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A characterization of symmetric domains

By

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Abstract

We prove that the Laplace-Beltrami operator commutes with the Berezin transform on a Kähler manifold if and only if it is a Hermitian symmetric space.

1. Introduction

Let Ω be a bounded domain in $\mathbf{C}^n$ (or biholomorphic to a bounded) and $G = \mathrm{Aut}(\Omega)$ the group of its holomorphic automorphisms (i.e. biholomorphic self-maps). The domain is called *homogeneous* if G acts transitively on it, i.e. for any $x, y \in \Omega$ there is $\omega \in G$ such that $\omega x = y$; and *symmetric* if for each $x \in \Omega$ there exists $s_x \in G$, $s_x^2 = \mathrm{id}$, which has x as an isolated fixed-point. Every symmetric domain is homogeneous, but not conversely; the first example of a non-symmetric homogeneous domain is due to Pyatetskii-Shapiro [28]. For this reason, it has been of interest to characterize the symmetric domains among the homogeneous ones. There are characterizations in terms of the defining data of the Siegel realization of the domain (Satake [31, Theorem V.3.5]; Dorfmeister [9, Theorem 3.3]) or in terms of the almost complex structure map on the tangent space belonging to the infinitesimal representation of the isotropy group [7]; further, a homogeneous bounded domain Ω is symmetric if and only if there are no nontrivial G-invariant vector fields on Ω; if and only if the algebra of all G-invariant differential operators is commutative; if and only if the isotropy group acts transitively on the Shilov boundary of Ω; if and only if all sectional curvatures of the Bergman metric on Ω are nonpositive; if and only if, finally, for every irreducible factor of Ω, the curvature operator of the Bergman metric has at most two distinct eigenvalues [7], [1].

The inspiration for this paper was yet another characterization, due recently to Nomura [24], [25]. Namely, let $K_\Omega(x, y)$ be the Bergman kernel of Ω, Δ the Laplace-Beltrami operator with respect to the Bergman metric, and B_Ω the integral operator

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$$B_\Omega f(x) := \int_\Omega f(y) \frac{|K_\Omega(x,y)|^2}{K_\Omega(x,x)}\, dy \tag{1.1}$$

where dy denotes the Lebesgue measure; B_Ω is called the *Berezin transform* on Ω. Nomura's result then says that

$$\Omega \text{ is symmetric } \iff B_\Omega \Delta = \Delta B_\Omega.$$

Actually, Nomura even proved this for certain weighted analogues of B_Ω, as well as for Laplace-Beltrami operators with respect to a bit more general metrics than the Bergman metric; see Section 2 below. His proofs rely on the theory of j-algebras, and make also contact with the work of Penney on Cayley's transforms for homogeneous domains [27].

Our main result is that the commutativity of the Berezin transform with Δ in fact characterizes the symmetric domains not only among the homogeneous ones, but actually among all domains in $\mathbf{C}^n$ (or even manifolds) with a Kähler metric. Let us make this more precise.

Consider, quite generally, a domain Ω in $\mathbf{C}^n$, equipped with a Kähler metric $g_{i\bar{j}}$. Let

$$d\mu(z) = \det[g_{i\bar{j}}(z)]\, dz$$

be the corresponding volume element and

$$\Delta = g^{\bar{j}i} \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \tag{1.2}$$

the corresponding Laplace-Beltrami operator; here $g^{\bar{j}i}$ is the inverse matrix to $g_{i\bar{j}}$, and we are using the usual summation convention. The condition that the metric be Kähler means that locally

$$g_{i\bar{j}} = \frac{\partial^2 \Phi}{\partial z_i \partial \overline{z}_j} \tag{1.3}$$

for some real-valued C^∞ function Φ on Ω, called the Kähler potential. Assume, for the moment, that Ω exists even globally (this will certainly be the case, for instance, whenever Ω is contractible), and consider the weighted Bergman space $L^2_{\mathrm{hol}}(\Omega, e^{-\Phi}\, d\mu)$ of all holomorphic functions on Ω square-integrable with respect to the measure $e^{-\Phi} d\mu$. Let $K(x,y)$ be its reproducing (i.e. weighted Bergman) kernel, and

$$Bf(x) := \int_\Omega f(y) \frac{|K(x,y)|^2}{K(x,x)}\, e^{-\Phi(y)}\, d\mu(y) \tag{1.4}$$

the corresponding Berezin transform. Here we are assuming that

$$K(z,z) > 0 \qquad \forall z \in \Omega, \tag{1.5}$$

so that the definition of B makes sense. Note that (1.5) is equivalent to the existence, for each $z \in \Omega$, of a function $f \in L^2_{\mathrm{hol}}(\Omega, e^{-\Phi}\, d\mu)$ for which $f(z) \neq 0$.

Observe that even though the potential Φ of the Kähler metric $g_{i\bar{j}}$ is not determined uniquely, the Berezin transform is independent of the choice of Φ, and thus depends only on the metric $g_{i\bar{j}}$. Indeed, if Φ is a real-valued solution to (1.3), then all other solutions are given by

$$\Phi' = \Phi + 2\operatorname{Re} F \tag{1.6}$$

with F a holomorphic function on Ω. However, then $f \mapsto fe^F$ is a Hilbert space isomorphism of $L^2_{\mathrm{hol}}(\Omega, e^{-\Phi}\,d\mu)$ onto $L^2_{\mathrm{hol}}(\Omega, e^{-\Phi'}\,d\mu)$, and from the reproducing property

$$f(x) = \int_\Omega f(y)K(x,y)e^{-\Phi(y)}\,d\mu(y) \qquad \forall f \in L^2_{\mathrm{hol}}(\Omega, e^{-\Phi}\,d\mu)$$

it follows that the reproducing kernel $K'(x,y)$ corresponding to Φ' is given by

$$K'(x,y) = e^{F(x)}K(x,y)\,e^{\overline{F(y)}}.$$

Consequently, the right-hand side of (1.4) remains unchanged when Φ and K are replaced by Φ' and K', respectively.

The same argument shows also that the function

$$m(x) := e^{-\Phi(x)}K(x,x) \tag{1.7}$$

does not change under the "gauge transformation" (1.6), and thus again depends only on the metric $g_{i\bar{j}}$. This function will turn out to be of crucial significance in the sequel.

It is clear from (1.4) that B is a continuous operator on $L^\infty(\Omega)$, and its range is contained in $C^\infty(\Omega)$. In particular, both $B\Delta$ and ΔB make sense on $C_0^\infty(\Omega)$, the subspace in $C^\infty(\Omega)$ of all functions with compact support.

Finally, let us agree to call the Kähler manifold $(\Omega, g_{i\bar{j}})$ *nondegenerate* [*1] if not only (1.5), but the following stronger condition is satisfied:

(1.8) for every point $z \in \Omega$ and vector $X \in T_z\Omega$, there exist functions $f, g \in L^2_{\mathrm{hol}}(\Omega, e^{-\Phi}\,d\mu)$ such that
$$Xf \cdot g(z) - Xg \cdot f(z) \neq 0.$$

This will always be the case, for instance, when the constants and the coordinate functions $z_1, \dots, z_n$ belong to $L^2_{\mathrm{hol}}(\Omega, e^{-\Phi}\,d\mu)$: one can then take $g = \mathbf{1}$ and $f = z_j$ for a suitable j. We will see below that (1.8) is actually equivalent to the matrix $[\partial^2 \log K(z,z)/\partial z_i \partial \overline{z}_j]_{i,j=1}^n$ being not only positive semidefinite (that is always the case), but actually positive definite. Again, the same argument as for (1.7) above shows that (1.8) is independent of the choice of the potential Φ, and so is indeed a property of Ω and $g_{i\bar{j}}$ themselves.

With these preparations, our main result is as follows. (The definitions of locally symmetric space and complete metric can be found in the next section.)

[*1] This is not a standard terminology!

Theorem (Main Theorem). *Let Ω be a domain in $\mathbf{C}^n$, $g_{i\bar{j}} = \partial\overline{\partial}\Phi$ a Kähler metric on Ω admitting a global potential Φ, $d\mu$ the associated volume element, $K(x,y)$ the Bergman kernel of $L^2_{hol}(\Omega, e^{-\Phi}\,d\mu)$, and suppose that $(\Omega, g_{i\bar{j}})$ is nondegenerate, so that, in particular, the corresponding Berezin transform B is defined.*

(i) *Assume that*

$$B\Delta = \Delta B$$

on $C_0^\infty(\Omega)$. Then the function m from (1.7) is constant, B is a bounded selfadjoint operator on $L^2(\Omega, d\mu)$, and $(\Omega, g_{i\bar{j}})$ is a locally symmetric space.

(ii) *If, in addition, $g_{i\bar{j}}$ is complete, then $(\Omega, g_{i\bar{j}})$ is a Hermitian globally symmetric space.*

Note that if Ω is homogeneous and $g_{i\bar{j}}$ is the Bergman metric, then it is easily seen, by comparing the transformation properties of both sides, that

$$\det[g_{i\bar{j}}] = ce^{\Phi}$$

for some positive constant c (see e.g. Helgason [17, Proposition VIII.3.6]). Consequently, $e^{-\Phi}d\mu$ is a constant multiple of the Lebesgue measure, and thus K will be (up to a constant factor) the ordinary Bergman kernel K_Ω and the Berezin transform B from (1.4) will coincide with the B_Ω from (1.1). Thus our theorem contains Nomura's result as a particular case.

We remark that the assumption that Ω be a domain and the potential Φ exist globally can be relaxed by passing from L^2-spaces of holomorphic functions to L^2-spaces of holomorphic sections of suitable line bundles; it is then enough that $(\Omega, g_{i\bar{j}})$ be any Kähler manifold such that the cohomology class determined by $g_{i\bar{j}}$ in $H^2(\Omega, \mathbf{R})$ is integral. See Section 5 below.

We also remark that the converse to part (ii) of the theorem is well known: on any Hermitian symmetric space, the Berezin transform commutes with the invariant Laplace operator. In fact, B is a function, in the sense of the functional calculus for commuting self-adjoint operators, of Δ and the "higher Laplacians" (generators of the (commutative) algebra of all invariant differential operators). See [32] and [34] for more information on these matters.

Our proof of the Main Theorem uses a completely different method than in [24], and exploits a relationship between the curvature tensor and the derivatives of the reproducing kernel. For the case of domains in $\mathbf{C}$, essentially the same idea was used in the author's earlier paper [10].

The paper is organized as follows. In Section 2, we review some preliminaries on symmetric spaces. The proof of part (i) of Main Theorem appears in Section 3, the proof of part (ii) in Section 4. The final Section 5 contains some concluding remarks and comments, and two open problems.

An earlier version of the Main Theorem — featuring a much weaker result, using however a rather elegant application of the Fuglede-Putnam theorem from operator theory — was presented at the Hayama Conference on Several Complex Variables in December 2003 [12]; the author takes this opportunity to thank the organizers for the nice time and stimulating research atmosphere he experienced there.

Notation. Throughout the paper, we will frequently abbreviate $\partial/\partial x_j$, $\partial/\partial\overline{z}_k$, etc., to just ∂_{x_j}, $\partial_{\overline{z}_k}$, etc.; and if there is no danger of confusion concerning the variable, even to ∂_j and $\overline{\partial}_k$.

2. Symmetric spaces

In this section we collect some useful facts on Riemannian symmetric spaces, see e.g. Helgason [17]; its main purpose is to recall some terminology and make the paper more self-contained.

Let Ω be a real manifold with a Riemannian metric $ds^2 = g_{ij}\,dx^i\,dx^j$. We denote by $T_x\Omega$ the tangent space at a point $x \in \Omega$, and by $\exp_x : T_x\Omega \to \Omega$ the exponential mapping, so that $t \mapsto \exp_x(tX)$, for t in some open interval containing the origin in $\mathbf{R}$, is the geodesic through x in the direction of X. Since the restriction of $\exp_x$ to a sufficiently small neighbourhood of $0 \in T_x\Omega$ is a diffeomorphism, we can define a mapping s_x, the *geodesic symmetry at x,* by

$$s_x : \exp_x(X) \mapsto \exp_x(-X), \qquad X \in T_x\Omega.$$

It follows from the definition that s_x preserves the distance from x; if s_x is actually an isometry in some neighbourhood of x, then the space (Ω, g_{ij}) is called *locally symmetric.* If s_x is actually defined on all of Ω and isometric there, (Ω, g_{ij}) is called a (*globally*) *symmetric* space.

As has already been mentioned in the Introduction, one can show (by employing the geodesic symmetries) that a symmetric space is homogeneous, i.e. for any $x, y \in \Omega$ there exists an isometry sending x into y; the converse, however, is false [29].

We will need the following characterizations of symmetry in terms of the curvature tensor R_{ijkl} of the metric [*2] g_{ij}; the proofs can be found in [17, Theorem IV.1.1 and IV.5.6].

Proposition 1. (a) *A Riemannian space is locally symmetric if and only if $\nabla R = 0$, i.e. the covariant derivatives of the curvature tensor vanish identically.*

(b) *A locally symmetric space is globally symmetric if it is complete* (*i.e.* Ω *is complete as a metric space with respect to the distance induced by* g_{ij}) *and simply connected.*

All the above implies, in particular, also in the case of complex (instead of only real) manifolds with Hermitian (instead of only Riemannian) metric $ds^2 = g_{i\overline{j}}\,dz^i\,d\overline{z}^j$, and (not only C^∞ but) holomorphic geodesic symmetries s_x.

[*2] More precisely: of the Riemannian connection determined by this metric.

One then speaks of *Hermitian* (*locally* or *globally*) *symmetric spaces.* In that case, the curvature tensor is uniquely determined by its components $R_{i\overline{j}k\overline{l}}$. (One has $R_{\overline{j}ik\overline{l}} = R_{i\overline{j}\overline{l}k} = -R_{\overline{j}i\overline{l}k} = -R_{i\overline{j}k\overline{l}}$, and all other components are zero.)

Thus, in particular, if Ω is a bounded domain in $\mathbf{C}^n$ (or biholomorphic to a bounded), then the Bergman metric

$$b_{i\overline{j}}(z) := \frac{\partial^2 \log K_\Omega(z,z)}{\partial z_i \partial \overline{z}_j},$$

where $K_\Omega(x,y)$ is the ordinary (i.e. unweighted) Bergman kernel of Ω, is a Hermitian (even Kähler) metric with respect to which any biholomorphic self-map of Ω is an isometry. Hence, the homogeneous (or symmetric) domains as discussed in the Introduction are also homogeneous (or symmetric) spaces in the sense of this section, when equipped with the Bergman metric.

Finally, for the homogeneous domains as discussed in the Introduction one sometimes also uses other metrics than the Bergman metric: namely, instead of the whole automorphism group $G = \mathrm{Aut}(\Omega)$, one considers only a suitable split solvable Lie subgroup $G_0 \subset G$ acting simply transitively on Ω. The Lie algebra $\mathbf{g}_0$ of G_0 has then a structure of normal j-algebra, and for any so-called *admissible* linear form $\gamma \in \mathbf{g}_0^*$, $(X,Y)_\gamma := \gamma([JX,Y])$, where J is the almost complex structure, defines a real inner product on $\mathbf{g}_0$. The corresponding Hermitian inner product

$$\langle X,Y\rangle_\gamma := (X,Y)_\gamma - i(JX,Y)_\gamma = \gamma[JX,Y] + i\gamma[X,Y] \tag{2.1}$$

thus defines, upon identifying Ω with G_0, a Hermitian metric on Ω, which turns out to be even Kähler. (See [15], pp. 35–38.) This time, not every holomorphic automorphism is an isometry, but only those in the subgroup $G_0 \subset G$ are; however, since G_0 still acts transitively, Ω is again a homogeneous Hermitian space also in the sense discussed above, when equipped with the metric (2.1) associated with an admissible form γ on $\mathbf{g}_0$. These are precisely the metrics considered by Nomura [24] [25] mentioned in the Introduction; the Bergman metric $b_{i\overline{j}}$ is obtained as a special case, by taking for γ the so-called Koszul form of Ω [21].

Remark. A quite general way of constructing invariant metrics on a bounded homogeneous domain $\Omega \subset \mathbf{C}^n$ is as follows. Let G_1 be an arbitrary subgroup of $G = \mathrm{Aut}(\Omega)$ acting transitively on Ω, x_0 some point of Ω, and K and $K_1 = K \cap G_1$ the stabilizers (isotropy groups) of x_0 in G and G_1, respectively. It is well known that K_1 is always compact (in the compact-open topology, see e.g. [17], Theorem IV.2.5(b)). Denoting by dk the normalized Haar measure on K_1, we can therefore define a K_1-invariant inner product on the tangent space at x_0 by

$$\langle X,Y\rangle_{x_0} := \int_{K_1} \langle k_* X, k_* Y\rangle_{\mathbf{C}^n}\, dk. \tag{2.2}$$

Using homogeneity, this can be transferred to all other points of Ω, by letting, for any $x \in \Omega$ and $X, Y \in T_x\Omega$,

$$\langle X, Y\rangle_x := \langle \omega_* X, \omega_* Y\rangle_{x_0}$$

for any $\omega \in G_1$ such that $\omega(x) = x_0$. The K_1-invariance of (2.2) guarantees that the right-hand side is independent of the choice of ω, so the definition is consistent. Thus we obtain a Hermitian metric for which any element of G_1 is an isometry, as desired. In particular, for $G_1 = G$ we obtain metrics on Ω invariant under any holomorphic automorphism.

Unfortunately, at the moment it is not clear whether these metrics are Kähler, so our theorem need not apply in general. It even seems to be unknown, for a given bounded homogeneous domain in $\mathbf{C}^n$, how big the isotropy group $K \subset G$ can be — for instance, whether it can reduce to the sole identity element. By Cartan's uniqueness theorem, the mapping $k \mapsto k_*$ is an injective homomorphism of K into $U(n)$, so the question is equivalent to characterizing the subgroups of $U(n)$ that can arise as images of this homomorphism. Clearly, Ω is symmetric if and only if this image contains $-I$. $\square$

We finish this section by the following proposition, which clarifies somewhat the notion of nondegeneracy. For $\mathcal{H}$ the ordinary Bergman space, this is a very standard assertion about the Bergman metric (see e.g. Helgason [17, Proposition VIII.3.4]); though the proof for the general case is the same, we include it here for completeness.

Proposition 2. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space of holomorphic functions on Ω, with reproducing kernel $K(x, y)$. Then for each $z \in \Omega$, the matrix*

$$\Big[\frac{\partial^2 \log K(z,z)}{\partial z_i \partial \overline{z}_j}\Big]_{i,j=1}^n$$

is always positive semidefinite; and is positive definite if and only if (1.8) *holds* (*with $\mathcal{H}$ in the place of $L^2_{hol}(\Omega, e^{-\Phi}\, d\mu)$*).

Proof. As $\mathcal{H}$ is a space of holomorphic functions, $K(x, y)$ is holomorphic in x and $\overline{y}$; thus $K(z, z)$ is continuous (in fact, real analytic) on Ω, hence, in particular, locally bounded. Since $K(z, z)^{1/2}$ is precisely the norm of the evaluation functional $f \mapsto f(z)$ on $\mathcal{H}$, it follows that the standard series defining the reproducing kernel in terms of an arbitrary orthonormal basis $\{\phi_k\}$ of $\mathcal{H}$,

$$K(z, z) = \sum_k \phi_k(z)\overline{\phi_k(z)}$$

converges uniformly on compact subsets of $\Omega \times \Omega$, and can be differentiated termwise any number of times. Consequently,

$$\begin{aligned}
&\frac{\partial^2 \log K}{\partial z_i \partial \overline{z}_j} \\
&\quad = \frac{K \cdot \partial_i \overline{\partial}_j K - \partial_i K \cdot \overline{\partial}_j K}{K^2} \\
&\quad = \frac{1}{K^2}\Big[\Big(\sum_k \phi_k \cdot \overline{\phi}_k\Big)\Big(\sum_l \partial_i \phi_l \cdot \overline{\partial_j \phi_l}\Big) - \Big(\sum_k \partial_i \phi_k \cdot \overline{\phi}_k\Big)\Big(\sum_l \phi_l \cdot \overline{\partial_j \phi_l}\Big)\Big] \\
&\quad = \frac{1}{K^2}\sum_{k>l} (\phi_k \cdot \partial_i \phi_l - \phi_l \cdot \partial_i \phi_k)\, \overline{(\phi_k \cdot \partial_j \phi_l - \phi_l \cdot \partial_j \phi_k)}.
\end{aligned}$$

(We are omitting the arguments z and (z,z) for clarity.) Consequently, for any $\xi_1, \dots, \xi_n \in \mathbf{C}$,

$$\sum_{i,j} \xi_i \overline{\xi}_j \, \partial_i \overline{\partial}_j \log K = \frac{1}{K^2} \sum_{k>l} \Big| \sum_i \xi_i (\phi_k \cdot \partial_i \phi_l - \phi_l \cdot \partial_i \phi_k) \Big|^2 \geq 0,$$

which proves the positive semidefiniteness. Equality can occur if and only if

$$\phi_k \cdot X\phi_l - \phi_l \cdot X\phi_k = 0 \qquad \forall k, l,$$

where $X := \sum_i \xi_i \partial_i$. Since $\{\phi_k\}$ is a basis for $\mathcal{H}$, this is actually equivalent to

$$f \cdot Xg - g \cdot Xf = 0 \qquad \forall f, g \in \mathcal{H},$$

and the claim about positive definiteness follows. □

For later use, we put down explicitly the following important corollary.

Corollary 3. *If $(\Omega, g_{i\overline{j}})$ is nondegenerate and $K(x,y)$ is the reproducing kernel of $L^2_{hol}(\Omega, e^{-\Phi}\, d\mu)$, then the matrix*

$$[\partial_i \overline{\partial}_j \log K(z,z)]_{i,j=1}^n$$

is invertible, for any $z \in \Omega$.

3. Proof of Main Theorem, part (i)

Recall that B is the integral operator

$$Bf(x) = \int_\Omega f(y)\, \beta(x,y)\, d\mu(y)$$

where we have introduced the notation

$$\beta(x,y) := \frac{|K(x,y)|^2}{K(x,x)}\, e^{-\Phi(y)}.$$

Clearly, ΔB is the integral operator with kernel $\Delta_x \beta(x,y)$. On the other hand, it is well known (and readily checked from (1.2)) that Δ is formally self-adjoint with respect to $d\mu$; thus we have for any f in $C_0^\infty(\Omega)$

$$\begin{aligned} B\Delta f(x) &= \int_\Omega \Delta f(y) \cdot \beta(x,y)\, d\mu(y) \\ &= \int_\Omega f(y)\, \Delta_y \beta(x,y)\, d\mu(y). \end{aligned}$$

Thus $\Delta B f = B \Delta f$ $\forall f \in C_0^\infty(\Omega)$ if and only if

$$\Delta_y \beta(x,y) = \Delta_x \beta(x,y),$$

or

$$F(x,y) := \frac{\Delta_y \beta(x,y) - \Delta_x \beta(x,y)}{\beta(x,y)} = 0 \qquad \forall x, y \in \Omega.$$

Since $K(x,x) > 0$ $\forall x$ by hypothesis, we can write

$$K(x,y) = \exp L(x,y)$$

for x, y near the diagonal, with some function $L(x,y)$ holomorphic in x and $\overline{y}$ and real-valued for $x = y$. Hence

$$\beta(x,y) = \exp[L(x,y) + L(y,x) - L(x,x) - \Phi(y)]. \tag{3.1}$$

At this point, it will be expedient to introduce some notation. First of all, let

$$u(x) := L(x,x) - \Phi(x) \tag{3.2}$$

be the logarithm of the function $m(x)$ from (1.7). Second, for the sake of brevity, let us denote derivatives of functions simply by subscripts, i.e. write u_k, $u_{i\overline{j}}$, etc., for $\partial_k u$, $\partial_i \overline{\partial}_j u$, etc., and similarly for Φ. Note that, by (1.3), we then have $\Phi_{i\overline{j}} = g_{i\overline{j}}$, $\Phi_{i\overline{j}k} = \partial_k g_{i\overline{j}}$, etc. For the function L, we will similarly also write, for instance, $L_{i\overline{j}a\overline{b}}(x,y)$ instead of $\partial_{x_i} \partial_{\overline{y}_j} \partial_{x_a} \partial_{\overline{y}_b} L(x,y)$; i.e. all barred indices apply to y-derivatives, and all unbarred ones to x-derivatives; since L is holomorphic in x and $\overline{y}$, this should cause no confusion, as all other derivatives are identically zero.

Now, using the formula

$$\begin{aligned} \frac{\Delta e^f}{e^f} &= g^{\overline{j}i} \Big(\frac{\partial^2 f}{\partial z_i \partial \overline{z}_j} + \frac{\partial f}{\partial z_i} \frac{\partial f}{\partial \overline{z}_j} \Big) \\ &\equiv g^{\overline{j}i} (f_{i\overline{j}} + f_i f_{\overline{j}}) \qquad \text{(in our shorthand notation)}, \end{aligned}$$

we have

$$\begin{aligned} F(x,y) &\equiv \frac{\Delta_y \beta(x,y) - \Delta_x \beta(x,y)}{\beta(x,y)} \\ &= -g^{\overline{j}i}(y) \Phi_{i\overline{j}}(y) + g^{\overline{j}i}(y) \cdot [L_i(y,x) - \Phi_i(y)] \cdot [L_{\overline{j}}(x,y) - \Phi_{\overline{j}}(y)] \\ &\quad + g^{\overline{j}i}(x) L_{i\overline{j}}(x,x) - g^{\overline{j}i}(x) \cdot [L_i(x,y) - L_i(x,x)] \cdot [L_{\overline{j}}(y,x) - L_{\overline{j}}(x,x)]. \end{aligned}$$

Using the fact that $g^{\bar{j}i}(y)\Phi_{i\bar{j}}(y) = g^{\bar{j}i}g_{i\bar{j}} = n = g^{\bar{j}i}(x)\Phi_{i\bar{j}}(x)$, and also the notation (3.2), this can be rewritten as

(3.3)

$$\begin{aligned} F(x,y) &= g^{\bar{j}i}(y)\,[L_i(y,x) - L_i(y,y) + u_i(y)]\,[L_{\bar{j}}(x,y) - L_{\bar{j}}(y,y) + u_{\bar{j}}(y)] \\ &\quad + g^{\bar{j}i}(x)u_{i\bar{j}}(x) - g^{\bar{j}i}(x)\,[L_i(x,y) - L_i(x,x)]\,[L_{\bar{j}}(y,x) - L_{\bar{j}}(x,x)] \\ &= g^{\bar{j}i}(x)u_{i\bar{j}}(x) + g^{\bar{j}i}(y)u_i(y)u_{\bar{j}}(y) \\ &\quad + g^{\bar{j}i}(y)u_i(y)[L_{\bar{j}}(x,y) - L_{\bar{j}}(y,y)] + g^{\bar{j}i}(y)u_{\bar{j}}(y)\,[L_i(y,x) - L_i(y,y)] \\ &\quad + g^{\bar{j}i}(y)\,[L_i(y,x) - L_i(y,y)]\,[L_{\bar{j}}(x,y) - L_{\bar{j}}(y,y)] \\ &\quad - g^{\bar{j}i}(x)\,[L_i(x,y) - L_i(x,x)]\,[L_{\bar{j}}(y,x) - L_{\bar{j}}(x,x)]. \end{aligned}$$

So this should vanish identically.

Our strategy now will be to extract information from the behaviour near the diagonal. Specifically, we have, first of all

$$F(x,x) = g^{\bar{j}i}(x)u_{i\bar{j}}(x) + g^{\bar{j}i}(x)u_i(x)u_{\bar{j}}(x);$$

thus $g^{\bar{j}i}u_{i\bar{j}} + g^{\bar{j}i}u_iu_{\bar{j}} = 0$, or

$$\Delta u = -g^{\bar{j}i}u_iu_{\bar{j}}. \tag{3.4}$$

Next, let us evaluate $\partial_{x_a}F(x,y)$ at a point on the diagonal. Clearly, the second and the fourth summands in (3.3) are killed by the differentiation. The fifth summand becomes $g^{\bar{j}i}(y)[L_i(y,x) - L_i(y,y)]L_{\bar{j}a}(x,y)$, which also disappears since the middle term vanishes for $x = y$. Similarly, upon differentiating the sixth summand, using the Leibniz rule, we get three terms, each of them containing either $L_i(x,y) - L_i(x,x)$ or $L_{ia}(x,y) - L_{ia}(x,x)$, which both disappear for $x = y$. Thus the only contribution comes from the first and the third summands in (3.3); the former contributes $(g^{\bar{j}i}u_{i\bar{j}})_a = (\Delta u)_a$, the latter $g^{\bar{j}i}u_iL_{\bar{j}a}$. Thus

$$\partial_{x_a}F\,\Big|_{y=x} = (\Delta u)_a + g^{\bar{j}i}u_iL_{\bar{j}a}$$

(for simplicity of notation, we are again omitting the arguments x at g, u and (x,x) at L, respectively). Thus if F vanishes identically then necessarily

$$(\Delta u)_a = -g^{\bar{j}i}u_iL_{\bar{j}a}. \tag{3.5}$$

We continue by giving a similar treatment to the second derivative $\partial_{y_a}\partial_{x_b}F|_{y=x}$. The first, the second and the fourth summands in (3.3) then disappear, being killed by the differentiation. The third summand yields

$$\partial_{y_a}\big[g^{\bar{j}i}(y)u_i(y)L_{\bar{j}b}(x,y)\big]\Big|_{y=x} = (g^{\bar{j}i}u_i)_aL_{\bar{j}b}.$$

For the fifth summand, we get

$$\partial_{y_a}\big[g^{\bar{j}i}(y)\,\big(L_i(y,x) - L_i(y,y)\big)\,L_{\bar{j}b}(x,y)\big]\Big|_{y=x} = 0,$$

since, again, using the Leibniz rule we get terms containing always either $L_i(y,x)-L_i(y,y)$ or $L_{ia}(y,x)-L_{ia}(y,y)$, which both vanish for $x=y$. Similarly, there is no contribution from the sixth summand in (3.3). Thus $\partial_{y_a}\partial_{x_b}F|_{y=x} = (g^{\overline{j}i}u_i)_a L_{\overline{j}b}$; so if F vanishes identically, then necessarily

$$(g^{\overline{j}i}u_i)_a L_{\overline{j}b} = 0.$$

However, since the matrix $L_{\overline{j}b}$ is invertible, owing to the nondegeneracy hypothesis (Corollary 3), this means that

$$(g^{\overline{j}i}u_i)_a = 0. \tag{3.6}$$

Now, we have

$$\begin{aligned}
0 &= (\Delta u + g^{\overline{j}i}u_i u_{\overline{j}})_a \qquad \text{by (3.4)}\\
&= (\Delta u)_a + (g^{\overline{j}i}u_i)_a u_{\overline{j}} + g^{\overline{j}i}u_i u_{\overline{j}a} \qquad \text{by Leibniz}\\
&= -g^{\overline{j}i}u_i L_{\overline{j}a} + g^{\overline{j}i}u_i u_{\overline{j}a} \qquad \text{by (3.5) and (3.6)}\\
&= -g^{\overline{j}i}u_i \Phi_{\overline{j}a} \qquad \text{by (3.2)}\\
&= -g^{\overline{j}i}u_i g_{\overline{j}a} = -u_a.
\end{aligned}$$

Since u is real-valued, this also implies that $u_{\overline{a}} = \overline{\overline{u_a}} = \overline{u_a} = 0$ for all $a = 1,\dots,n$. Consequently, u must be constant, and so must be $m = e^u$. This proves the first claim in part (i) of the Main Theorem.

To prove the second claim in part (i), note that, by (1.4) and the reproducing property of $K(x,y)$,

$$B\mathbf{1} = \mathbf{1}$$

where $\mathbf{1}$ denotes the function constant one. In other words,

$$\int_\Omega \beta(x,y)\,d\mu(y) = 1, \qquad \forall x \in \Omega.$$

However, from the constancy of the function m just proved it follows that $\beta(x,y)$ is symmetric in x and y; thus also

$$\int_\Omega \beta(x,y)\,d\mu(x) = 1, \qquad \forall y \in \Omega.$$

By the classical Schur test (see e.g. [16, Theorem 5.2]), this implies that B is bounded on $L^2(\Omega, d\mu)$, with $\|B\| \le 1$ (i.e. even a contraction). The self-adjointness of B then follows from the symmetry of $\beta(x,y)$. This settles the second claim in (i).

To establish the last — and main — claim in (i), we return again to the derivatives of F at the diagonal, and compute this time the fifth order derivative

$$\partial_{x_a}\partial_{\overline{x}_b}\partial_{y_c}\partial_{\overline{y}_d}\partial_{x_e}F\big|_{y=x}.$$

Note that since we now know u to be constant, the formula (3.3) assumes the simpler form

$$(3.7)\qquad \begin{aligned} F(x,y) = g^{\overline{j}i}(y)\,[L_i(y,x)-L_i(y,y)]\,[L_{\overline{j}}(x,y)-L_{\overline{j}}(y,y)] \\ - g^{\overline{j}i}(x)\,[L_i(x,y)-L_i(x,x)]\,[L_{\overline{j}}(y,x)-L_{\overline{j}}(x,x)]. \end{aligned}$$

Consequently,

$$\begin{aligned} \partial_{x_a}\partial_{\overline{x}_b}\partial_{y_c}\partial_{\overline{y}_d}\partial_{x_e}F(x,y) &= \partial_{y_c}\partial_{\overline{y}_d}[g^{\overline{j}i}(y)\,L_{i\overline{b}}(y,x)\,L_{\overline{j}ae}(x,y)] \\ &\quad - \partial_{x_a}\partial_{\overline{x}_b}\partial_{x_e}[g^{\overline{j}i}(x)\,L_{i\overline{d}}(x,y)\,L_{\overline{j}c}(y,x)]. \end{aligned}$$

In view of the constancy of u, we also have by (3.2) $L_{i\overline{j}}(x,x)=\Phi_{i\overline{j}}(x)$ $(=g_{i\overline{j}}(x))$, etc. Using the Leibniz rule, we thus get

$$(3.8)\qquad \begin{aligned} \partial_{x_a}\partial_{\overline{x}_b}\partial_{y_c}\partial_{\overline{y}_d}\partial_{x_e}F\Big|_{y=x} &= (g^{\overline{j}i})_{c\overline{d}}\Phi_{i\overline{b}}\Phi_{\overline{j}ae}+(g^{\overline{j}i})_c\Phi_{i\overline{b}}\Phi_{\overline{j}ae\overline{d}} \\ &\quad + (g^{\overline{j}i})_{\overline{d}}\Phi_{i\overline{b}c}\Phi_{\overline{j}ae}+g^{\overline{j}i}\Phi_{i\overline{b}c}\Phi_{\overline{j}ae\overline{d}} \\ &\quad - (g^{\overline{j}i})_{a\overline{b}e}\Phi_{i\overline{d}}\Phi_{\overline{j}c}-(g^{\overline{j}i})_{a\overline{b}}\Phi_{i\overline{d}e}\Phi_{\overline{j}c} \\ &\quad - (g^{\overline{j}i})_{ae}\Phi_{i\overline{d}}\Phi_{\overline{j}c\overline{b}}-(g^{\overline{j}i})_{\overline{b}e}\Phi_{i\overline{d}a}\Phi_{\overline{j}c} \\ &\quad - (g^{\overline{j}i})_{\overline{b}}\Phi_{i\overline{d}ae}\Phi_{\overline{j}c}-(g^{\overline{j}i})_a\Phi_{i\overline{d}e}\Phi_{\overline{j}c\overline{b}} \\ &\quad - (g^{\overline{j}i})_e\Phi_{i\overline{d}a}\Phi_{\overline{j}c\overline{b}}-g^{\overline{j}i}\Phi_{i\overline{d}ae}\Phi_{\overline{j}c\overline{b}}. \end{aligned}$$

To evaluate this expression, it is convenient to pass to a different local coordinate system. Namely, fix for the moment some point $z_0\in\Omega$ and let ϕ be a biholomorphic map defined in some neighbourhood U of z_0 and fixing z_0. The potential $\Phi\circ\phi=:\widetilde{\Phi}$ then defines some metric $\widetilde{g}_{i\overline{j}}$ on $\widetilde{U}:=\phi(U)$; and since the definition of the Laplace-Beltrami operator is coordinate independent, the operators Δ and $\widetilde{\Delta}$ corresponding to $g_{i\overline{j}}$ and $\widetilde{g}_{i\overline{j}}$, respectively, satisfy

$$(3.9)\qquad \Delta(f\circ\phi)=(\widetilde{\Delta}f)\circ\phi \qquad \forall f\in C^\infty(\widetilde{U}).$$

If now L is a (quite arbitrary) function on $U\times U$, $\beta(x,y)=\exp[L(x,y)+L(y,x)-L(x,x)-L(y,y)]$, $F(x,y)=\frac{\Delta_y\beta(x,y)-\Delta_x\beta(x,y)}{\beta(x,y)}$, and $\widetilde{\beta}$ and $\widetilde{F}$ are similarly associated to $\widetilde{L}(x,y):=L(\phi^{-1}(x),\phi^{-1}(y))$, then (3.9) implies that

$$\widetilde{F}(x,y)=F(\phi^{-1}(x),\phi^{-1}(y)).$$

Thus F vanishes identically on $U\times U$ if and only if $\widetilde{F}$ does; and, consequently, the right-hand side of (3.8) will then vanish also if we put tildes over everything. (The fact that L is actually the logarithm of the reproducing kernel is not needed in this argument.)

Recall now that at any point z_0 on an arbitrary Kähler manifold, there exists a *geodesic* (other names: *normal, Bochner*) local coordinate system around it, in which the following equalities hold at z_0:

$$\begin{aligned}
&\widetilde{\Phi}_{i\bar{j}} \equiv \widetilde{g}_{i\bar{j}} = \delta_{ij}; \\
&\widetilde{\Phi}_{i\bar{j}k} \equiv (\widetilde{g}_{i\bar{j}})_k = 0; \qquad \widetilde{\Phi}_{i\bar{j}\bar{k}} \equiv (\widetilde{g}_{i\bar{j}})_{\bar{k}} = 0; \\
&\widetilde{\Phi}_{i\bar{j}k\bar{l}} = \widetilde{R}_{i\bar{j}k\bar{l}}, \qquad \text{the curvature tensor; and} \\
&\widetilde{\Phi}_{i\bar{j}k\bar{l}m} = \widetilde{R}_{i\bar{j}k\bar{l}/m}, \qquad \text{its covariant derivative.}
\end{aligned}$$

(See e.g. [14, Lemma 3.7.1]; or [4, Chapter VIII].) These equalities further imply that at z_0,

$$(\widetilde{g}^{\bar{j}i})_k = (\widetilde{g}^{\bar{j}i})_{\bar{k}} = 0 \qquad \text{and} \qquad (\widetilde{g}^{\bar{j}i})_{k\bar{l}m} = -\widetilde{R}^{\bar{j}i}{}_{\cdot\cdot\, k\bar{l}/m}.$$

Switching to these coordinates, all terms on the right-hand side of (3.8) therefore disappear except for the fifth one, which becomes

$$\widetilde{R}^{\bar{j}i}{}_{\cdot\cdot\, a\bar{b}/e}(z_0)\delta_{i\bar{d}}\delta_{\bar{j}c} = \widetilde{R}_{a\bar{b}c\bar{d}/e}(z_0).$$

Thus if F vanishes identically, then

$$\widetilde{R}_{a\bar{b}c\bar{d}/e} = 0.$$

Similarly $\widetilde{R}_{a\bar{b}c\bar{d}/\bar{e}} = 0$. Thus $\nabla\widetilde{R} = 0$, i.e. $\nabla R = 0$. By Proposition 1, part (a), $(\Omega, g_{i\bar{j}})$ is locally symmetric. This completes the proof.

4. Proof of Main Theorem, part (ii)

We are actually going to prove a little more: we show that if the function m is constant and $(\Omega, g_{i\bar{j}})$ is locally symmetric and complete, then it has already to be globally symmetric. Since the constancy of m and the local symmetry are guaranteed by part (i), the desired assertion will follow.

So assume that Ω is locally symmetric and complete and that m is constant. Let $\mathcal{X}$ be the universal cover of Ω, and $\pi : \mathcal{X} \to \Omega$ the covering map. Then $\mathcal{X}$ is locally symmetric, complete and simply connected; hence, by Proposition 1, part (b), it is a globally symmetric space.

Let us now recall several facts about Hermitian globally symmetric spaces. First of all, any such space is (biholomorphic to) a Cartesian product of irreducible ones. Further, irreducible Hermitian symmetric spaces are of three types: noncompact, compact and Euclidean.

An irreducible Hermitian symmetric space of *noncompact type* is biholomorphic to a bounded symmetric domain $D \subset \mathbf{C}^d$, which can be chosen to be circular (i.e. $z \in D$, $\theta \in \mathbf{R}$ imply $e^{i\theta}z \in D$) and convex, with the metric given by the potential $\Psi(z) = c \log K_D(z,z)$, where K_D is the ordinary Bergman kernel of D (i.e. with respect to the Lebesgue measure) and c is a positive constant. Note that the circularity of D implies that $K_D(z,0) = K_D(0,z) = 1/\operatorname{vol}(D)$, for all $z \in D$. The volume element $d\mu(z)$ then coincides with (a constant multiple of) $K_D(z,z)\,dz$; and, further, it is known that the measure $e^{-\alpha\Psi}\,d\mu$ is finite as soon as $\alpha c > 1 - \frac{1}{p}$, where the positive integer p is the so-called *genus* of D, and the reproducing kernel of the weighted Bergman space $L^2_{\rm hol}(D, e^{-\alpha\Psi}\,d\mu)$ is then equal to $c_\alpha K_D^{\alpha c}$ for some positive constant c_α. Note that in view of the boundedness of D, the finiteness of the measure $e^{-\alpha\Psi}\,d\mu$ implies that the last space $L^2_{\rm hol}$ contains all polynomials.

An irreducible Hermitian symmetric space of *Euclidean type* is biholomorphic to $\mathbf{C}^d$ equipped with the usual (Euclidean) metric, i.e. given by the potential $c\|z\|^2$ for some $c > 0$; that is, by the potential $\Psi(z) = c\log K_{\rm Fock}(z,z)$, where $K_{\rm Fock}$ is the reproducing kernel of the *Fock* (or *Segal-Bargmann*) space $L^2_{\rm hol}(\mathbf{C}^d, e^{-\|z\|^2}\pi^{-1}\,dz)$. The volume element $d\mu(z)$ is simply (a constant multiple of) the Lebesgue measure, and, again, it is well known (and easily verified) that for any $\alpha > 0$, the reproducing kernel of the space $L^2_{\rm hol}(\mathbf{C}^d, e^{-\alpha\Psi}\,d\mu)$ equals to $e^{\alpha c\langle x,y\rangle} = K_{\rm Fock}(x,y)^{\alpha c}$, up to a constant factor. Also, the last space again contains all polynomials.

Finally, an irreducible Hermitian symmetric space of *compact type* is a compact manifold D, which however admits a dense open subset D' biholomorphic to $\mathbf{C}^d$ (i.e. a local chart), such that $D \setminus D'$ has zero measure, and on D' the metric $g_{i\bar j}$ is given by the potential $\Psi(z) = -c\log K_{\widetilde D}(z,-z)$, where c is a positive constant and $K_{\widetilde D}$ is the ordinary (i.e. unweighted) Bergman kernel of a certain bounded symmetric domain $\widetilde D \subset \mathbf{C}^d$ (called the *dual* of D). The restriction of $d\mu$ to D' coincides with a constant multiple of $K_{\widetilde D}(z,-z)\,dz$, and for any $\alpha \geq 0$ such that $\alpha\tilde p c$ is an integer (where $\tilde p$ is the genus of $\widetilde D$), the reproducing kernel of the space $L^2_{\rm hol}(\mathbf{C}^d, e^{-\alpha\Psi}\,d\mu)$ coincides, up to a constant factor, with $K_{\widetilde D}(x,-y)^{-\alpha c}$. Also, the last space $L^2_{\rm hol}$ is finite-dimensional and consists of all polynomials of degree less than $N(\alpha)$, where $N(\alpha) \to \infty$ as $\alpha \to \infty$; in particular, it contains both the constants and the coordinate functions $z_1, \dots, z_d$ on $\mathbf{C}^d$ as soon as α is sufficiently large.

Combining the information above, we thus see that for any Hermitian globally symmetric space $\mathcal{X}$, there exists an open dense subset $\mathcal{X}' \subset \mathcal{X}$ (biholomorphic to $\mathbf{C}^d \times D \subset \mathbf{C}^n$ for some integer $d \geq 0$ and circular convex bounded symmetric domain $D \subset \mathbf{C}^{n-d}$), such that $\mathcal{X} \setminus \mathcal{X}'$ has zero measure, and

- on $\mathcal{X}'$ the metric is defined by the potential
$$\Psi(z) = c\log H(z,z)$$
where H is, up to a constant factor, the reproducing kernel of the space $L^2_{\rm hol}(\mathcal{X}', e^{-\Psi/c}\,d\mu)$ for some $c > 0$;
- the function $H(\,\cdot\,,0)$ is constant;
- for any integer $\alpha \geq 1$, the reproducing kernel of the weighted Bergman space $L^2_{\rm hol}(\mathcal{X}', e^{-\alpha\Psi/c}\,d\mu)$ coincides, up to a constant factor, with H^α;
- and, finally, if α is also sufficiently large, then the last space $L^2_{\rm hol}$ contains the constants as well as the coordinate functions $z_1, \dots, z_n$ on $\mathbf{C}^n$.

Example. The unit disc $\mathbf{D} \subset \mathbf{C}$ equipped with the Poincaré metric $g_{1\overline{1}}(z) = 2c(1-|z|^2)^{-2}$ is a symmetric space of noncompact type, with potential $\Psi(z) = -c\log[\pi(1-|z|^2)^2] = c\log K_{\mathbf{D}}(z,z)$; its genus is $p=2$, and for $\alpha c > 1-\frac{1}{p} = \frac{1}{2}$ the reproducing kernel of $L^2_{\mathrm{hol}}(\mathbf{D}, e^{-\alpha\Psi}\,d\mu) = L^2_{\mathrm{hol}}(\mathbf{D}, 2c\pi^{c\alpha}(1-|z|^2)^{2c\alpha-2}\,dz)$ is equal to $\frac{2c\alpha-1}{2c\pi^{c\alpha+1}}(1-x\overline{y})^{-2\alpha c} = \frac{2c\alpha-1}{2\pi c}\,K_{\mathbf{D}}(x,y)^{\alpha c}$.

The complex plane $\mathbf{C}$ with the Euclidean metric $g_{1\overline{1}}(z) = c$ is a symmetric space of Euclidean type, with potential $\Psi(z) = c|z|^2 = c\log K_{\mathrm{Fock}}(z,z)$; for any $\alpha > 0$, the reproducing kernel of $L^2_{\mathrm{hol}}(\mathbf{C}, e^{-\alpha\Psi}\,d\mu) = L^2_{\mathrm{hol}}(\mathbf{C}, ce^{-\alpha c|z|^2}\,dz)$ is $\frac{\alpha}{\pi}e^{\alpha c\langle x,y\rangle} = \frac{\alpha}{\pi}K_{\mathrm{Fock}}(x,y)^{\alpha c}$.

The Gauss sphere $\mathbf{G} = \mathbf{C}\cup\{\infty\}$, equipped with the invariant metric, whose restriction to $D' = \mathbf{C}$ coincides with $g_{1\overline{1}}(z) = 2c(1+|z|^2)^{-2}$, is a symmetric space of compact type, with potential on D' given by $\Psi(z) = c\log[\pi(1+|z|^2)^2] = -c\log K_{\mathbf{D}}(z,-z)$; the genus of its dual $\mathbf{D}$ is $\tilde{p} = 2$, and for any $\alpha \geq 0$ such that $\alpha\tilde{p}c = 2\alpha c$ is an integer, the reproducing kernel of the space $L^2_{\mathrm{hol}}(\mathbf{C}, e^{-\alpha\Psi}\,d\mu) = L^2_{\mathrm{hol}}(\mathbf{C}, \frac{2c}{\pi^{\alpha c}}(1+|z|^2)^{-2\alpha c-2}\,dz)$ is equal to $\frac{2\alpha c+1}{2c\pi^{1-\alpha c}}(1+x\overline{y})^{2\alpha c} = \frac{2\alpha c+1}{2\pi c}K_{\mathbf{D}}(x,-y)^{-\alpha c}$. Further, the last space L^2_{hol} consists of all polynomials of degree not exceeding $2\alpha c$; in particular, it contains the constants for any $\alpha \geq 0$, and the coordinate function z as soon as $2\alpha c \geq 1$.

Finally, the product space $\mathcal{X} = \mathbf{D}\times\mathbf{C}\times\mathbf{G}$, with the metric given on $\mathcal{X}' = \mathbf{D}\times\mathbf{C}\times\mathbf{C} \subset \mathcal{X}$ by $g_{1\overline{1}}(z) = 2c_1(1-|z_1|^2)^{-2}$, $g_{2\overline{2}}(z) = c_2$, $g_{3\overline{3}}(z) = 2c_3(1+|z_3|^2)^{-2}$, and $g_{i\overline{j}} = 0$ for $i \neq j$, is a reducible symmetric space admitting the potential $\Psi(z) = c\log H(z,z)$, where

$$H(z,z) = \left[\frac{1}{\pi^{c_1}(1-|z_1|^2)^{2c_1}}\cdot e^{c_2|z_2|^2}\cdot\frac{1}{\pi^{c_3}(1+|z_3|^2)^{2c_3}}\right]^{1/c}$$

is, up to the constant factor $\frac{2\frac{c_1}{c}-1}{2\pi c_1}\cdot\frac{1}{\pi c}\cdot\frac{2\frac{c_3}{c}+1}{2\pi c_3}$, the reproducing kernel of the space $L^2_{\mathrm{hol}}(\mathcal{X}', e^{-\Psi/c}\,d\mu)$, whenever $\frac{c_1}{c} > 1-\frac{1}{2} = \frac{1}{2}$ and $\frac{2c_3}{c}$ is an integer, i.e. for $c = \frac{2c_3}{m}$ with any integer $m > \frac{c_3}{c_1}$. Further, for any integer $\alpha \geq 1$, the reproducing kernel of $L^2_{\mathrm{hol}}(\mathcal{X}', e^{-\alpha\Psi/c}\,d\mu)$ coincides, up to a similar constant factor as above, with $H(x,y)^\alpha$; and the last space L^2_{hol} contains all polynomials on $\mathbf{C}^3$ of degree not exceeding $2\alpha c_3/c = \alpha m$. □

In particular, the last conclusion applies to our covering space $\mathcal{X}$ of Ω. Since the covering map π is a local isometry, the potentials Ψ and Φ of the spaces $\mathcal{X}$ and Ω must be related by $\pi^*(\partial\overline{\partial}\Phi) = \partial\overline{\partial}\Psi$; so

$$c\log H(x,x) = \Phi(\pi x) + F(x) + \overline{F(x)} \qquad \forall x \in \mathcal{X}'$$

for some holomorphic function F on $\mathcal{X}'$.

On the other hand, if m is constant, then

$$\Phi(z) = \log K(z,z) + \mathrm{const.}$$

Absorbing the last constant into F, we thus see that

$$H(x,x)^c = K(\pi x,\pi x)\,e^{F(x)+\overline{F(x)}}.$$

Since a function of two variables x, y which is holomorphic in x and $\overline{y}$ is uniquely determined by its restriction to the diagonal $x = y$ (see e.g. [3, Proposition II.4.7]), the last equality implies that even

$$H(x,y)^c = K(\pi x, \pi y) e^{F(x)+\overline{F(y)}} \qquad \forall x, y \in \mathcal{X}'. \tag{4.1}$$

(Note that since $\mathcal{X}'$ is simply connected, there is no problem with the definition of the c-th power.)

Let now x, x' be any two points of $\mathcal{X}'$ such that $\pi x = \pi x'$. Then (4.1) implies that

$$H(x,y)^c e^{-F(x)} = H(x',y)^c e^{-F(x')} \qquad \forall y \in \mathcal{X}'.$$

But we know that $H(\cdot, 0)$ is constant (and nonzero); thus taking $y = 0$ yields $e^{-F(x)} = e^{-F(x')}$. Consequently,

$$H(x,y)^c = H(x',y)^c \qquad \forall y \in \mathcal{X}'.$$

Since H^α is (up to a constant factor) the reproducing kernel of $L^2_{\text{hol}}(\mathcal{X}', e^{-\alpha\Psi/c} d\mu_{\mathcal{X}})$, for any integer $\alpha \geq 1$, it follows that

$$f(x) = \epsilon f(x') \qquad \forall f \in L^2_{\text{hol}}(\mathcal{X}', e^{-\alpha\Psi/c}\, d\mu_{\mathcal{X}})$$

for some unimodular number ϵ (not depending on f). However, we know that if α is taken large enough, then the last space contains the constants as well as all the coordinate functions. Thus necessarily $\epsilon = 1$ and $x = x'$.

In other words, we have proved that π is injective on the open dense subset $\mathcal{X}'$ of $\mathcal{X}$. Since π is a covering map, it follows that π is a biholomorphism and $\Omega \cong \mathcal{X}$. Thus Ω is globally symmetric, q.e.d.

5. Concluding remarks

5.1. Manifolds

Although we have so far assumed that Ω is a domain and $g_{i\overline{j}}$ a Kähler metric for which the potential Φ exists globally, everything can easily be adapted to arbitrary Kähler manifolds $(\Omega, g_{i\overline{j}})$, as long as they satisfy an appropriate *integrality condition.* Namely, let $\{U_\alpha\}$ be a covering of Ω by contractible local charts; then on each U_α there exists a local potential Φ_α for $g_{i\overline{j}}$ (by the Kählerness condition). On any nonempty intersection $U_\alpha \cap U_\beta$ of two charts, it follows from $\partial\overline{\partial}\Phi_\alpha = [g_{i\overline{j}}] = \partial\overline{\partial}\Phi_\beta$ that $\Phi_\alpha = \Phi_\beta + \operatorname{Re} f_{\alpha\beta}$ for some holomorphic function $f_{\alpha\beta}$. Hence $e^{-\Phi_\alpha} = e^{-\Phi_\beta} \cdot |e^{f_{\alpha\beta}}|^{-2}$. Consequently, if the functions $e^{f_{\alpha\beta}}$ satisfy the cocycle condition

$$e^{f_{\alpha\beta}} e^{f_{\beta\gamma}} = e^{f_{\alpha\gamma}},$$

that is,

$$f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 2\pi i n_{\alpha\beta\gamma} \qquad \text{for some } n_{\alpha\beta\gamma} \in \mathbf{Z},$$

whenever $U_\alpha \cap U_\beta \cap U_\gamma$ is nonempty, then the local metric coefficients $e^{-\Phi_\alpha}$ on U_α can be glued together into a holomorphic Hermitian line bundle $\mathcal{L}$ over Ω. It is known that this happens if and only if the cohomology class $\frac{1}{2\pi i}[g_{i\bar{j}}]$ determined by $g_{i\bar{j}}$ in $H^2(\Omega, \mathbf{R})$ is integral, i.e. belongs in fact to $H^2(\Omega, \mathbf{Z})$:

$$\frac{1}{2\pi i}[g_{i\bar{j}}] \in H^2(\Omega, \mathbf{Z}). \tag{5.1}$$

In that case, one can consider, in place of $L^2_{\mathrm{hol}}(\Omega, e^{-\Phi}\, d\mu)$, the space $L^2_{\mathrm{hol}}(\mathcal{L}, d\mu)$ of all holomorphic sections of $\mathcal{L}$ square-integrable with respect to $d\mu$; and in place of the weighted Bergman kernel $K(x,y)$ the reproducing kernel of $L^2_{\mathrm{hol}}(\mathcal{L}, d\mu)$, which is a holomorphic section of the product bundle $\mathcal{L} \times \overline{\mathcal{L}}$ over $\Omega \times \Omega$, with $\overline{\mathcal{L}}$ being the complex conjugate of $\mathcal{L}$ (i.e. the line bundle with transition functions $e^{\overline{f_{\alpha\beta}}}$). The Berezin transform, defined by an obvious analogue of the formula (1.4), turns out to be defined — due to cancellation of the corresponding transition functions in the analogue of the expression $|K(x,y)|^2 K(x,x)^{-1} e^{-\Phi(y)}$ — not on sections of any bundle, but again on honest functions on Ω. Further details can be found e.g. in Peetre [26].

Thus it again makes sense to speak of the commutativity of B and Δ on $C_0^\infty(\Omega)$, and the whole Main Theorem extends to this setting.

Theorem (Main Theorem for manifolds). *Let* $(\Omega, g_{i\bar{j}})$ *be a Kähler manifold satisfying the integrality condition* (5.1) *and* B *the associated Berezin transform. Then*:

(i) *if* $B\Delta = \Delta B$ *on* $C_0^\infty(\Omega)$, *then the function* m (*defined by* (1.7) *in any local chart*) *is constant,* B *is a bounded self-adjoint operator on* $L^2(\Omega, d\mu)$, *and* $(\Omega, g_{i\bar{j}})$ *is locally symmetric*;

(ii) *if in addition the metric* $g_{i\bar{j}}$ *is complete, then* $(\Omega, g_{i\bar{j}})$ *is a Hermitian globally symmetric space.*

Proof. For part (i), the same proof still works, without any need for modifications, since all our arguments there were in fact of purely local nature. Thus we only need to show that the argument from Section 4 can be extended to the situation when the potential exists only locally. This is easily done by observing that the function $K(x,y)e^{F(x)+\overline{F}(y)}$ is globally defined (possibly after adding to the F's some purely imaginary constants in each local chart), even if $K(x,y)$ and the potential are not; the argument then applies without changes.

In more detail, let $\{U_\alpha\}_\alpha$ be a covering of Ω by contractible local coordinate charts; and for each α, let $\{V_{\alpha j}\}_j$ be the connected components of $\pi^{-1}U_\alpha \cap \mathcal{X}'$. Upon breaking some $V_{\alpha j}$ into smaller (slightly overlapping) pieces if necessary, we may also assume that each $V_{\alpha j}$ is contractible. Finally let, as above, Φ_α and $f_{\alpha\beta}$ be the local potentials and the transition functions for $\mathcal{L}$, respectively. Then there again exist functions $F_{\alpha j}$ holomorphic on $V_{\alpha j}$ such that

$$c \log H(x,x) = \Phi_\alpha(\pi x) + F_{\alpha j}(x) + \overline{F_{\alpha j}(x)} \qquad \forall x \in V_{\alpha j}. \tag{5.2}$$

If $x \in V_{\alpha j} \cap V_{\beta k}$, it follows that

$$|e^{F_{\beta k}(x)-F_{\alpha j}(x)-f_{\alpha\beta}(\pi x)}|^2 = 1.$$

Thus

$$F_{\beta k}(x) - F_{\alpha j}(x) - f_{\alpha\beta}(\pi x) = c_{\alpha j,\beta k} \tag{5.3}$$

for some purely imaginary constants $c_{\alpha j,\beta k}$. If in addition $x \in V_{\gamma l}$, then it follows from the integrality condition $e^{f_{\alpha\beta}} e^{f_{\beta\gamma}} e^{f_{\gamma\alpha}} = 1$ that $e^{c_{\alpha j,\beta k}} e^{c_{\beta k,\gamma l}} e^{c_{\gamma l,\alpha j}} = 1$. Since $\mathcal{X}'$ is contractible, there exist numbers $d_{\alpha j}$ such that

$$e^{c_{\alpha j,\beta k}} = \frac{d_{\alpha j}}{d_{\beta k}}.$$

The fact that $c_{\alpha j,\beta k}$ are purely imaginary implies that $|d_{\alpha j}| = |d_{\beta k}|$, so replacing $d_{\alpha j}$ by $d_{\alpha j}/|d_{\alpha j}|$ if necessary we can assume that

$$|d_{\alpha j}| = 1 \qquad \forall \alpha, j.$$

Introducing the (nonvanishing, holomorphic) functions

$$G_{\alpha j}(x) := d_{\alpha j} e^{F_{\alpha j}(x)},$$

we can therefore rewrite (5.2) as

$$H(x,x)^c = e^{\Phi_\alpha(x)} |G_{\alpha j}(x)|^2 \qquad \forall x \in V_{\alpha j}. \tag{5.4}$$

Recall now that the reproducing kernel $K_{\alpha z}$ in a local chart U_α at a point $z \in U_\alpha$ is defined by the requirement that

$$\phi_\alpha(z) = \langle \phi, K_{\alpha z} \rangle$$

for all square-integrable holomorphic sections ϕ. Since the values of a section ϕ on the intersection of two local charts are related by $\phi_\alpha = \phi_\beta e^{f_{\alpha\beta}}$, necessarily

$$K_{\alpha z} = K_{\beta z} e^{\overline{f_{\alpha\beta}(z)}} \qquad \forall z \in U_\alpha \cap U_\beta.$$

Thus if $x \in V_{\alpha j} \cap V_{\beta k}$ and $y \in V_{\gamma l} \cap V_{\delta m}$, then

$$\begin{aligned}
&\langle K_{\gamma,\pi y}, K_{\alpha,\pi x} \rangle \, e^{F_{\alpha j}(x) + \overline{F_{\gamma l}(y)}} \\
&\qquad = \langle K_{\delta,\pi y}, K_{\beta,\pi x} \rangle \, e^{F_{\alpha j}(x) + f_{\alpha\beta}(\pi x) + \overline{F_{\gamma l}(y) + f_{\gamma\delta}(\pi y)}} \\
&\qquad = \langle K_{\delta,\pi y}, K_{\beta,\pi x} \rangle \, e^{F_{\beta k}(x) - c_{\alpha j,\beta k} + \overline{F_{\delta m}(y) - c_{\gamma l,\delta m}}} \quad \text{by (5.3).}
\end{aligned}$$

This means that

$$\langle K_{\gamma,\pi y}, K_{\alpha,\pi x} \rangle \, G_{\alpha j}(x) \overline{G_{\gamma l}(y)} = \langle K_{\delta,\pi y}, K_{\beta,\pi x} \rangle \, G_{\beta k}(x) \overline{G_{\delta m}(y)} =: L(x,y)$$

is a globally defined function on $\mathcal{X}' \times \mathcal{X}'$, holomorphic in x and $\overline{y}$.

Since, finally, $e^{-\Phi_\alpha(\pi x)}\langle K_{\alpha,\pi x}, K_{\alpha,\pi x}\rangle = m(\pi x) = \epsilon$, a constant, by hypothesis, we see from (5.4) that

$$H(x,x)^c = \frac{1}{\epsilon} L(x,x) \qquad \forall x \in \mathcal{X}'.$$

Thus by the uniqueness principle

$$H(x,y)^c = \frac{1}{\epsilon} L(x,y) \qquad \forall x,y \in \mathcal{X}'.$$

Let now $x \in V_{\alpha j}$, $x' \in V_{\alpha n}$ be two points in $\mathcal{X}'$ such that $\pi x = \pi x'$. If $y \in V_{\gamma l}$, then from

$$\langle K_{\gamma,\pi y}, K_{\alpha,\pi x}\rangle = \langle K_{\gamma,\pi y}, K_{\alpha,\pi x'}\rangle$$

it follows that

$$L(x,y)G_{\alpha j}(x)^{-1}\overline{G_{\gamma l}(y)}^{-1} = L(x',y)G_{\alpha n}(x')^{-1}\overline{G_{\gamma l}(y)}^{-1},$$

so

$$H(x,y)^c G_{\alpha j}(x)^{-1} = H(x',y)^c G_{\alpha n}(x')^{-1} \qquad \forall y \in \mathcal{X}'.$$

Taking $y = 0$ gives $G_{\alpha j}(x)^{-1} = G_{\alpha n}(x')^{-1}$; so even $H(x,y)^c = H(x',y)^c$ for all y, and the rest of the argument is the same as at the end of Section 4. □

We remark that the integrality condition (5.1) is well known also in geometric quantization, under the name of *prequantization condition*; see e.g. [20]. It is automatically fulfilled for any bounded homogeneous domain in $\mathbf{C}^n$, since such domains are biholomorphic to a Siegel domain (see e.g. [29]), hence contractible, and thus $H^2(\Omega, \mathbf{R}) = 0$. On the other hand, for a compact symmetric space, for instance, (5.1) need not be satisfied in general, but can always be achieved upon replacing the metric $g_{i\bar{j}}$ by its multiple $N\nu g_{i\bar{j}}$, where N may be any positive integer and ν is a certain positive real number.

We also note that the curvature of the line bundle $\mathcal{L}$, as can readily be seen from the construction, coincides — up to a constant factor, involving π and i, depending on the conventions used — with the Kähler form $g^{\bar{j}i}\, dz_i \wedge d\bar{z}_j$. We may thus equivalently state our main theorem also solely in terms of line bundles:

Theorem (Main Theorem in terms of line bundles). *Let $\mathcal{L}$ be a holomorphic Hermitian line bundle over a complex manifold Ω whose curvature form $\omega_{\mathcal{L}}$ is positive* (*i.e. a positive line bundle*)*. Let $g_{i\bar{j}}$ be the Kähler metric on Ω determined by $\omega_{\mathcal{L}}$, $d\mu = (\omega_{\mathcal{L}})^n$ the associated volume element, $L^2_{hol}(\mathcal{L}, d\mu)$ the space of all square-integrable holomorphic sections of $\mathcal{L}$, K its reproducing kernel, and B the corresponding Berezin transform. Then the assertions* (i) *and* (ii) *above hold.*

5.2. Nondegeneracy

The simplest example of a Kähler manifold which is not nondegenerate is the unit disc $\mathbf{D}$ with the invariant metric

$$g_{1\bar{1}}(z) = \frac{\nu}{(1-|z|^2)^2}$$

for $0 < \nu \leq 1$. Indeed, in that case $\Phi(z) = \log(1-|z|^2)^{-\nu}$, and thus

$$L^2_{\mathrm{hol}}(\Omega, e^{-\Phi}\, d\mu) = L^2_{\mathrm{hol}}(\mathbf{D}, (1-|z|^2)^{\nu-2}),$$

which contains only the zero function if $\nu \leq 1$.

This suggests the following conjecture, reminiscent perhaps of the relationship between positive and very ample line bundles in Kodaira's embedding theorem (see e.g. [36, Section 8.3]).

Conjecture. *For any Kähler manifold* $(\Omega, g_{i\bar{j}})$ *satisfying the integrality condition* (5.1)*, there exists* $\nu_0 \in \mathbf{R}$ *such that for any integer* $\nu \geq \nu_0$*, the rescaled metric* $\nu g_{i\bar{j}}$ *is nondegenerate.*

We do not know if the Main Theorem can be extended also to degenerate manifolds. Our proof of the constancy of the function m in part (i) then certainly breaks down: an example of a Kähler potential $\Phi(x)$ and a function $L(x,y)$ such that the function $F(x,y) = (\Delta_y\beta(x,y) - \Delta_x\beta(x,y))/\beta(x,y)$, with $\beta(x,y)$ given by (3.1), vanishes identically but $u(z) = \log m(z) = L(z,z) - \Phi(z)$ is nonconstant, is

$$\begin{aligned} \Phi &= -\log(\operatorname{Re} h), \\ L &= 0, \end{aligned}$$

on an arbitrary domain $\Omega \subset \mathbf{C}$ and for any holomorphic function h on Ω with positive real part.

5.3. Boundedness of the Berezin transform

If the function m is not constant, then the Schur test used in Section 3 and the property $B\mathbf{1} = \mathbf{1}$ imply that B will be bounded on $L^2(\Omega, d\mu)$ whenever $B^*\mathbf{1} \leq c\mathbf{1}$, i.e. whenever

$$B\frac{1}{m} \leq \frac{c}{m}$$

for some finite constant c. This, in turn (in view of $B\mathbf{1} = \mathbf{1}$), will certainly be the case whenever both $\frac{1}{m}$ and m are bounded; that is, whenever

$$K(z,z) \asymp e^{\Phi(z)} \qquad \text{as } z \to \partial\Omega.$$

Using Fefferman's theorem on the boundary behaviour of the Bergman kernel [13], the last property can be shown to hold, for instance, whenever Ω is a smoothly bounded strictly pseudoconvex domain in $\mathbf{C}^n$ and $e^{-\Phi} \asymp \operatorname{dist}(\cdot, \partial\Omega)^c$

for some integer $c > n$. (See e.g. [11, Corollary 6].) The boundedness of B on $L^2(\Omega, d\mu)$ in general seems to be an interesting open problem.

5.4. Balanced metrics

There are various canonical metrics associated to bounded domains in $\mathbf{C}^n$: the Bergman metric [2], the Cheng-Yau metric (the solution of the Monge-Ampere equation — only on pseudoconvex domains) [6], [23], the Wu metrics [33], etc. We conclude by mentioning an open problem, concerning another canonical metric, inspired by the constancy of our function m.

Problem. *On a given contractible domain Ω, does there exist a Kähler metric $g_{i\bar{j}}$ for which $m \equiv$const.?*

In other words: for any given Kähler metric $g_{i\bar{j}}$ on Ω, let Φ be its potential, $d\mu = \det[g_{i\bar{j}}(x)]\,dx$ the volume element, and consider the measure $e^{-\Phi}\,d\mu$, the weighted Bergman space $L^2_{\text{hol}}(\Omega, e^{-\Phi}\,d\mu)$, and its reproducing kernel $K(x,y)$; and let $g^*_{i\bar{j}}(z) := \partial^2 \log K(z,z)/\partial z_i \partial \overline{z}_j$. It is again readily seen that $g^*_{i\bar{j}}$ is independent of the choice of the potential Φ, and thus is uniquely determined by the original metric $g_{i\bar{j}}$. Can $g_{i\bar{j}}$ be chosen so that $g^*_{i\bar{j}} = g_{i\bar{j}}$?

The problem makes, of course, sense also on complex manifolds Ω, subject to the integrality condition (5.1). Note that the analogue of the function $m(x) = e^{-\Phi(x)}K(x,x)$ can then be simply written as

$$m(x) = \sum_j \|\phi_j(x)\|_x^2, \tag{5.5}$$

where $\{\phi_j\}$ is an arbitrary orthonormal basis of $L^2_{\text{hol}}(\mathcal{L}, d\mu)$, and $\|\cdot\|_x$ stands for the norm in the fiber $\mathcal{L}_x$. This function has appeared in the literature under different names, the earliest ones being probably the *η-function* of Rawnsley [30] (later renamed to *ϵ-function* [5]) for arbitrary Kähler manifolds, or the *distortion function* of Kempf [19] and Ji [18] for the special case of Abelian varieties, and of Zhang [35] for complex projective varieties; and the metrics for which m is constant are called *critical* [35] or *balanced* [8]. Thus a more general formulation of our problem is as follows:

Problem. *Given a complex manifold Ω, does there exist a Kähler metric $g_{i\bar{j}}$ on Ω which satisfies the integrality condition* (5.1) *and is balanced?*

We remark that balanced metrics are of significance in the problem of existence of constant curvature metrics, and in some questions concerning (semi)stability of projective algebraic varieties; see e.g. [22].

Our Main Theorem implies that if B commutes with Δ, then the metric has to be balanced. Currently, the only noncompact manifolds on which complete balanced metrics are known to exist are the symmetric spaces, where the Bergman metric is balanced; in that case, this metric is not unique, since $cg_{i\bar{j}}$ is then also balanced, for any constant $c \geq 1$. In the nonsymmetric setting, the problem of existence of balanced metrics seems to be open even for domains of dimension 1 (i.e. in **C**). For compact manifolds, the existence of balanced metrics has been studied in a recent paper by Donaldson [8].

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