\eta = v_1^\eta - \Pi_1 \left(\sum_{i=1}^s \beta_i^\eta x_i^\eta \right)$$

as before, this entails that

$$w_1^\eta * u_1 > 0 \quad \text{and} \quad |w_1^\eta| < |v_1^\eta|$$

along this subsequence. Thus, reasoning exactly as above, we can write

$$v_L^\eta = \sum_{i=1}^{r+s} \beta_i^\eta x_i^\eta + \left(w_1^\eta - \sum_{i=1}^r \beta_{s+i}^\eta x_{s+i}^\eta \right) + \left(v_2^\eta - \Pi_1^\perp \left(\sum_{i=1}^s \beta_i^\eta x_i^\eta \right) \right),$$

where $r \leq d$ and $x^\eta_{s+1}, \ldots, x^\eta_{s+r}$ (resp. $\beta^\eta_{s+1}, \ldots, \beta^\eta_{s+r}$) are elements of $\mathcal{S}^\eta$ (resp. minimizing integers) such that (24) and (25) hold. Clearly together (24), (25), (27), and (28) yield (15) and (16), which completes the proof of the Theorem.

Suppose finally that

$$\left| \Pi_1 \left(\sum_{i=1}^s \beta^\eta_i x^\eta_i \right) \right| \geq |v^\eta_1| \tag{29}$$

along the original subsequence. To treat this very last situation we will remove some mass from $\operatorname{Supp} \nu$ in the L_1^+-direction, digging some subset of $\mathcal{S}^\eta$, and use Lemma 2 together with Skorohod's absolute continuity theorem - see e.g. Theorem 33.1 in [22]. Actually, we will need less vectors than in the above situation.

We first appeal to Lemma 2 with $\nu_0 = \nu$ and introduce $\eta_0 > 0$ such that for every $\eta < \eta_0$, every subset Ξ_η of $\{|z| \leq \eta\}$ and every Lévy measure $\bar{\nu} \leq \nu$,

$$\mathbb{P}[\|\tilde{\bar{Z}}^\eta\|_{T,p} < \varepsilon/2] > 1/2,$$

where $\bar{\mu}$ is the Poisson measure over $\mathbb{R}^+ \times \mathbb{R}^d$ with intensity $ds \otimes \bar{\nu}(dz)$, $\tilde{\bar{\mu}} = \bar{\mu} - ds \otimes \bar{\nu}$, and

$$\tilde{\bar{Z}}^\eta_t = \int_0^t \int_{\Xi_\eta} z\, \tilde{\bar{\mu}}(ds, dz) + t \int_{|z| \leq \eta} z_K\, \nu(dz)$$

for every $t > 0$. Choose $\eta < \eta_0$ in the subsequence, distinct $x^\eta_1, \ldots, x^\eta_s \in \mathcal{S}^{\rho_\eta} \cap \{0 < |z_K| < z_2\}$, and minimizing integers $\beta^\eta_1, \ldots, \beta^\eta_s$ such that (27) and (28) hold. Set

$$\lambda_\eta = \inf\{|x^\eta_1|, \ldots, |x^\eta_s|\}/2 \quad \text{and} \quad \mu_\eta = \left| \Pi_1 \left(\sum_{i=1}^s \beta^\eta_i x^\eta_i \right) \right| - |v^\eta_1|.$$

We may rewrite (27) as

$$\left| v^\eta_L + \mu_\eta u_1 - \sum_{i=1}^s \beta^\eta_i x^\eta_i \right| < \varepsilon/8T. \tag{30}$$

Consider now the restriction of ν to $\{|z| \leq \lambda_\eta\}$. Because of the degeneracy of $\mathcal{D}$, we see that for every $\rho > 0$,

$$\int_{|z| \leq \lambda_\eta} |z|\, \mathbf{1}_{\{|z|^2 \leq (1+\rho^2) z_1^2\}}\, \nu(dz) = +\infty. \tag{31}$$

Take ρ such that $\rho\mu_\eta < \varepsilon/8T$. From (31), it is clear that we can find $\lambda_\rho \in]0, \lambda_\eta[$ and a positive measure $\bar{\nu} \leq \nu$ on $\{|z| \leq 1\}$, with $\bar{\nu}$ equivalent to ν and $\bar{\nu}$ equal to ν on $\{|z| \leq \lambda_\rho\} \cup \{|z| \geq \lambda_\eta\}$, such that

$$|u_\eta - \mu_\eta u_1| < \varepsilon/8T, \tag{32}$$

where we set

$$u_\eta = \int_{\lambda_\rho \le |z| \le \lambda_\eta} z(\nu - \bar{\nu})(dz) = \int_{|z|\le 1} z(\nu - \bar{\nu})(dz).$$

It follows from (30) and (32) that

$$\left| v_L^\eta + u_\eta - \sum_{i=1}^{s} \beta_i^\eta x_i^\eta \right| < \varepsilon/4T. \tag{33}$$

Introduce the Lévy process $\bar{Z}$ given by

$$\bar{Z}_t = \int_0^t \int_{|z|\le 1} z_K \, \bar{\mu}(ds, dz) + \int_0^t \int_{|z|\le 1} z_L \, \widetilde{\bar{\mu}}(ds, dz) - t u_\eta$$

for every $t > 0$. Take $\Xi_\eta = \{|z| \le \eta\} \cap (\mathcal{V}_1^\eta \cup \cdots \cup \mathcal{V}_s^\eta)^c$, where the $\mathcal{V}_i^\eta$'s are respective neighborhoods of the x_i^η's in $\{\lambda_\eta \le |z| \le \eta\}$ such that

$$\mathbb{P}[\|\bar{Z}^\eta\|_{T,p} < \varepsilon/2] > 0,$$

having set

$$\bar{Z}_t^\eta = \int_0^t \int_{\Xi_\eta^c} z \, \bar{\mu}(ds, dz) - t\left(u_\eta + \int_{\Xi_\eta^c} z \, \bar{\nu}(dz) \right)$$

for every $t > 0$ (this is clearly possible because of (28), (33), and the reasoning in Case A which led to (13) and (14)). Lemma 2 entails that

$$\mathbb{P}[\|\bar{\bar{Z}}^\eta\|_{T,p} < \varepsilon/2] > 0,$$

so that since $\bar{Z} = \bar{Z}^\eta + \bar{\bar{Z}}^\eta$ with $\bar{Z}^\eta$ and $\bar{\bar{Z}}^\eta$ independent, we finally get

$$\mathbb{P}[\|\bar{Z}\|_{T,p} < \varepsilon] > 0$$

by the triangle inequality. But now by Skorohod's absolute continuity theorem, the law of $\bar{Z}$ and $\tilde{Z}$ are equivalent for every $\eta > 0$. Hence

$$\mathbb{P}[\|\tilde{Z}\|_{T,p} < \varepsilon] > 0,$$

which completes the proof of the Theorem in the case $\operatorname{Dim} L = 2$.

4.4. $\operatorname{Dim} L > 2$

We briefly describe how this higher dimensional situation can be handled. First, it is clear that we just need to prove (15) and (16) along some subsequence $\{\eta\}$ tending to 0. Again, we can make a choice of strict positivity and suppose that $\operatorname{Supp}\nu$ is included in a quadrant of $\mathbb{R}^d$. Take an asymptotic direction $L_1 = \operatorname{Vect}\{u_1\}$ and a corresponding subsequence $\{\eta\}$. In order to control the projections of our approximating vectors and to preserve (16), we need to refine our choice of positivity: consider the projection of $\operatorname{Supp}\nu$ onto $L_1^\perp$, take an orthonormal basis of $L_1^\perp$ and divide $L_1^\perp$ accordingly into 2^{d-1} quadrants $Q_1, \ldots, Q_{2^{d-1}}$. Set

$$\nu_{1,i} = \nu \mathbf{1}_{\{z_1^\perp \in Q_i\}} \quad \text{and} \quad v_{1,i}^\eta = \int_{\eta \le |z| \le 1} z_1^\perp \, \nu_{1,i}(dz)$$

for $i = 1, \ldots, 2^{d-1}$. It is clear that

$$\lim_{\eta \downarrow 0} \frac{|v_1^\eta|}{|v_{1,i}^\eta|} = +\infty$$

along the subsequence. Take a sub-subsequence along which either

$$\lim_{\eta \downarrow 0} \frac{v_{1,i}^\eta}{|v_{1,i}^\eta|} = u_{2,i} \in L_1^\perp, \quad \text{or } \liminf_{\eta \downarrow 0} |v_{1,i}^\eta| = 0,$$

for every $i = 1, \ldots, 2^{d-1}$, and set $L_{2,i} = \mathrm{Vect}\{u_{2,i}\}$.

Suppose first that $\mathrm{Dim}\, L = 3$. Because of our choice of strict positivity for each $\nu_{1,i}$, we can reason as in the situation $\mathrm{Dim}\, L = 2$, Case A or B, and prove that there exists a sub-subsequence $\{\eta\}$, $x_{1,i}^\eta, \ldots, x_{r_i,i}^\eta \in \mathrm{Supp}\, \nu_{1,i} \cap \{|z| \le \eta\}$, $\beta_{1,i}^\eta, \ldots, \beta_{r_i,i}^\eta$ minimizing integers such that for every $i = 1, \ldots, 2^{d-1}$

$$\left| v_{1,i}^\eta + \rho_{1,i}^\eta u_{2,i} - \Pi_1^\perp \left(\sum_{j=1}^{r_i} \beta_{j,i}^\eta x_{j,i}^\eta \right) \right| < \varepsilon / 2^d T,$$

where $\rho_{i,1}^\eta \ge 0$ and

$$\lim_{\eta \downarrow 0} \beta_{j,i}^\eta |x_{j,i}^\eta|^p = 0$$

for every $j = 1, \ldots, r_i$. Writing

$$v_L^\eta = v_1^\eta + \sum_{i=1}^{2^{d-1}} v_{1,i}^\eta$$

and reasoning as in the end of Case B leads to

$$\left| v_L^\eta + \rho_\eta u_1 + \sum_{i=1}^{2^{d-1}} \rho_{1,i}^\eta u_{2,i} - \sum_{i=1}^{r} \beta_i^\eta x_i^\eta \right| < \varepsilon / 2T, \tag{34}$$

where $\rho_\eta \ge 0$ and

$$\lim_{\eta \downarrow 0} \beta_i^\eta |x_i^\eta|^p = 0, \tag{35}$$

for some fixed integer r, $x_i^\eta, \ldots, x_i^\eta \in \mathrm{Supp}\, \nu \cap \{|z| \le \eta\}$, and $\beta_i^\eta, \ldots, \beta_r^\eta$ minimizing integers. The approximation (34) is not exactly (15), but the (positive) perturbing term

$$\rho_\eta u_1 + \sum_{i=1}^{2^{d-1}} \rho_{1,i}^\eta u_{2,i}$$

can be canceled as above by an absolutely continuous transformation of the law of $\tilde{Z}$, after removing some mass from $\operatorname{Supp}\nu_{1,i} \cap \{|z| \leq \lambda_\eta\}$ for each $i = 1, \ldots, 2^{d-1}$, with $\lambda_\eta = \inf\{|x_i^\eta|\}/2$. This transformation leads to the small deviation property for the original process $\tilde{Z}$.

When $\operatorname{Dim} L$ gets higher, we need to refine again and again our decomposition of $\operatorname{Supp}\nu$, dividing first each orthogonal of $u_{2,i}$ into 2^{d-2} quadrants $R_1, \ldots, R_{2^{d-2}}$ and introducing

$$\nu_{2,i,j} = \nu \mathbf{1}_{\{z_1^\perp \in Q_i, z_{2,i}^\perp \in R_j\}} \quad \text{and} \quad v_{2,i,j}^\eta = \int_{\eta \leq |z| \leq 1} z_{2,i}^\perp \nu_{2,i,j}(dz)$$

for $i = 1, \ldots, 2^{d-1}$, $j = 1, \ldots, 2^{d-2} \ldots$ etc. Setting $k = \operatorname{Dim} L$ and writing

$$v_L^\eta = v_1^\eta + \sum_{i=1}^{2^{d-1}} v_{2,i}^\eta + \cdots + \left(\sum_{i_1=1}^{2^{d-1}} \cdots \sum_{i_{k-2}=1}^{2^{d-(k-2)}} v_{k-1,i_1,\ldots,i_{k-2}}^\eta \right)$$

leads to an approximation of type (34) together with the control (35), which finishes the proof of the Theorem.

5. Proof of the Corollaries

5.1. Small deviations around continuous curves

The following proposition shows that the small deviation property for $\tilde{Z}$ also holds around L-valued curves with finite regular p-variation. Of course, this would be a direct consequence of the Theorem if one had some kind of Cameron-Martin formula for $\tilde{Z}$ as for Brownian motion. But here $\tilde{Z}$ has no Gaussian part and it is well-known, for example, that the law of $\tilde{Z}^u : t \mapsto \tilde{Z}_t + tu$ is not absolutely continuous (and even singular) with respect to the law of $\tilde{Z}$ if $u \neq 0$ —see again Theorem 33.1. in [22]. To prove this proposition we will need Lemma 13 as well as a slight perturbation of the Poisson measure, which replaces in some sense the density transformation.

Proposition 15. *Let $1 \leq p < 2$ and Z be a Lévy process with finite p-variation and parameters (α, ν). For every $\varepsilon > 0$, $T > 0$ and $\phi_L : \mathbb{R}^+ \to L$ with finite regular p-variation over compact sets,*

$$\mathbb{P}[\|\tilde{Z} - \phi_L\|_{T,p} < \varepsilon] > 0.$$

Proof. Clearly, we can suppose that $\phi_L(0) = 0$ and that the jumps of Z are bounded by 1. In particular

$$\tilde{Z}_t = \int_0^t \int_{|z| \leq 1} z \widetilde{\mu}(ds, dz) + t \int_{|z| \leq 1} z_K \nu(dz)$$

for every $t \geq 0$. Fix $\varepsilon > 0$, $T > 0$ and $\phi_L : \mathbb{R}^+ \to L$ with finite regular p-variation over compact sets. By Lemma 13, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$

$$\|\phi_L - \phi_L^n\|_{T,p} < \frac{\varepsilon}{3},$$

where ϕ_L^n is the polygonal approximation of ϕ_L over $[0,T]$ with step T/n. Fix $n \geq n_0$. Let $v_0 = 0$ and $v_1, \ldots, v_n$ be the vectors of L defining ϕ_L^n:

$$\phi_L^n(t) = \frac{T}{n}(v_0 + \cdots + v_j) + (t - s_j)v_{j+1} \qquad \text{if} \quad s_j \leq t \leq s_{j+1},$$

where again we set $s_j = jT/n$. Set, for every $j = 0, \ldots, n$ and $s_j \leq t \leq s_{j+1}$,

$$\tilde{Z}_t^j = \tilde{Z}_t - \tilde{Z}_{s_j} - (t - s_j)v_{j+1}$$

and

$$Y_t^j = \int_{s_j}^t \int_{|z|\leq 1} z\widetilde{\mu}_j(ds, dz) + (t - s_j) \int_{|z|\leq 1} z_K \nu_j(dz),$$

where we wrote

$$\nu_j(dz) = \nu(dz) + 2|v_{j+1}|\delta_{\frac{v_{j+1}}{2|v_{j+1}|}}(dz)$$

for every $j = 0, \ldots, n$ (with the notation $v_j/|v_j| = 0$ if $|v_j| = 0$), and where $\widetilde{\mu}_j$ is the compensated measure of μ_j, the Poisson measure with intensity $ds \otimes \nu_j$. Notice that clearly,

$$Y_t^j = \int_{s_j}^t \int_{|z|\leq 1} z\, \widetilde{\mu}_j(ds, dz) + (t - s_j) \int_{|z|\leq 1} z_K\, \nu(dz),$$

so that for every $0 < \eta < 1/2$ and $j = 0, \ldots, n$,

$$\{\|Y^j\|_{[s_j, s_{j+1}],p} < \eta\} \subset \{Y_t^j = \tilde{Z}_t^j \;\; \forall t \in [s_j, s_{j+1}]\}.$$

In particular

$$\{\|Y^j\|_{[s_j, s_{j+1}],p} < \eta\} = \{\|\tilde{Z}^j\|_{[s_j, s_{j+1}],p} < \eta\}.$$

Now, by the Theorem,

$$\mathbb{P}[\|Y^j\|_{[s_j, s_{j+1}],p} < \eta] > 0$$

for every $\eta > 0$ and $j = 0, \ldots, n$. Hence we get, by independence of the increments of $\tilde{Z}$,

$$\mathbb{P}[\|\tilde{Z}^j\|_{[s_j, s_{j+1}],p} < \eta \text{ for every } j = 0, \ldots, n] > 0 \tag{36}$$

for every $0 < \eta < 1/2$. Introduce now the following function:

$$\widetilde{\phi}_L^n(t) = \phi_L^n(t) + \sum_{k=0}^{j-1} \tilde{Z}_{s_{k+1}}^k \qquad \text{if} \quad s_j \leq t < s_{j+1},$$

for every $j = 0, \ldots, n$. $\widetilde{\phi}_L^n$ is a discontinuous perturbation of ϕ_L^n such that $\widetilde{\phi}_L^n - \phi_L^n$ is a step-function and $\widetilde{\phi}_L^n(s_j) = \tilde{Z}_{s_j}$ for every $j = 0, \ldots, n$. On the one hand, reasoning as in Lemma 13, we can choose η sufficiently small such that

$$\{\|\tilde{Z}^j\|_{[s_j, s_{j+1}],p} < \eta \text{ for every } j = 0, \ldots, n\} \subset \left\{\|\tilde{Z} - \widetilde{\phi}_L^n\|_{T,p} < \frac{\varepsilon}{3}\right\}. \tag{37}$$

On the other hand, since according to Lemma 8

$$\|\phi_L^n - \widetilde{\phi}_L^n\|_{T,p}^p \le n^{p+1} \max_{0\le k\le n-1} |\tilde{Z}^k_{s_{k+1}}|^p,$$

we also have

$$\{\|\tilde{Z}^j\|_{[s_j,s_{j+1}],p} < \eta \text{ for every } j=0,\dots,n\} \subset \left\{\|\phi_L^n - \widetilde{\phi}_L^n\|_{T,p} < \frac{\varepsilon}{3}\right\} \tag{38}$$

for η small enough. Putting (36), (37), (38) together and using the triangle inequality complete the proof of the Proposition. □

5.2. Proof of Corollary A

(a) By the Theorem we just need to prove the reverse inclusion. Suppose $\alpha_\nu \neq 0$. Since

$$Z_t = \alpha_\nu + \sum_{s\le t} \Delta Z_s$$

for every $t \ge 0$, we see by Lemma 12 that

$$\mathbb{P}[\|Z\|_{1,1} < \varepsilon] = 0$$

as soon as $\varepsilon < |\alpha_\nu|$.

(b) This follows readily from Proposition 15.

(c) Fix $\varepsilon, T > 0$. Since $\alpha \in \Pi_K^{-1}(\mathcal{A}_K)$, there exists $\alpha_L \in L$ such that

$$\beta = \alpha_L - \alpha_\nu \in \mathcal{C}.$$

Hence, for every $\eta > 0$, there exists $x_1^\eta, \dots, x_d^\eta \in \operatorname{Supp}\nu \cap \{|z| \le \eta\}$ and $\alpha_1^\eta, \dots, \alpha_d^\eta$ minimizing integers, such that

$$\left|\beta - \sum_{i=1}^d \alpha_i^\eta x_i^\eta\right| \le \varepsilon/4T. \tag{39}$$

Besides, since $\mathcal{C}$ is strictly convex, it is clear that there exists $c > 0$ independent of η such that

$$|\alpha_i^\eta x_i^\eta| \le c|\beta| \tag{40}$$

for every $i = 1, \dots, d$. Introduce now $\rho_\eta = \inf\{|x_i^\eta|,\ i = 1,\dots,d\}/2$, and decompose Z into

$$Z = \tilde{Z}^\eta + Z^\eta$$

where we set

$$\tilde{Z}_t^\eta = \int_0^t \int_{|z|\le\rho_\eta} z_K \mu(ds,dz) + \int_0^t \int_{|z|\le\rho_\eta} z_L \widetilde{\mu}(ds,dz) + t\left(\alpha_L - \int_{\rho_\eta\le|z|\le 1} z_L \nu(dz)\right)$$

and

$$\tilde{Z}_t^\eta = \sum_{s \le t} \Delta Z_s \mathbf{1}_{\{|\Delta Z_s| > \rho_\eta\}} - t\beta$$

for every $t > 0$. The Theorem and Proposition 15 yield readily

$$\mathbb{P}[\|\tilde{Z}^\eta\|_{T,p} < \varepsilon/2] > 0.$$

Hence, by independence and the triangle inequality, it suffices to show that

$$\mathbb{P}[\|Z^\eta\|_{T,p} < \varepsilon/2] > 0.$$

It is now clear that the latter can be done through (39), (40), and the same approximation procedure which we used repeatedly during the proof of the Theorem.

(d) This follows readily from the main Theorem in [26] and from the inequality

$$\mathbb{P}[\|Z^\eta\|_{T,p} < \varepsilon] \le \mathbb{P}[\|Z^\eta\|_{T,\infty} < \varepsilon]$$

for every $T, \varepsilon > 0$.

5.3. Proof of Corollary B

We first quote a lemma which is a direct consequence of Lyons' continuity theorem [19].

Lemma 16. *Let $\{x_t^i,\ 0 \le t \le T\}_{i=1,2}$ be the solutions to the following rough differential equations on $\mathbb{R}^m$:*

$$x_t^i = x_i + \int_0^t f(x_s^i) dz_s,$$

where z is a function with regular finite p-variation and f an α-Lipschitz vector field with $\alpha > p$. Then there exists a constant K (depending on T and f) such that

$$\|x^1 - x^2\|_{T,p} \le K|x_1 - x_2|.$$

We can now proceed to the proof of Corollary B, which will mimic that of the Theorem in [25]. The first inclusion $\operatorname{Supp} X \subset \overline{\mathcal{S}}$ is an easy consequence of the fact that for every $n \ge 1$

$$\lim_{\eta \to 0} \|X - X^\eta\|_{n,p} = 0,$$

where X^η is the solution to (1) with ν replaced by $\mathbf{1}_{|z| \ge \eta}\nu(dz)$—which follows readily from Lemma 1 and Theorem 7, and of the usual routine which may be found e.g. in [30].

The second inclusion $\overline{\mathcal{S}} \subset \operatorname{Supp} X$ will be a consequence of Theorem 7 and Proposition 15, as in [25]. Fix $n \in \mathbb{N}^*$, $\varepsilon > 0$, $u \in \mathbf{U}$, and $\phi_L : \mathbb{R}^+ \to L$ with regular p-variation. Let ψ be the solution of (2) given by u and ϕ_L. Let $N_n \in \mathbb{N}^*$ be such that

$$t_0 = 0 < t_1 < \cdots < t_{N_n} \le n + 1 < t_{N_n + 1}$$

are the successive jumping times of ψ. Introduce

$$\eta = \inf\{|z_i|,\ i = 1, \dots, N_n\}/2 \quad \text{and} \quad Z_t^\eta = \int_0^t \int_{|z|\geq\eta} z\mu(ds, dz)$$

for every $t \geq 0$. Set $\{T_q\}$ for the sequence of Z^η's successive jumping times, and $\widetilde{\psi}$ for the solution of (2) where $\{t_q\}$ is replaced by $\{T_q\}$. For every $\rho > 0$, the event

$$\left\{ \sup_{1\leq q\leq N_n+1} |T_q - t_q| < \rho \right\}$$

has a positive probability. We now introduce λ, the only piecewise linear change of time transforming t_q into T_q for each $q = 1, \dots, N_n$, and whose right derivative takes its values in $\{1/2, 1, 2\}$. Thanks to the continuity of ψ (resp. of $\widetilde{\psi}$) on each $]t_i, t_{i+1}[$ (resp. on each $]T_i, T_{i+1}[$) and to a repeated use of Lemma 16, it is easy to see that

$$\left\{ \sup_{1\leq q\leq N_n+1} |T_q - t_q| < \rho \right\} \subset \{\|\psi \circ \lambda - \widetilde{\psi}\|_{[0,n+1],p} < \varepsilon/2\}$$

for $\rho > 0$ small enough. Hence

$$\left\{ \sup_{1\leq q\leq N_n+1} |T_q - t_q| < \rho \right\} \subset \{\mathbf{d}_p^n(\psi, \widetilde{\psi}) < \varepsilon/2\}$$

for $\rho > 0$ small enough. On the other hand, Proposition 15 entails easily that

$$\mathbb{P}[\|\tilde{Z}^\eta - \phi^L\|_{n+1,p} < \rho] > 0$$

for every $\rho > 0$, where we set $\tilde{Z}^\eta = Z - Z^\eta$. Using now Theorem 7 and reasoning exactly as in the proof of the Theorem of [25] (under the p-variation norm) entail

$$\mathbb{P}[\|X - \widetilde{\psi}\|_{n+1,p} < \varepsilon/2,\ \mathbf{d}_p^n(\psi, \widetilde{\psi}) < \varepsilon/2] > 0,$$

which finishes the proof since obviously

$$\{\|X - \widetilde{\psi}\|_{n+1,p} < \varepsilon/2,\ \mathbf{d}_p^n(\psi, \widetilde{\psi}) < \varepsilon/2\} \subset \{\mathbf{d}_p^n(X, \psi) < \varepsilon\}.$$

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References

[1] P. Baldi and B. Roynette, *Some exact equivalents for the Brownian motion in Hölder norm.* Probab. Theory Related Fields **93**-4 (1992), 457–484.

[2] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular Variation*, Cambridge University Press, Cambridge, 1987.

[3] J. Bertoin, *Lévy processes*, Cambridge University Press, Cambridge, 1996.

[4] J. Bretagnolle, *p-variation de fonctions aléatoires* I *et* II, In: Sém. Prob. **VI**, Springer, Berlin, 1971, pp. 51–71.

[5] V. V. Chistyakov and O. E. Galkin, *On maps of bounded p-variation with* $p > 1$, Positivity **2**-1 (1998), 19–45.

[6] R. M. Dudley and R. Norvaiša, *An introduction to p-variation and Young integrals. With emphasis on sample functions of stochastic processes*, MaPhySto Lectures Notes **1**, Univ. of Aarhus, 1998.

[7] M. Errami, F. Russo and P. Vallois, *Itô formula for* $\mathcal{C}^{1,\lambda}$ *functions of a càdlàg process and related calculus*, Probab. Theory Related Fields **122**-2 (2002), 191–221.

[8] B. E. Fristedt and S. J. Taylor, *Strong variation for the sample function of a stable process*, Duke Math. J. **40**-1 (1973), 259–278.

[9] T. Fujiwara, *Stochastic differential equations of jump type on manifolds and Lévy flows*, J. Math. Kyoto Univ. **31**-1 (1991), 99–119.

[10] T. Fujiwara and H. Kunita, *Canonical stochastic differential equations based on semimartingales with spatial parameters* I *and* II, Kyushu J. Math. **53** (1999), 265–331.

[11] Y. Ishikawa, *Existence of the density for a singular jump process and its short time properties*, Kyushu J. Math. **55** (2001), 267–299.

[12] J. Jacod and A. N. Shiryaev, *Limit theorems for stochastic processes*, Springer-Verlag, Berlin, 1987.

[13] H. Kunita, *Supports of diffusion processes and controllability problems*, In: Proceedings of the International Symposium on Stochastic Differential Equations (Kyoto 1976), Wiley, Brisbane, 1978, pp. 163–185.

[14] ———, *Stochastic differential equations with jumps and stochastic flows of diffeomorphisms*, In: Ikeda et al., (ed.), Itô's stochastic calculus and probability theory, Springer, Berlin, 1996, pp. 197–211.

[15] ———, *Canonical stochastic differential equations based on Lévy processes and their supports*, In: Crauel et al., (ed.), Stochastic dynamics, Springer, New York, 1999, pp. 283–304.

[16] T. G. Kurtz, E. Pardoux and Ph. Protter, *Stratonovich stochastic differential equations driven by general semimartingales*, Ann. Inst. H. Poincaré Prob. Statist. **31** (1995), 351–377.

[17] M. Ledoux, Z. Qian and T. Zhang, *Large deviations and support theorem for diffusion processes via rough paths*, Stochastic Process. Appl. **102**-2 (2002), 265–283.

[18] M. A. Lifshits and T. Simon, *Small deviations for fractional stable processes*, Submitted to Annales de l'Institut Henri Poincaré, 2003.

[19] T. J. Lyons, *Differential equations driven by rough signals* (I): *An extension of an inequality of L. C. Young*, Math. Res. Lett. **1**-4 (1994), 451–464.

[20] ———, *Differential equations driven by rough signals*, Rev. Mat. Iberoamericana **14**-2 (1999), 215–310.

[21] S. I. Marcus, *Modelling and approximation of stochastic differential equations driven by semi-martingales*, Stochastics **4** (1981), 223–245.

[22] K.-I. Sato, *Lévy processes and infinitely divisible distributions*, Cambridge University Press, Cambridge, 1999.

[23] M. J. Sharpe, *Support of convolution semi-groups and densities*, In: Heyer (ed.), Probability measures on groups and related structures **XI**, World Scientific Publishing, River Edge, 1995, pp. 364–369.

[24] T. Simon, *Support theorem for jump processes*, Stochastic Process. Appl. **89**-1 (2000), 1–30.

[25] T. Simon, *Support of a Marcus equation in dimension* 1, Electron. Comm. Probab. **5** (2000), 149–157.

[26] ———, *Sur les petites déviations d'un processus de Lévy*, Potential Anal. **14**-2 (2001), 155–173.

[27] ———, *Support d'une équation d'Itô en dimension* 1, Sém. Prob. **36** (2002), 314–330.

[28] ———, *Small ball estimates in p-variation for stable processes*, Submitted to Journal of Theoretical Probability, 2003.

[29] W. Stolz, *Une méthode élémentaire pour l'évaluation de petites boules browniennes*, C. R. Acad. Sci. Paris Sér. I **316**-11 (1993), 1217–1220.

[30] D. W. Stroock and S. R. S. Varadhan, *On the support of diffusion processes with applications to the strong maximum principle*, In: Proc. 6th Berkeley Symp. Math. Stat. Prob. **III**, Univ. California Press, Berkeley, 1972, pp. 333–359.

[31] H. J. Sussmann, *On the gap between deterministic and stochastic ordinary differential equations*, Ann. Probab. **6** (1978), 19–41.

[32] S. J. Taylor, *Sample path properties of a transient stable process*, J. Math. Mech. **16** (1967), 1229–1246.

[33] ______, *Exact asymptotic estimates of Brownian path variation*, Duke Math. J. **39**-1 (1972), 219–241.

[34] A. Tortrat, *Le support des lois indéfiniment divisibles dans un groupe Abélien localement compact*, Math. Z. **197** (1988), 231–250.

[35] D. R. E. Williams, *Differential equations driven by discontinuous paths*, Ph.D. Thesis, Imperial College of London, 1998.

[36] ______, *Path-wise solutions of stochastic differential equations driven by Lévy processes*, Rev. Mat. Iberoamericana **17**-2 (2001), 295–329.