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Lacunary A_p-summable sequence spaces defined by Orlicz functions*

By

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Abstract

In this paper we introduce some new sequence spaces combining a lacunary sequence, an infinite matrix, a bounded sequence and an Orlicz function. We discuss some topological properties and establish some inclusion relations between these spaces. It is also shown that if a sequence is lacunary A_p-convergent with respect to an Orlicz function then it is lacunary strongly $S^{\theta}(A)$-statistically convergent.

1. Introduction

Let w be the spaces of all real or complex sequence $x = (x_k)$. ℓ_∞ and c denote the Banach spaces of real bounded and convergent sequences $x = (x_i)$ normed by $\|x\| = \sup_i |x_i|$, respectively.

A sequence of positive integers $\theta = (k_r)$ is called "lacunary" if $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $u_r = k_r/k_{r-1}$. The space of lacunary strongly convergent sequence N_θ was defined by Freedman et al [7] as:

$$N_\theta = \left\{ x : \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} |x_i - l| = 0, \text{ for some, } l \right\}$$

Lindentrauss and Tzafirir [11] used the idea of Orlicz function to defined the following sequence spaces.

$$l_M = \left\{ x : \sum_{i=1}^{\infty} M\left(\frac{|x_i|}{\rho}\right) < \infty, \ \rho > 0 \right\}$$

which is called an Orlicz sequence spaces l_M is a Banach space with the norm,

$$\|x\| = \left\{ \inf \rho > 0 : \sum_{i=1}^{\infty} M\left(\frac{|x_i|}{\rho}\right) \le 1 \right\}.$$

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An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

It is well known that if M is a convex function and $M(0) = 0$; then $M(tx) \leq tM(x)$ for all t with $0 < t < 1$.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u, if there exists a constant $L > 0$ such that $M(2u) \leq LM(u), u \geq 0$.

It is also easy to see that always $L > 2$. The Δ_2-condition equivalent to the satisfaction of inequality $M(Du) \leq LDM(u)$ for all values of u and for all $D > 1$ (see, Krasnoselskii and Rutitsky [10]).

In the later stage different Orlicz sequence spaces were introduced and studied by Parashar and Choudhary [12], Bhardwaj and Singh [3], Bilgin [2], Güngör et al [9], and many others.

The main purpose of this paper is to give some new sequence spaces combining the concept of an Orlicz function and lacunary convergence by using an Infinite matrix and a bounded sequence. Also we will investigate inclusion relations between these new spaces:

The following well known inequality will be used troughout the paper;

$$|a_i + b_i|^{p_i} \leq T(|a_i|^{p_i} + |b_i|^{p_i}) \tag{1}$$

where a_i and b_i are complex numbers, T $= \max(1, 2^{H-1})$, and H $= \sup p_i < \infty$.

Let $A = (a_{ki})$ be an infinite matrix of complex numbers. We write $Ax = (A_k(x))$ if $A_k(x) = \sum_{i=1}^{\infty} a_{ki}x_i$ converges for each k.

We now introduce the generalizations of the spaces of lacunary strongly convergent sequences.

Let M be an Orlicz function, $A = (a_{ki})$ be an infinite matrix of complex numbers, and $p = (p_k)$ be a bounded sequence of positive real numbers such that $0 < h = \inf p_i \leq p_i \leq \sup p_i = H < \infty$. We define the following sequence spaces:

$$N_\theta^0(A, M, p) = \left\{ \text{x} : \lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} M\left(\frac{|A_k(x)|}{\rho}\right)^{p_k} = 0, \text{ for some } \rho > 0 \right\}$$

$$N_\theta(A, M, p) = \left\{ \text{x} : \lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} M\left(\frac{|A_k(x) - s|}{\rho}\right)^{p_k} = 0, \right.$$
$$\left. \text{for some } s \text{ and } \rho > 0 \right\}$$

$$N_\theta^\infty(A, M, p) = \left\{ \text{x} : \sup_r h_r^{-1} \sum_{k\in I_r} M\left(\frac{|A_k(x)|}{\rho}\right)^{p_k} < \infty, \text{ for some } \rho > 0 \right\},$$

where for convenince, we put $M(\frac{|A_k(x)|}{\rho})^{p_k}$ instead of $[M(\frac{|A_k(x)|}{\rho})]^{p_k}$. If x $\in N_\theta(A, M, p)$, we say that x is lacunary A_p-convergence to s with respect to the Orlicz function M.

In the case $M(x) = x$, then we write the spaces $N_\theta^0(A,p)$, $N_\theta(A,p)$, and $N_\theta^\infty(A,p)$ in place of the spaces $N_\theta^0(A,M,p)$, $N_\theta(A,M,p)$, and $N_\theta^\infty(A,M,p)$, respectively.

If $a_{ki} = \left\{ \begin{array}{ll} 1, & k=i \\ -1, & k=i+1 \end{array} \right\}$, the spaces $N_\theta^0(A,M,p)$, $N_\theta(A,M,p)$, and $N_\theta^\infty(A,M,p)$ reduce to $w_0^\theta(M,p)_\triangle$, $w^\theta(M,p)_\triangle$, and $w_\infty^\theta(M,p)_\triangle$ (See Bilgin [2]).

If A = I, $N_\theta^0(A,M,p)$, $N_\theta(A,M,p)$, and $N_\theta^\infty(A,M,p)$ reduce to $[N_\theta,M,p]_0$, $[N_\theta,M,p]$, and $[N_\theta,M,p]_\infty$ (See Bhardwaj and Singh [3]).

2. Inclusion theorems

In this section we examine some topological properties of $N_\theta^0(A,M,p)$, $N_\theta(A,M,p)$, and $N_\theta^\infty(A,M,p)$ spaces and investigate some inclusion relations between these spaces.

Theorem 2.1. *The spaces $N_\theta^0(A,M,p)$, $N_\theta(A,M,p)$, and $N_\theta^\infty(A,M,p)$ are linear spaces over C (the set of complex numbers).*

Proof. We just prove only for $N_\theta^0(A,M,p)$. The others follow similar lines. Let $x,y \in N_\theta^0(A,M,p)$ and $\alpha,\beta \in C$. Then there exist some positive numbers ρ_1 and ρ_2 such that

$$\lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} M\left(\frac{|A_k(x)|}{\rho_1}\right)^{p_k} = 0 \text{ and } \lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} M\left(\frac{|A_k(y)|}{\rho_2}\right)^{p_k} = 0$$

Let $\rho_3 = \max(2\,|\alpha|\,\rho_1, 2\,|\beta|\,\rho_2)$. Since M is non-decreasing and convex and A is a linear transformation, by using inequality (1), we have

$$\begin{aligned} h_r^{-1} \sum_{k\in I_r} M\left(\frac{|A_k(\alpha x+\beta y)|}{\rho_3}\right)^{p_k} &\le h_r^{-1} \sum_{k\in I_r} M\left(\frac{\alpha\,|A_k(x)|}{\rho_3}\right) + \frac{\beta\,|A_k(y)|}{\rho_3}\Bigg)^{p_k} \\ &\le T\left[h_r^{-1} \sum_{k\in I_r} M\left(\frac{|A_k(x)|}{\rho_1}\right)^{p_k} + h_r^{-1} \sum_{k\in I_r} M\left(\frac{|A_k(y)|^{p_k}}{\rho_2}\right)\right] \to 0 \end{aligned}$$

as $r \to \infty$. This proves that $N_\theta^0(A,M,p)$ is linear. $\square$

Theorem 2.2. *Let $Ax \to \infty$ as $x \to \infty$, then the space $N_\theta^0(A,M,p)$ is a paranormed space (not totally paranormed), paranormed by*

$$g(x) = \inf\left\{\sigma^{p_r/H} : \left(h_r^{-1} \sum_{k\in I_r} M\left(\frac{|A_k(x)|}{\rho}\right)^{p_k}\right)^{1/H} \le 1; r = 1,2,3,\ldots\right\}$$

The proof of Theorem 2.2.used the ideas similar to those used in proving Thorem 2.2 of Bhardwaj and Singh [3].

Theorem 2.3. *Let M be an Orlicz function which satisfies the Δ_2-condition. Then $N_\theta^0(A,M,p) \subset N_\theta(A,M,p) \subset N_\theta^\infty(A,M,p)$ and the inclusions are strict.*

Proof. The first inclusion is obvious. Thus, we need to prove only $N_\theta(A,M,p) \subset N_\theta^\infty(A,M,p)$. Now let $x \in N_\theta(A,M,p)$. Then there exists some positive $\rho_1 > 0$ such that

$$\lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} M\left(\frac{|A_k(x)-s|}{\rho_1}\right)^{p_k} = 0.$$

Define $\rho = 2\rho_1$. Since M is non decreasing and convex, by using inequality (1), we have

$$\begin{aligned}\sup_r h_r^{-1} \sum_{k\in I_r} M\left(\frac{|A_k(x)|}{\rho}\right)^{p_k} &= \sup_r h_r^{-1} \sum_{k\in I_r} M\left(\frac{|A_k(x)-s+s|}{\rho}\right)^{p_k} \\ &\le T\left\{\sup_r h_r^{-1} \sum_{k\in I_r} M\left(\frac{|A_k(x)-s|}{\rho_1}\right)^{p_k} + \sup_r h_r^{-1} \sum_{k\in I_r} M\left(\frac{|s|}{\rho_1}\right)^{p_k}\right\} \\ &\le T\left\{\sup_r h_r^{-1} \sum_{k\in I_r} M\left(\frac{|A_k(x)-s|}{\rho_1}\right)^{p_k} + \left[L\left\{\frac{|s|}{\rho_1}/\delta\right\} M(2)\right]^H\right\} < \infty\end{aligned}$$

Hence $x \in N_\theta^\infty(A,M,p)$. This completes the proof. □

The proof of the following result is a consequence of Theorem 2.3.

Corollary 2.4. *$N_\theta^0(A,M,p)$ and $N_\theta(A,M,p)$ are nowhere dense subsets of $N_\theta^\infty(A,M,p)$.*

Let X be a sequence space. Then X is called

i) *Solid (or normal) if $(\alpha_k x_k) \in X$ whenever $(x_k) \in X$ for all sequences (α_k) of scalars with $\|\alpha_k\| \le 1$; for all $k \in N$;*

ii) *Monotone provided X contains the canonical preimages of all its step-spaces.*

If X is normal, then X is monotone.

Theorem 2.5. *If A = I, the sequence spaces $N_\theta^0(A,M,p)$ and $N_\theta^\infty(A,M,p)$ are solid and as such monotone.*

Proof. Let $\alpha = (\alpha_k)$ be sequence of scalars such that $\|\alpha_k\| \le 1$; for all $k \in N$. Since M is non-decreasing and A = I, we get

$$\begin{aligned}h_r^{-1} \sum_{k\in I_r} M\left(\frac{|A_k(\alpha x)|}{\rho}\right)^{p_k} &= h_r^{-1} \sum_{k\in I_r} M\left(\frac{|\alpha_k x_k|}{\rho}\right)^{p_k} \\ &\le h_r^{-1} \sum_{k\in I_r} M\left(\frac{|x_k|}{\rho}\right)^{p_k}\end{aligned}$$

Then the result follows from the above inequality. □

Theorem 2.6. *Let M, M_1, and M_2 be Orlicz functions which satisfies Δ_2-condition. We have*

i) $N_\theta(A, M_1, p) \subseteq N_\theta(A, MoM_1, p)$, $N_\theta^0(A, M_1, p) \subseteq N_\theta^0(A, MoM_1, p)$, *and* $N_\theta^\infty(A, M_1, p) \subseteq N_\theta^\infty(A, MoM_1, p)$,

ii) $N_\theta(A, M_1, p) \cap N_\theta(A, M_2, p) \subseteq N_\theta(A, M_1 + M_2, p)$, $N_\theta^0(A, M_1, p) \cap N_\theta^0(A, M_2, p) \subseteq N_\theta^0(A, M_1 + M_2, p)$, *and* $N_\theta^\infty(A, M_1, p) \cap N_\theta^\infty(A, M_2, p) \subseteq N_\theta^\infty(A, M_1 + M_2, p)$

Proof. We shal prove only for first inclusions. The others can be treated similarly.

i) Let $x \in N_\theta(A, M_1, p)$ and $\varepsilon > 0$ be given and choose δ with $0 < \delta < 1$ such that $M(t) < \varepsilon$ for $0 \le t \le \delta$. Writ

$$h_r^{-1} \sum_{k \in I_r} M\left(M_1\left(\frac{|A_k(x) - s|}{\rho}\right)^{p_k} = h_r^{-1} \sum_1 M\left(M_1\left(\frac{|A_k(x) - s|}{\rho}\right)\right)^{p_k} + h_r^{-1} \sum_2 M\left(M_1\left(\frac{|A_k(x) - s|}{\rho}\right)\right)^{p_k}$$

where the first summation is over $M_1(\frac{|A_k(x)-s|}{\rho}) \le \delta$ and the second summation is over $M_1(\frac{|A_k(x)-s|}{\rho}) > \delta$.

Let $M_1(\frac{|A_k(x)-s|}{\rho}) > \delta$, we have

$$M_1\left(\frac{|A_k(x) - s|}{\rho}\right) < M_1\left(\frac{|A_k(x) - s|}{\rho}\right)/\delta \le 1 + M_1\left(\frac{|A_k(x) - s|}{\rho}\right)/\delta$$

Since M is non-decreasing and satisfies Δ_2-condition, then there exists $L \ge 1$ such that

$$\begin{aligned} & M\left(M_1\left(\frac{|A_k(x) - s|}{\rho}\right)\right) < M\left(1 + M_1\left(\frac{|A_k(x) - s|}{\rho}\right)/\delta\right) \\ & \le \frac{1}{2}M(2) + \frac{1}{2}M\left(2M_1\left(\frac{|A_k(x) - s|}{\rho}\right)/\delta\right) \\ & \le \frac{1}{2}L\left\{M_1\left(\frac{|A_k(x) - s|}{\rho}\right)/\delta\right\}M(2) + \frac{1}{2}L\left\{M_1\left(\frac{|A_k(x) - s|}{\rho}\right)/\delta\right\}M(2) \\ & = L\left\{M_1\left(\frac{|A_k(x) - s|}{\rho}\right)/\delta\right\}M(2) \end{aligned}$$

Then

$$h_r^{-1} \sum_2 M\left(M_1\left(\frac{|A_k(x) - s|}{\rho}\right)\right)^{p_k} \le \max(1, (L\delta^{-1}M(2))^H) h_r^{-1} \sum_2 M_1\left(\frac{|A_k(x) - s|}{\rho}\right)^{p_k}$$

Since M is continuous, we have

$$h_r^{-1} \sum_1 M\left(\frac{|A_k(x) - s|}{\rho}\right)^{p_k} < \max(\varepsilon, \varepsilon^h).$$

Thuse

$$h_r^{-1}\sum_{k\in I_r} M\left(M_1\left(\frac{|A_k(x)-s|}{\rho}\right)\right)^{p_k} \le \max(\varepsilon,\varepsilon^h)$$
$$+\max(1,(L\delta^{-1}M(2))^H)h_r^{-1}\sum_2 M_1\left(\frac{|A_k(x)-s|}{\rho}\right)^{p_k}$$

By taking the limit as $\varepsilon \to 0$ and $r \to \infty$ we obtain $x \in N_\theta(A, MoM_1, p)$. This completes the proof.

ii) Let $x \in N_\theta(A, M_1, p) \cap N_\theta(A, M_2, p)$. Then using inequality (1) it can be show that $x \in N_\theta(A, M_1 + M_2, p)$.

The method of the proof of Theorem 2.6(i) shows that, for any Orlicz function M which satisfies Δ_2-condition, we have

$$N_\theta^0(A,p) \subset N_\theta^0(A,M,p),\ N_\theta(A,p) \subset N_\theta(A,M,p) \quad \text{and } N_\theta^\infty(A,p) \subset N_\theta^\infty(A,M,p).$$

□

Theorem 2.7. *Let M be an Orlicz function. Then*

i) *For* $\liminf_r u_r > 1$ *we have* $c^\theta(A,M,p) \subset N_\theta(A,M,p)$

ii) *For* $\limsup_r u_r < \infty$ *we have* $N_\theta(A,M,p) \subset c^\theta(A,M,p)$

iii) $c^\theta(A,M,p) = N_\theta(A,M,p)$ *if* $1 < \liminf_r u_r \le \limsup_r u_r < \infty$, *where*

$$c^\theta(A,M,p) = \left\{x \in w(X) : \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} M\left(\frac{|A_k(x)-s|}{\rho}\right)^{p_k} = 0,\ \textit{for some } \rho > 0\right\}$$

Proof. i) Let $x \in c^\theta(A,M,p)$ and $\liminf_r u_r > 1$. There exist $\delta > 0$ such that $u_r = (k_r/k_{r-1}) \ge 1+\delta$ for sufficiently large r. We have, for sufficiently large r, that $(h_r/k_r) \ge \delta/(1+\delta)$ and $(k_r/h_r) \le (1+\delta)/\delta$. Hence we get

$$\begin{aligned}
k_r^{-1}\sum_{k=1}^{k_r} M\left(\frac{|A_k(x)-s|}{\rho}\right)^{p_k} &\ge k_r^{-1}\sum_{k\in I_r} M\left(\frac{|A_k(x)-s|}{\rho}\right)^{p_k}\\
&= (h_r/k_r)h_r^{-1}\sum_{k\in I_r} M\left(\frac{|A_k(x)-s|}{\rho}\right)^{p_k}\\
&\ge \delta/(1+\delta)h_r^{-1}\sum_{k\in I_r} M\left(\frac{|A_k(x)-s|}{\rho}\right)^{p_k}
\end{aligned}$$

which implies just $x \in N_\theta(A,M,p)$.

ii) If $\limsup_r u_r < \infty$ then there exists $K > 0$ such that $u_r < K$ for every r.

Now suppose that $x \in N_\theta(A, M, p)$ and $\varepsilon > 0$. There exists m_0 such that for every i and $m \geq m_0$,

$$H_m = h_m^{-1} \sum_{k \in I_m} M\left(\frac{|A_k(x) - s|}{\rho}\right)^{p_k} < \varepsilon$$

We can also find $R > 0$ such that $H_m \leq R$ for all m. Let n be any integer with $k_r \geq n > k_{r-1}$. Now write

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n} M\left(\frac{|A_k(x) - s|}{\rho}\right)^{p_k} \leq k_{r-1}^{-1}\sum_{k=1}^{k_r} M\left(\frac{|A_k(x) - s|}{\rho}\right)^{p_k} \\
&= k_{r-1}^{-1}\left(\sum_{k=1}^{m_0} + \sum_{k=m_0+1}^{k_r}\right)\sum_{k\in I_m} M\left(\frac{|A_k(x) - s|}{\rho}\right)^{p_k} \\
&= k_{r-1}^{-1}\sum_{k=1}^{m_0}\sum_{k\in I_m} M\left(\frac{|A_k(x) - s|}{\rho}\right)^{p_k} + k_{r-1}^{-1}\sum_{k=m_0+1}^{k_r}\sum_{k\in I_m} M\left(\frac{|A_k(x) - s|}{\rho}\right)^{p_k} \\
&\leq k_{r-1}^{-1}\sum_{k=1}^{m_0}\sum_{k\in I_m} M\left(\frac{|A_k(x) - s|}{\rho}\right)^{p_k} + \varepsilon(k_r - k_{m_0})k_{r-1}^{-1} \\
&\leq k_{r-1}^{-1} \sup_{1 \leq k \leq m_0} H_k k_{m_0} + \varepsilon K < R k_{r-1}^{-1} k_{m_0} + \varepsilon K
\end{aligned}$$

from which we obtain $x \in c^\theta(A, M, p)$.

iii) It follows from (i) and (ii).

We consider that (p_k) and (t_k) are any bounded sequences of strictly positive real numbers.

We are able to prove $N_\theta(A, M, t) \subset N_\theta(A, M, p)$ only under additional conditions.

Using the same technique in the Thorem 2.6 of Bhardwaj and Singh [3] it is easy to prove the following theorem. □

Theorem 2.8. *Let $0 < p_k \leq t_k$ for all k and let (t_k/p_k) be bounded. Then $N_\theta(A, M, t) \subset N_\theta(A, M, p)$.*

Corollary 2.9. i) *If $0 < \inf p_k \leq 1$ for all k, then $N_\theta(A, M) \subset N_\theta(A, M, p)$.*

ii) $1 \leq p_k \leq \sup p_k = H < \infty$, *then $N_\theta(A, M, p) \subset N_\theta(A, M)$*

Proof. i) Follows from Theorem 2.8 $t_k = 1$ for all k.

ii) Follows from Theorem 2.8 $p_k = 1$ for all k. □

3. $S^\theta(A)$-statistical convergence

In this section we introduce the concept of lacunary strongly $S^\theta(A)$-statistical convergence and give some inclusion relations related to this sequence space.

In [5], Fast introduced the idea of statistical convergence. These idea were later studied in Connor [4], Freedman and Sember [8], Salat [13], and other authors independently.

A complex number sequence $x = (x_i)$ is said to be statistically convergent to the number l if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n}|\{i \le n : |x_i - l| \ge \varepsilon\}| = 0$$

The set of statistically convergent sequences is denoted by S.

Recently, Fridy and Orhan [6] introduced the concept of lacunary Statistical convergence as follows;

Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_i)$ is said to be lacunary statistically convergent to s if for any $\varepsilon > 0$

$$\lim_{r\to\infty} h_r^{-1} |\{k \in I_r : |x_k - s| \ge \varepsilon\}| = 0,$$

The set of lacunary statistically convergent sequences is denoted by S_θ.

A sequence $x = (x_i)$ is said to be lacunary strongly $S^\theta(A)$-statistically convergent to s if for any $\varepsilon > 0$

$$\lim_{r\to\infty} h_r^{-1} |\{k \in I_r : |A_k(x) - s| \ge \varepsilon\}| = 0,$$

uniformly in i. The set of all lacunary strongly $S^\theta(A)$-statistically convergent sequences is denoted by $S^\theta(A)$ (Bilgin [1]).

We now establish inclusion relations between $S^\theta(A)$ and $N_\theta(A, M, p)$.

Theorem 3.1. *Let M be Orlicz function. Then $N_\theta(A, M, p) \subset S^\theta(A)$.*

Proof. $x \in N_\theta(A, M, p)$. There exists some positive $\rho > 0$ such that

$$\lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} M\left(\frac{|A_k(x) - s|}{\rho}\right)^{p_k} = 0$$

Then

$$\begin{aligned} h_r^{-1} &\sum_{k\in I_r} M\left(\frac{|A_k(x) - s|}{\rho}\right)^{p_k} \\ &= h_r^{-1} \sum_1 M\left(\frac{|A_k(x) - s|}{\rho}\right)^{p_k} + h_r^{-1} \sum_2 M\left(\frac{|A_k(x) - s|}{\rho}\right)^{p_k} \\ &\ge h_r^{-1} \sum_1 M\left(\frac{|A_k(x) - s|}{\rho}\right) \\ &\ge h_r^{-1} \sum_1 M\left(\varepsilon/\rho\right)^{p_k} \\ &\ge h_r^{-1} \sum_1 Min\{M\left(\varepsilon/\rho\right)^h,\ M\left(\varepsilon/\rho\right)^H\} \\ &\ge h_r^{-1} |\{k \in I_r : |A_k(x) - s| \ge \varepsilon\}|\, Min\{M\left(\varepsilon/\rho\right)^h,\ M\left(\varepsilon/\rho\right)^H\} \end{aligned}$$

where the first summation is over $|A_k(x) - s| \geq \varepsilon$ and the second summation is over $|A_k(x) - s| < \varepsilon$. Hence $x \in S^\theta(A)$. $\square$

Theorem 3.2. *Let M be Orlicz function and A be a limitation method. Then $\ell_\infty \cap N_\theta(A, M, p) = \ell_\infty \cap S^\theta(A)$,*

Proof. By Theorem 3.1, we need only show that $\ell_\infty \cap S^\theta(A) \subset \ell_\infty \cap N_\theta(M, p, q)$. Let $x \in \ell_\infty \cap S^\theta(A)$. Since $x \in \ell_\infty$ and A is limitation method, so there exists $K > 0$ such that $M(\frac{|A_k(x)-s|}{\rho}) \leq K$.

Then for a given $\varepsilon > 0$, we have

$$\begin{aligned} &h_r^{-1} \sum_{k \in I_r} M\left(\frac{|A_k(x) - s|}{\rho}\right)^{p_k} \\ &\quad = h_r^{-1} \sum_1 M\left(\frac{|A_k(x) - s|}{\rho}\right)^{p_k} + h_r^{-1} \sum_2 M\left(\frac{|A_k(x) - s|}{\rho}\right)^{p_k} \\ &\quad \leq K h_r^{-1} \left|\{k \in I_r : |A_k(x) - s| \geq \varepsilon\}\right| + Max\left\{M\left(\varepsilon/\rho\right)^h, M\left(\varepsilon/\rho\right)^H\right\} \end{aligned}$$

where the first summation is over $|A_k(x) - s| \geq \varepsilon$ and the second summation is over $|A_k(x) - s| < \varepsilon$.

Taking the limit as $\varepsilon \to 0$ and $r \to \infty$, it follows that $x \in N_\theta(A, M, p)$. Hence $x \in \ell_\infty \cap N_\theta(A, M, p)$. This completes the proof. $\square$

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