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The k-Buchsbaum property for some polynomial ideals

By

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Introduction

In order to define the topic of the title, we always assume that R is a standard graded ring over a field k and $\mathfrak{m}$ is the maximal homogeneous ideal. k-Buchsbaum graded modules M over R can be defined as having their local cohomology modules $H^i_{\mathfrak{m}}(M),\ 0 \leq i \leq d$, annihilated by $\mathfrak{m}^k$, where $d+1$ is the Krull-dimension of M. (For undefined terminology see [E].) They are natural generalizations of Cohen-Macaulay modules, which have $H^i_{\mathfrak{m}}(M) = 0,\ 0 \leq i \leq d$. A more workable definition for k-Buchsbaum ideals $\mathfrak{a} \subset K[x_0, \dots, x_r] =: R_{r+1}$, where $\mathfrak{a}$ is a homogeneous ideal ($\delta(x_i) :=$ degree$(x_i) = 1,\ 0 \leq i \leq r$), is given below. An algorithm to test if such an ideal is perfect (i.e. $R_{r+1}/\mathfrak{a}$ is Cohen-Macaulay) or Buchsbaum (i.e. $R_{r+1}/\mathfrak{a}$ and $R_{r+1}/(\mathfrak{a}, F_0, \dots, F_i),\ 0 \leq i \leq d$, are 1-Buchsbaum for any system of parameters (s.o.p.) $\{F_0, \dots, F_d\}$) was given in [BV1] and [BV2]. Thus both of these papers deal with a fixed $k \in \{0, 1\}$ and do not address the question of an upper bound k, if $\mathfrak{a}$ is to be k-Buchsbaum for some k. The purpose of the present paper is to investigate this question without explicit computation of Ext-modules or local cohomology modules. We obtain an algorithm for certain binomial ideals. Although in [BH] it was shown, that no conclusive information about the k-Buchsbaum property of $\mathfrak{a}$ can be obtained from in$(\mathfrak{a})$ (the ideal of initial terms), our algorithm is based on the Gröbner bases calculations.

1. Homogeneous k-Buchsbaum ideals

We assume $R_{r+1} := K[x_0, \dots, x_r],\ K$ an infinite field, $\mathfrak{a} \subset R_{r+1}$ a homogeneous ideal (with respect to the standard grading), $\dim(\mathfrak{a}) =$ Krull-dim$(R_{r+1}/\mathfrak{a})$ $= d+1$, without loss of generality $\{x_0, \dots, x_d\}$ a s.o.p. for $\mathfrak{a}$ since K is infinite (i.e. the images $\{\bar{x_0}, \dots, \bar{x_d}\}$ form a s.o.p. in $R_{r+1}/\mathfrak{a}$).

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Definition 1.1. Let $\{y_0, \dots, y_d\}$ be any s.o.p. for $\mathfrak{a}$. For a $\mathfrak{m}$-primary ideal $\mathfrak{q}$, $\{y_0, \dots, y_d\}$ is said to be a $\mathfrak{q}$-weak sequence for $\mathfrak{a}$, if $\mathfrak{a} : y_0 \subseteq \mathfrak{a} : \mathfrak{q}$, $(\mathfrak{a}, y_0, \dots, y_{i-1}) : y_i \subseteq (\mathfrak{a}, y_0, \dots, y_{i-1}) : \mathfrak{q}$, $0 \leq i \leq d$.

Definition 1.2. For $k \geq 0$ $\mathfrak{a}$ is said to be k-Buchsbaum if every s.o.p. $\{y_0, \dots, y_d\} \subseteq \mathfrak{m}^{2k}$ for $\mathfrak{a}$ is an $\mathfrak{m}^k$-weak sequence for $\mathfrak{a}$.

It is clear from the definition and the theorem below, that if $\mathfrak{a}$ is k-Buchsbaum then it is k'-Buchsbaum for all $k' \geq k$, and $\mathfrak{a}$ is a generalized Cohen-Macaulay ideal (i.e. all local cohomology modules $H^i_{\mathfrak{m}}(R_{r+1}/\mathfrak{a})$, $i \leq d$, are of finite length) iff $\mathfrak{a}$ is k-Buchsbaum for $k \gg 0$.

Theorem 1.3 ([FV]). *The following are equivalent*:

(i) $\mathfrak{a}$ *is k-Buchsbaum.*

(ii) *For every s.o.p.* $\{y_0, \dots, y_d\} \subseteq \mathfrak{m}^{2k}$ *for* $\mathfrak{a}$, $\{y_0, \dots, y_d\}$ *is an* $\mathfrak{m}^k$*-weak sequence for* $\mathfrak{a}$.

(iii) *For a s.o.p.* $\{y_0, \dots, y_d\} \subseteq \mathfrak{m}^{2k}$ *for* $\mathfrak{a}$, $\{y_0, \dots, y_d\}$ *is an* $\mathfrak{m}^k$*-weak sequence for* $\mathfrak{a}$.

(iv) $\mathfrak{m}^k H^i_{\mathfrak{m}}(R_{r+1}/\mathfrak{a}) = 0$, $0 \leq i \leq d$, $H^i_{\mathfrak{m}}(R_{r+1}/\mathfrak{a})$ *the i-th local cohomology module of* $R_{r+1}/\mathfrak{a}$ *with respect to* $\mathfrak{m}$.

Proof. See [FV]. □

Theorem 1.4. *Assume* $\mathfrak{a}$ *is as before,* $\{x_0, \dots, x_d\}$ *a s.o.p. for* $\mathfrak{a}$. *Let* $j = (j_0, \dots, j_d)$ *be an arbitrary, but fixed vector of non-negative integers,* $0 \leq i \leq d$. *The following are equivalent*:

(i) $\mathfrak{a}$ *is k-Buchsbaum.*

(ii) $(\mathfrak{a}, x_0^{2k+j_0}, \dots, x_{i-1}^{2k+j_{i-1}}) : x_i^{2k+\gamma_i} \subseteq (\mathfrak{a}, x_0^{2k+j_0}, \dots, x_{i-1}^{2k+j_{i-1}}) : \mathfrak{m}^k$, $0 \leq i \leq d$, *for all* $\gamma_i \in \mathbf{N}$.

(iii) $(\mathfrak{a}, x_0^{2k+j_0}, \dots, x_{i-1}^{2k+j_{i-1}}) : x_i^k = (\mathfrak{a}, x_0^{2k+j_0}, \dots, x_{i-1}^{2k+j_{i-1}}) : x_i^{k+1} = (\mathfrak{a}, x_0^{2k+j_0}, \dots, x_{i-1}^{2k+j_{i-1}}) : \mathfrak{m}^k = (\mathfrak{a}, x_0^{2k+j_0}, \dots, x_{i-1}^{2k+j_{i-1}}) : \mathfrak{m}^{k+1}$, $0 \leq i \leq d$.

Proof. (i) iff (ii) by Theorem 1.3, (iii) implies (ii) is immediate.
(ii) ⇒ (iii). For short, let $\mathfrak{A} = (\mathfrak{a}, x_0^{2k+j_0}, \dots, x_{i-1}^{2k+j_{i-1}})$.

Let $k = 0$. Then $\mathfrak{A} : \mathfrak{m} \subseteq \mathfrak{A} : x_i = \mathfrak{A}$ (by (ii)) $\subseteq \mathfrak{A} : \mathfrak{m}$. Hence $\mathfrak{A} = \mathfrak{A} : x_i = \mathfrak{A} : \mathfrak{m}$.

Assume $k \geq 1$. Then $\mathfrak{A} : \mathfrak{m}^k \subseteq \mathfrak{A} : x_i^k \subseteq \mathfrak{A} : x_i^{k+1} \subseteq \mathfrak{A} : x_i^{2k} \subseteq \mathfrak{A} : \mathfrak{m}^k$ (by (ii)). Hence $\mathfrak{A} : \mathfrak{m}^k = \mathfrak{A} : x_i^k = \mathfrak{A} : x_i^{k+1} \supseteq \mathfrak{A} : \mathfrak{m}^{k+1}$, which implies $\mathfrak{A} : \mathfrak{m}^k = \mathfrak{A} : \mathfrak{m}^{k+1}$. □

Definition 1.5. Let $\mathfrak{b}$ and $\mathfrak{c}$ be ideals in R_{r+1}, $S = \{z_1, \dots, z_s\} \subseteq R_{r+1}$ and for $t \geq 1$, $S^t = \{z_1^t, \dots, z_s^t\}$. $(\mathfrak{b}, S^t)$ is said to stabilize with respect to $\mathfrak{c}$, if there exist positive integers T and k such that $(\mathfrak{b}, S^t) : \mathfrak{c}^k = (\mathfrak{b}, S^t) : \mathfrak{c}^{k+1}$ for all $t \geq T$. For such a pair (T, k), we also will say that $(\mathfrak{b}, S^t) : \mathfrak{c}^u$ *stabilizes at* (T, k) (here t and u are integer variables).

Clearly, if $(\mathfrak{b}, S^t) : \mathfrak{c}^u$ stabilizes at (T, k), then it stabilizes at (T', k') for any $T' \geq T$ and $k' \geq k$. Moreover, in this case we have

$$(\mathfrak{b}, S^t) : \mathfrak{c}^u = (\mathfrak{b}, S^t) : \mathfrak{c}^\infty := \cup_{i=1}^\infty (\mathfrak{b}, S^t) : \mathfrak{c}^i,$$

for all $t \geq T$ and $u \geq k$.

Theorem 1.6. *Assume $\mathfrak{b} \subseteq R_{r+1}$ is a homogeneous ideal and $\{x_0, \dots, x_d\}$ a s.o.p. for $\mathfrak{b}$. If $\mathfrak{b}$ is k-Buchsbaum for some k, then for $T \geq 2k$*

a) $(\mathfrak{b}, x_0^T, \dots, x_{i-1}^T) : x_i^k = (\mathfrak{b}, x_0^T, \dots, x_{i-1}^T) : x_i^{k+1}$, $0 \leq i \leq d$.

b) $(\mathfrak{b}, x_0^T, \dots, x_{i-1}^T) : x_i^k \subseteq (\mathfrak{b}, x_0^T, \dots, x_{i-1}^T) : x_l^k$, $0 \leq i, l \leq d$.

a') $(\mathfrak{b}, x_{j_0}^t, \dots, x_{j_{i-1}}^t) : x_{j_i}^u$ *stabilizes at* (T, k) *for all parts of all permutations* $(j_0, \dots, j_d)$ *of* $(0, \dots, d)$, $0 \leq i \leq d$,

b') $(\mathfrak{b}, x_{j_0}^t, \dots, x_{j_{i-1}}^t) : x_{j_i}^k \subseteq (\mathfrak{b}, x_{j_0}^t, \dots, x_{j_{i-1}}^t) : x_l^k$ *for all parts of all permutations* $(j_0, \dots, j_d)$ *of* $(0, \dots, d)$, $0 \leq i, l \leq d$, *and for all* $t \geq T$.

Conversely, if a) *and* b) (*or* a') *and* b')) *hold for some* $T \geq 2k \in \mathbf{N}$, *then* $\mathfrak{b}$ *is* γ*-Buchsbaum, where* γ *is the least integer such that* $\mathfrak{m}^\gamma \subseteq (x_0^k, \dots, x_d^k) + \mathfrak{b}$.

Proof. Note that for all permutations $(j_0, \dots, j_d)$ of $(0, \dots, d)$, $x_{j_0}, \dots, x_{j_d}$ is again a s.o.p for $\mathfrak{b}$. Hence, the necessity of a) and a') follows from Theorem 1.4 (iii), then the necessity of b) and b') from a) and a'), respectively, and from Theorem 1.3 (ii). Clearly a') and b') imply a) and b). Assume therefore a) and b). Let

$$\mathfrak{q} = (x_0^k, \dots, x_d^k) \quad \text{and} \quad \mathfrak{b}(i-1, T) := (\mathfrak{b}, x_0^T, \dots, x_{i-1}^T).$$

Since $T \geq 2k$, $x_0^T, \dots, x_d^T \subseteq \mathfrak{q}^2$. For all $0 \leq i \leq d$ we have:

$$\begin{aligned} \mathfrak{b}(i-1, T) : x_i^T &= \mathfrak{b}(i-1, T) : x_i^k && \text{(by a))} \\ &\subseteq \mathfrak{b}(i-1, T) : x_l^k, \ 0 \leq l \leq d && \text{(by b))}. \end{aligned}$$

Hence $\mathfrak{b}(i-1, T) : x_i^T = \mathfrak{b}(i-1, T) : \mathfrak{q}$. This means $\{x_0^T, \dots, x_{i-1}^T\}$ is a $\mathfrak{q}$-weak sequence. By Proposition 13 in the Appendix of [SV], $\mathfrak{q} H_{\mathfrak{m}}^i(R_{r+1}/\mathfrak{b}) = 0$, $0 \leq i < d$. Let γ be an integer such that $\mathfrak{m}^\gamma \subseteq \mathfrak{q} + \mathfrak{b}$. Then $\mathfrak{m}^\gamma H_{\mathfrak{m}}^i(R_{r+1}/\mathfrak{b}) = 0$, $0 \leq i < d$. By Theorem 1.3 (iv), $\mathfrak{b}$ is γ-Buchsbaum. $\square$

Example 1.7. Let $\mathfrak{b} = (x_1^2 - x_0x_2, x_3x_2 - x_0x_2, x_2^2 - x_0x_2)$. A s.o.p. for $\mathfrak{b}$ is $\{x_0, x_3\}$. Let $>$ be the reverse lexicographical term order with $x_2 > x_1 > x_3 > x_0$. A Gröbner basis calculation gives $\{x_1^2 - x_0x_2, x_3x_2 - x_0x_2, x_2^2 - x_0x_2\}$ as the reduced Gröbner basis of $\mathfrak{b}$ and $\{x_1^2 - x_0x_2, x_3x_2 - x_0x_2, x_2^2 - x_0x_2, x_3^t, x_0^t x_2\}$ as the reduced Gröbner basis of $(\mathfrak{b}, x_3^t)$. Hence

$$(\mathfrak{b}, x_3^t) : x_0^\infty = (x_1^2, x_2, x_3^t) \neq (\mathfrak{b}, x_3^t) : x_0^k$$

for all $t > k$. Thus $(\mathfrak{b}, x_3^t)$ does not stabilize with respect to x_0.

In this paper the reduced Gröbner basis always means a minimal Gröbner basis $\{g_1, \dots, g_s\}$ such that all $\mathrm{in}(g_1), \dots, \mathrm{in}(g_s)$ are terms and no term of g_i is divisible by $\mathrm{in}(g_j)$, $j \neq i$.

Example 1.8. Let $\mathfrak{b} = (x_1, x_2^2, x_3^2) \cap (x_2, x_3) \subseteq R_4 = K[x_0, x_1, x_2, x_3]$. A s.o.p. for $\mathfrak{b}$ is $\{x_0, x_1\}$ since $\mathfrak{b} = (x_1x_2, x_1x_3, x_2^2, x_3^2)$. $\mathfrak{b}$ is not k-Buchsbaum for any k, since $\mathfrak{b}$ has a nontrivial embedded component ([SV, Lemma I.2.2]). $\mathfrak{b} : x_0 = \mathfrak{b} : x_0^2 = \mathfrak{b}$ and $(\mathfrak{b}, x_0^t) : x_1 = (x_2, x_3, x_0^t) = (\mathfrak{b}, x_0^t) : x_1^2$, thus $\mathfrak{b}$ stabilizes with respect to x_0 and $(\mathfrak{b}, x_0^t)$ with respect to x_1. *However one condidtion in* b) *is not satisfied*: $(\mathfrak{b}, x_0^t) : x_1^k \not\subseteq (\mathfrak{b}, x_0^t) : x_0^k$ *for all* $k > 0$.

Remark 1.9. If $\mathfrak{b}$ is 1-dimensional, then a) in Theorem 1.6 is vacuously true and b) readily obtained algorithmically. Thus in the sequel $\dim \mathfrak{b} \geq 2$.

2. Stabilization of binomial ideals

Definition 2.1. A monomial in R_{r+1} is a polynomial $m = cx_0^{\alpha_0} \cdots x_r^{\alpha_r}$, $c \neq 0$. A term is a monomial with $c = 1$. The set of terms in R_{r+1} is denoted by T_{r+1}.

Definition 2.2. For $m_1 = c_1 x_0^{\alpha_{01}} \cdots x_r^{\alpha_{r1}},\ m_2 = c_2 x_0^{\alpha_{02}} \cdots x_r^{\alpha_{r2}},\ c_1 c_2 \neq 0$, g.c.d.$(m_1, m_2) = x_0^{\delta_0} \cdots x_r^{\delta_r},\ \delta_i = \min\{\alpha_{i1}, \alpha_{i2}\},\ 0 \leq i \leq r$.

Definition 2.3. An ideal $0 \neq \mathfrak{b} \subset R_{r+1}$ (for us) is a binomial ideal if:
(i) $\mathfrak{b}$ is generated by binomials and monomials,
(ii) If b is a binomial generator of $\mathfrak{b}$, assume $b = m_1 - m_2,\ m_1$ a term, m_2 a monomial,
(iii) $\mathfrak{b}$ is homogeneous with respect to some nonnegative grading,
(iv) For an admissible term order $>$, we assume $m_1 > (1/c_2)m_2,\ c_2 \in K \setminus \{0\}$, and we will write $\text{in}(b) = m_1$. $B \cup M$ is the reduced Gröbner basis of $\mathfrak{b}$, where B consists of binomials and M consists of terms.
(v) The variables $x_0, \ldots, x_d$ form a s.o.p. for $\mathfrak{b}$.

Note. We also use $\{X, Y\}$ and $\{X, u, v, \ldots\}$ to denote the sets $X \cup Y$ and $X \cup \{u, v, \ldots\}$, respectively.

Definition 2.4. Assume $>$ is a term order, $b = m_1 - m_2$ a binomial, $m_1 > (1/c_2)m_2,\ m_1, m \in T_{r+1}$. Let $m_1 = q_1 d,\ m = qd,\ d = \text{g.c.d.}(m_1, m)$. $s(b, m) = qm_2$ is the *successor polynomial* of b and m. (By abuse of notation, we sometimes identify $s(b, m)$ and $(1/c_2)s(b, m) \in T_{r+1}$).

Note that the s-polynomial formation above is only a particular case of the s-polynomial formation of an arbitrary pair of polynomials in Buchberger's algorithm, defined as $s(f_1, f_2) = m_2 f_1 - m_1 f_2$, where $m_1 = \text{in}(f_1)/m,\ m_2 = \text{in}(f_2)/m$ and $m = \text{g. c. d}(\text{in}(f_1), \text{in}(f_2))$. We will use this remark in the proof of Lemma 2.10.

Definition 2.5. Assume m is a monomial, $b = m_1 - m_2$ a binomial, $m_1 \in T_{r+1}$ such that $m_1 \mid m$, i.e. $m = qm_1$. $qm_2 = \tilde{m}$ is said to be a *reduction* of $m \bmod b$ and m is said to *reduce to* $\tilde{m} \bmod b$. We write

$$m \xrightarrow{b} \tilde{m}.$$

For a sequence of reductions

$$m \xrightarrow{b_1} \tilde{m}_1 \xrightarrow{b_2} \cdots \xrightarrow{b_n} \tilde{m}_n,\ \{b_1, \ldots, b_n\} \subseteq B,$$

we write

$$m \xrightarrow{B} \tilde{m}_n$$

and say $\tilde{m_n}$ is a *reduction* of $m \bmod B$ and m *reduces* to $\tilde{m_n}$. Monomials not reducible $\bmod b$ (respectively not reducible for any $b \in B$) are said to be *irreducible* $\bmod b$ (respectively *irreducible* $\bmod B$). If, in addition, they are not divisible by a monomial of M, they are said to be *irreducible* $\bmod B \cup M$. We write irr. $\bmod b$ (respectively irr. $\bmod B$ and irr. $\bmod B \cup M$).

Lemma 2.6. *Fix a term order on R_{r+1} and a variable $y \in \{x_0, \dots, x_r\}$. Let $\mathfrak{b}$ be a binomial ideal with the reduced Gröbner basis $B \cup M$ with*

$$\Delta = \max_{b \in B \cup M} \{\delta(b)\}.$$

Then, for an arbitrary integer $\tau \geq \Delta$, the reduced Gröbner basis of the ideal $\mathfrak{a}(\tau) = (\mathfrak{b}, y^\tau)$ has the form

$$\{B \cup M, G(\tau)\},$$

where $G(\tau) \subseteq T_{r+1} \setminus M$.

Proof. Note that $\mathfrak{a}(\tau) = (B \cup M, y^\tau)$. Since $B \cup M$ is a Gröbner basis (of $\mathfrak{b}$) and $\tau \geq \Delta$, all new elements in Buchberger's algorithm, applied to $\{B, M, y^\tau\}$ are monomials of degree at least τ. Hence, no term of $b \in B$ and no monomial of M is divisible by a new monomial. Since $B \cup M$ is already the reduced Gröbner basis, the set $G(\tau)$ of all new terms together with $B \cup M$ forms the reduced Gröbner basis of $\mathfrak{a}(\tau)$. Clearly $G(\tau) \cap M = \emptyset$. □

Using the notation $G(\tau)$, $\tau \geq \Delta$, we will give a criterion for the stabilization of ideals of the type $\mathfrak{a}(\tau)$ with respect to an ideal generated by a certain single variable.

Lemma 2.7. *Assume $>$ is a term order, $\mathfrak{b}$, $B \cup M$ as specified. Fix an integer $t_0 \geq \Delta$ and let $G(t_0)$ denote the corresponding set of terms given by the previous lemma. Then*

(1) *$m \in G(t_0)$ implies $\delta(m) \geq \delta(b)$ for all $b \in B \cup M$.*

(2) *$m \in G(t_0)$ implies $m = y^{t_0}$ or $\delta_y(m) < t_0$ ($\delta_y(m)$ denotes the degree of m in y).*

(3) *Let*

$$\Delta_y = \max_{b \in B \cup M} \{\delta_y(\mathrm{in}(b)\}.$$

Assume if $m \in G(t_0)$, then $\delta_y(m) \geq \Delta_y$. Then for $t \geq 0$, $G(t_0 + t) = y^t G(t_0)$.

(4) *Assume the monomial $m \in (B \cup M, y^{t_0})$, $b = m_1 - m_2 \in B$, $\mathrm{in}(b) = m_1$, $\delta_y(m) \geq \delta_y(m_1)$. Let $t \geq 0$. Then $y^t m$ is irr. $\bmod B \cup M$ implies $s(b, y^t m) \to 0 \mod\{B, y^t G(t_0)\}$.*

Proof. (1) follows from the proof of Lemma 2.6.

(2) As in (1), for any term m obtained in the Buchberger algorithm, starting with $\{B \cup M, y^{t_0}\}$, $\delta(m) \geq \delta(y^{t_0})$. Therefore if y^{t_0} is irr. $\bmod B \cup M$, then $y^{t_0} \in G(t_0)$ and the conclusion in (2) follows. If y^{t_0} is not irr. $\bmod B \cup M$, then any term m, such that $y^{t_0} \mid m$, is not irr. $\bmod B \cup M$, thus $m \notin G(t_0)$.

(3) Let $m \in G(t_0)$. Write $m = y^{\delta_y(m)}\bar{m}$. For $b \in B$, let

$$b = m_1 - m_2,\ \mathrm{in}(b) = m_1 = y^{\delta_y(m_1)}\bar{m_1},\ \bar{m} = \bar{q}d,\ \bar{m_1} = \bar{q_1}d,\ d = \text{g.c.d}(\bar{m}, \bar{m_1}).$$

Then

$$s(b, m) = \bar{q}y^{\delta_y(m)-\delta_y(m_1)}m_2 \xrightarrow{B} \tilde{m}$$

such that there exists $m^* \in G(t_0) \cup M$ with the property $m^* \mid \tilde{m}$. We have

$$s(b, y^t m) = \bar{q}y^{\delta_y(m)-\delta_y(m_1)+t}m_2 \xrightarrow{B} y^t\tilde{m}.$$

If $m^* \in M$, $s(b, y^t m)$ reduces to 0 w.r.t. $B \cup M$ (in the Buchberger algorithm). If $m^* \in G(t_0)$, then $y^t m^* \mid y^t\tilde{m}$, $y^t m^* \in y^t G(t_0)$, and $s(b, y^t m)$ reduces to 0 w.r.t. $B \cup M, y^t G(t_0)$. Thus $\{B \cup M, y^t G(t_0)\}$ is a Gröbner basis, which is reduced since each $y^t m \in y^t G(t)$ is clearly irr. mod $B \cup M$ (from the conditions $m \in G(t_0)$ and $\delta_y(m) \geq \Delta_y$).

(4) The conclusion in (4) follows immediately from the proof in (3). $\square$

Definition 2.8. Assume $\mathfrak{b}$, $B \cup M$, t_0, $G(t_0)$ are as before. $m \in G(t_0)$ is said to be *absolutely irreducible* (*a. irr.*) mod $B \cup M$ if $y^t m$ is irr. mod $B \cup M$ for all $t \geq 0$. $m \in G(t_0)$ is said to be *stable* if $y^t m \in G(t_0 + t)$, $t \geq 0$.

From now on in this section (unless otherwise specified), $>$ is the reverse lexicographical term order (rev. lex.) with $y > x$ the smallest linear terms, $\{x, y\} \subseteq \{x_0, \dots, x_r\}$. $\delta_x(m)$ denotes the degree of a monomial m in x.

Lemma 2.9. *For m a monomial and an element $b \in B$, $\delta_x(s(b, m)) \geq \delta_x(m)$.*

Proof. Let $b = m_1 - m_2$. Since $m_1 = \mathrm{in}(b)$ (see Definition 2.3) with respect to the term order rev. lex., always $\delta_x(m_2) \geq \delta_x(m_1)$. Let $m_1 = q_1 d$, $m = qd$, $d = \text{g.c.d}(m_1, m)$. Then $qm_1 = q_1 m$ and $\delta_x(m_2) \geq \delta_x(m_1)$ imply $\delta_x(s(b, m)) = \delta_x(qm_2) \geq \delta_x(qm_1) = \delta_x(q_1 m) \geq \delta_x(m)$. $\square$

Lemma 2.10. *If there is $N \in \mathbf{N}$ such that $\delta_x(m) \leq N$ for all $m \in G(t)$ and for all $t \geq T \geq \Delta$, then $\delta_y(m) \geq \Delta_y$ for all $m \in G(t_0)$ and $t_0 \geq \max\{T,\ \Delta_y(N+1)\}$.*

Proof. Assume $t_0 \geq \max\{T,\ \Delta_y(N+1)\}$. Let $m \in G(t_0)$. Then there is a sequence of reductions:

$$n_0 = y^{t_0},\ n_1 = s(b_0, n_0), \dots, n_p = s(b_{p-1}, n_{p-1}) = m;\ b_1, \dots, b_p \in B.$$

We first show that for all $i \geq 0$ in this sequence we have:

$$\delta_y(n_i)/\Delta_y + \delta_x(n_i) \geq N + 1. \tag{1}$$

Induction on i. The case $i = 0$ is trivial. Assume $\delta_y(n_i)/\Delta_y + \delta_x(n_i) \geq N + 1$ and $i < p$. Let $b_i = m_1 - m_2$ (see Definition 2.3).

If $\delta_y(m_1) \le \delta_y(m_2)$, then $\delta_y(n_{i+1}) \ge \delta_y(n_i)$. By Lemma 2.9 and induction we have

$$\delta_y(n_{i+1})/\Delta_y + \delta_x(n_{i+1}) \ge \delta_y(n_i)/\Delta_y + \delta_x(n_i) \ge N+1.$$

If $\delta_y(m_1) > \delta_y(m_2)$, then since $y > x$, we have $\delta_x(m_1) < \delta_x(m_2)$. Analyzing the proof of Lemma 2.9 we even have $\delta_x(n_{i+1}) \ge \delta_x(n_i) + 1$. Moreover if $n_i = dn_i'$, $d = \mathrm{g.c.d}(n_i, m_1)$, then

$$\delta_y(n_{i+1}) \ge \delta_y(n_i') = \delta_y(n_i) - \delta_y(d) \ge \delta_y(n_i) - \Delta_y.$$

Hence

$$\delta_y(n_{i+1})/\Delta_y + \delta_x(n_{i+1}) \ge \delta_y(n_i)/\Delta_y - 1 + \delta_x(n_i) + 1 \ge N+1.$$

The induction is completed.

Now, by assumption, $\delta_x(m) \le N$ for all $m \in G(t_0)$. Hence, by (1), $\delta_y(m) \ge \Delta_y$. □

Theorem 2.11. *Consider the following conditions*:
(i) *For all $m \in G(t)$ and for all $t \ge T \ge \Delta$, $\delta_x(m) \le N \in \mathbf{N}$.*
(ii) *For all $m \in G(t)$ and for all $t \ge t_0 \ge \Delta$, m is stable.*
Then (i) *implies* (ii) *for* $t_0 = \max\{T,\ \Delta_y(N+1)\}$; *and* (ii) *implies* (i) *for* $T = t_0 + \Delta_y$ *and* $N = \max\{\delta_x(m);\ m \in G(t_0)\}$.

Proof. (ii) $\Rightarrow$ (i). Let $T = t_0 + \Delta_y$ and $m \in G(t_0)$. Then $y^{\Delta_y} m \in (B \cup M, G(T))$. By hypothesis $y^t y^{\Delta_y} m = y^{t+\Delta_y} m \in G(t + \Delta_y + t_0) = G(T+t)$. In particular $y^t y^{\Delta_y} m$ is irr. $\bmod B \cup M$. Hence, by Lemma 2.7 (4) elements $y^t y^{\Delta_y} m$, $m \in G(t_0)$ and $B \cup M$ form a Gröbner basis of $(B \cup M, y^{T+t})$, i.e. $G(T+t) = y^{t+\Delta_y} G(t_0)$ for all $t \ge 0$. From this $G(T+t) = y^t G(T)$, $t \ge 0$, which implies (i).

(i) $\Rightarrow$ (ii) Let $t \ge t_0 := \max\{T,\ \Delta_y(N+1)\}$. In the proof of Lemma 2.10 we have shown that for all $m \in G(t_0)$, $\delta_y(m) \ge \Delta_y$. By Lemma 2.7 (3), $G(t+t_0) = y^t G(t_0))$ which implies (ii). □

Lemma 2.12. *Let $\mathcal{M} = (m_1, \ldots, m_s) \subseteq R_{r+1}$ be a monomial ideal, $m_i \in T_{r+1}$, $1 \le i \le s$, $x \in \{x_0, \ldots, x_r\}$. Then*

$$\mathcal{M} : x^k = (m_1/\mathrm{g.c.d}(m_1, x^k), \ldots, m_s/\mathrm{g.c.d}(m_s, x^k)).$$

In particular for $k \ge \max\{\delta_x(m_i);\ 1 \le i \le s\}$,

$$\mathcal{M} : x^k = (m_1/x^{\delta_x(m_1)}, \ldots, m_s/x^{\delta_x(m_s)}) = (\mathcal{M}|_{x=1}),$$

and $k = \max\{\delta_x(m_i)\ 1 \le i \le s\}$ *is minimal such that* $\mathcal{M} : x^k = \mathcal{M} : x^{k+1}$. *Here* $(\mathcal{M}|_{x=1})$ *mean the variable* x *is replaced by* 1 *in all monomials of* $\mathcal{M}$.

Proof. This is immediate by [KR, Satz 5']. □

Definition 2.13. Assume $\mathfrak{b}$, $B\cup M$, t_0, $G(t_0)$ are as specified before. If for some $\mathfrak{a}(t_0) = (\mathfrak{b}, y^{t_0})$ with reduced Gröbner basis $\{B\cup M, G(t_0)\}$, $\mathfrak{a}(t_0+t) = (\mathfrak{b}, y^{t_0+t})$ has reduced Gröbner basis $\{B\cup M, y^t G(t_0)\}$ (i.e. $G(t+t_0) = y^t G(t_0)$) for all $t \geq 0$, then we say that the Buchberger algorithm stabilizes for $\mathfrak{a}(t)$ at t_0.

The Buchberger algorithm stabilizes at t_0 only if every $m \in G(t_0)$ is stable, and it stabilizes at any $t_0' \geq t_0$.

Theorem 2.14. *Let $t_0, t^* \geq \Delta$ and*

$$k \geq \Delta_x := \max_{b\in B\cup M}\{\delta_x(\mathrm{in}(b)\}.$$

Consider the following conditions:

(i) $\mathfrak{a}(t)$ *stabilizes at* (t_0, k) *with respect to* x, *i.e.* $\mathfrak{a}(t_0+t) : x^k = \mathfrak{a}(t_0+t) : x^{k+1}$ *for all* $t \geq 0$.

(ii) $\delta_x(m) \leq k \in \mathbf{N}$ *for all* $m \in G(t)$ *and all* $t \geq t_0$.

(iii) *The Buchberger algorithm stabilizes for* $\mathfrak{a}(t)$ *at* t^*.

Then

a) (i) *is equivalent to* (ii).

b) (ii) *implies* (iii) *for* $t^* = \max\{t_0, \Delta_y(k+1)\}$.

c) (iii) *implies* (ii) *for* $t_0 = t^* + \Delta_y$ *and* $k = \max\{\delta_x(m);\ m \in G(t^*)\}$. *Moreover if* $\delta_y(m) \geq \Delta_y$ *for all* $m \in G(t^*)$ *we may take* $t_0 = t^*$.

Proof. b) and c) follow from Theorem 2.11. If already $\delta_y(m) \geq \Delta_y$ for all $m \in G(t^*)$, then $\delta_y(m) \geq \delta_y(m_1)$ for all $b = m_1 - m_2 \in B$. Hence, from the proof of (ii) $\Rightarrow$ (i) of Theorem 2.11, every $m \in G(t^*)$ is stable.

We show a). For an ideal $\mathfrak{c}$ let $\mathfrak{c} : x^\infty := \cup_{n\geq 1}\mathfrak{c} : x^n$. We always have $\mathfrak{c} : x^k \subseteq \mathfrak{c} : x^\infty$. Recall that the term order under consideration is rev. lex. and x is the smallest term. By Proposition 15.12 and its application in Exercise 15.41 a. in [E], $\mathfrak{a}(t_0+t) : x^k = \mathfrak{a}(t_0+t) : x^\infty$ iff $\mathrm{in}(\mathfrak{a}(t_0+t)) : x^k = \mathrm{in}(\mathfrak{a}(t_0+t)) : x^\infty$. Since $\mathrm{in}(\mathfrak{a}(t_0+t))$ is minimally generated by $\{\mathrm{in}(b);\ b \in B\} \cup M \cup G(t_0+t)$, by Lemma 2.12 $\mathrm{in}(\mathfrak{a}(t_0+t)) : x^k = \mathrm{in}(\mathfrak{a}(t_0+t)) : x^\infty$ iff $k \geq \max\{\delta_x(\mathrm{in}(b)),\ \delta_x(m);\ b \in B,\ m \in M \cup G(t^*)\}$. By hypothesis $k \geq \Delta_x = \max_{b\in B\cup M}\{\delta_x(\mathrm{in}(b))\}$. Hence (i) iff (ii). □

Below is a criterion for the stabilization of the Buchberger algorithm:

Proposition 2.15. *If the Buchberger algorithm stabilizes at* t_0 *then for all* $m \in G(t_0+\Delta_y)$, $\delta_y(m) \geq \Delta_y$. *Conversely, if* $\delta_y(m) \geq \Delta_y$ *for all* $m \in G(t_0)$, *then the Buchberger algorithm stabilizes at* t_0.

Proof. $\Rightarrow$ is immediate from the formula $G(t_0 + \Delta_y) = y^{\Delta_y} G(t_0)$. $\Leftarrow$. Assume if $m \in G(t_0)$, then $\delta_y(m) \geq \Delta_y$. Then, as before by Lemma 2.10 (3), $G(t_0+t) = y^t G(t_0)$. □

Remark 2.16. Assume $\mathfrak{b} : y = \mathfrak{b}$. Then one can show that the Buchberger algorithm does not stabilize at any t_0 iff for every $t \geq \Delta$ there exists $m \in G(t)$ which is not divisible by y. Thus, the stability conditions of the above proposition can be weakened in this case.

We collect the preceding into:

Algorithm A. Let $t_0 = \Delta$ and $n = 0$.
(i) Calculation of $G(t_0 + n\Delta_y)$ from B and $y^{\Delta_y} G(t_0 + (n-1)\Delta_y)$.
(ii) If $\delta_y(m) \geq \Delta_y$ for all $m \in G(t_0 + n\Delta_y)$: stop;
Otherwise increase n by one and repeat (i).

Thus if Algorithm A stops at the n-th step, the Buchberger algorithm stabilizes at $\Delta + n\Delta_y$, and by Theorem 2.14, $\mathfrak{a}(t)$ stabilizes at $(\Delta + n\Delta_y, k)$, where

$$k = \max\{\Delta_x,\ \delta_x(m); m \in G(\Delta + n\Delta_y)\}.$$

From the proof of Theorem 2.14 we also get that this is the smallest possible value of k for the stabilization of $\mathfrak{a}(t)$.

In the last section we will determine the number of steps needed to decide if Algorithm A stops at some n, or will never stop.

3. The k-Buchsbaum property for some binomial ideals

Assume $\mathfrak{b}$ is as before. In this section we will relate the stabilization of ideals $\mathfrak{a}_j(t) := (\mathfrak{b}, x_j^t)$ with respect to x_i^k, $0 \leq i \neq j \leq d$, to the k-Buchsbaum property of $\mathfrak{b}$.

Lemma 3.1. *Assume $\mathcal{M}$ is a set of monomials and $\bar{\mathcal{M}}$ a generating set for the ideal of all monomials in $(\mathfrak{b}, \mathcal{M})$. For a monomial $m \in R_{r+1}$ we have*

$$(\mathfrak{b}, \bar{\mathcal{M}}) : m = (\mathfrak{b} + (\bar{\mathcal{M}})) : m = \mathfrak{b} : m + (\bar{\mathcal{M}}) : m.$$

Proof. $(\mathfrak{b} + (\bar{\mathcal{M}})) : m \supseteq \mathfrak{b} : m + (\bar{\mathcal{M}}) : m$ follows trivially. Let $v \in (\mathfrak{b} + (\bar{\mathcal{M}})) : m$. Write $v = v_1 + \cdots + v_s + v_1' + \cdots + v_t'$, $s, t \geq 0$, such that $mv_1, \ldots, mv_s \notin \mathfrak{b} + (\bar{\mathcal{M}})$ and $mv_1', \ldots, mv_t' \in \mathfrak{b} + (\bar{\mathcal{M}})$. Since $(\bar{\mathcal{M}})$ is the ideal of all monomials in $(\mathfrak{b}, \mathcal{M})$, $mv_1', \ldots, mv_t' \in (\bar{\mathcal{M}})$, which yields $v_1', \ldots, v_t' \in (\bar{\mathcal{M}}) : m$. On the other hand, by Proposition 1.10 in [ES], $mv_1 + \cdots + mv_s \in \mathfrak{b}$, which implies $v_1 + \cdots + v_s \in \mathfrak{b} : m$. Thus $v \in \mathfrak{b} : m + (\bar{\mathcal{M}}) : m$, as required. □

Proposition 3.2. *For $\mathfrak{b}$ a binomial ideal as defined, $(\mathfrak{b}, x_{s_0}^t, \ldots, x_{s_{i-1}}^t) : x_{s_i}^{k_i}$ stabilizes at (T, k) for all parts of permutations $(s_0, \ldots, s_d)$ of $(0, \ldots, d)$, $0 < i \leq d$ iff $(\mathfrak{b}, x_j^t) : x_l^{k_l}$ stabilizes at (T, k) for each l, j such that $0 \leq j \neq l \leq d$.*

Proof. The implication $\Rightarrow$ is trivial by taking $i = 1$, $s_0 = j$, $s_1 = l$. We show the converse. W.l.o.g. one may assume $(s_0, \ldots, s_d) = (0, \ldots, d)$. Let $(\bar{\mathcal{M}}_j(t))$, $0 \leq j \leq i-1$, be the ideal of all monomials in $(\mathfrak{b}, x_j^t)$. Then $\sum_{j=0}^{i-1}(\bar{\mathcal{M}}_j(t)) = (\cup_{j=0}^{i-1}\bar{\mathcal{M}}_j(t))$ is the ideal of all monomials in $(\mathfrak{b}, x_0^t, \ldots, x_{i-1}^t)$

([ES, Corollary 1.6 (b)]). Therefore

$$\begin{aligned}
(\mathfrak{b}, x_0^t, \dots, x_{i-1}^t) : x_i^{k_i} &= \left(\mathfrak{b}, \sum_{j=0}^{i-1} (\bar{\mathcal{M}}_j(t))\right) : x_i^{k_i} \\
&= \mathfrak{b} : x_i^{k_i} + \left[\sum_{j=0}^{i-1} (\bar{\mathcal{M}}_j(t)) : x_i^{k_i}\right] \qquad \text{(by Lemma 3.1)} \\
&= \mathfrak{b} : x_i^{k_i} + \left[\sum_{j=0}^{i-1} (\bar{\mathcal{M}}_j(t) : x_i^{k_i})\right] \qquad \text{(by [KR, Satz 5])} \\
&= \sum_{j=0}^{i-1} ((\mathfrak{b} + (\bar{\mathcal{M}}_j(t))) : x_i^{k_i}) \qquad \text{(by Lemma 3.1)} \\
&= \sum_{j=0}^{i-1} ((\mathfrak{b}, \bar{\mathcal{M}}_j(t)) : x_i^{k_i}) \\
&= \sum_{j=0}^{i-1} ((\mathfrak{b}, x_j^t) : x_i^{k_i}).
\end{aligned}$$

Since $(\mathfrak{b}, x_j^t) : x_i^{k_i}$ stabilizes at (T, k) for each $j,\ 0 \le j \le i-1$, from the above equality it follows that $(\mathfrak{b}, x_0^t, \dots, x_{i-1}^t) : x_i^{k_i}$ stabilizes at (T, k). □

Recall that $\mathfrak{b}(i-1, T) := (\mathfrak{b}, x_0^T, \dots, x_{i-1}^T)$. The following result clarifies the relationship between the stabilization considered in the previous section and being l-Buchsbaum for $l \gg 0$.

Theorem 3.3. *Assume that the ideal $\mathfrak{a}_j(t) := (\mathfrak{b}, x_j^t)$ stabilizes at (T, k) with respect to x_i for all $0 \le i \ne j \le d$ and for some $T \ge 2k$. Then $\mathfrak{b}$ is l-Buchsbaum for $l \gg 0$ if and only if the following conditions are satisfied*:

$$\mathfrak{b}(i-1, T) : x_i^k \subseteq \mathfrak{b}(i-1, T) : x_j^k \qquad \text{for all} \quad 0 \le i \ne j \le d.$$

In this case $\mathfrak{b}$ is already γ-Buchsbaum, where γ is the least integer such that $\mathfrak{m}^\gamma \subseteq (x_0^k, \dots, x_d^k) + \mathfrak{b}$.

Proof. $\Leftarrow$. By the stabilization and Proposition 3.2, $\mathfrak{b}(i-1, T) : x_i^k = \mathfrak{b}(i-1, T) : x_i^{k+1}$. We also have

$$\begin{aligned}
\mathfrak{b} : x_i^k &= (\cap_{t \ge T} (\mathfrak{b}, x_j^t)) : x_i^k = \cap_{t \ge T} ((\mathfrak{b}, x_j^t) : x_i^k) \\
&= \cap_{t \ge T} ((\mathfrak{b}, x_j^t) : x_i^{k+1}) \qquad \text{(by stabilization)} \\
&= (\cap_{t \ge T} (\mathfrak{b}, x_j^t)) : x_i^{k+1} = \mathfrak{b} : x_i^{k+1}.
\end{aligned}$$

Thus $\mathfrak{b}(i-1, T) : x_i^k = \mathfrak{b}(i-1, T) : x_i^{k+1}$ holds for all $0 \le i \le d$ (where $\mathfrak{b}(-1, T) := \mathfrak{b}$), i.e. the condition a) of Theorem 1.6 is satisfied. $\mathfrak{b}(i-1, T)$:

$x_i^k \subseteq \mathfrak{b}(i-1,T) : x_j^k$ for all $0 \le i \ne j \le d$ is exactly b) in Theorem 1.6. Hence $\mathfrak{b}$ is γ-Buchsbaum with γ as above.

$\Rightarrow$. Assume $\mathfrak{b}$ is l-Buchsbaum. We may assume that $l \ge k$. and $2l \ge T$. Then for all $0 \le i \ne j \le d$ we have

$$\begin{aligned}\mathfrak{b}(i-1,2l) : x_i^k &= \mathfrak{b}(i-1,2l) : x_i^{2l} \qquad \text{(by stabilization)}\\ &\subseteq \mathfrak{b}(i-1,2l) : \mathfrak{m}^l \qquad \text{(by Theorem 1.3 (ii))}\\ &\subseteq \mathfrak{b}(i-1,2l) : x_j^l,\ 0 \le j \le d\\ &\subseteq \mathfrak{b}(i-1,2l) : x_j^k,\ 0 \le j \le d \qquad \text{(by stabilization)}.\end{aligned}$$

As shown in the proof of Theorem 1.6, this implies that $(x_0^k, \dots, x_d^k)H_{\mathfrak{m}}^i(R_{r+1}/\mathfrak{b}) = 0,\ 0 \le i < d$. Again by Proposition 13 in the Appendix of [SV], we get $\mathfrak{b}(i-1,T) : x_i^k \subseteq \mathfrak{b}(i-1,T) : x_j^k$ (since $T \ge 2k$). □

To formulate the following result we need some more notation. For all $i, j,\ 0 \le j < i$, $\{B_{ij} \cup M_{ij}, G_{ij}(t)\}$, $G_{ij}(t) \subseteq T_{r+1}$, is a reduced Gröbner basis of $\mathfrak{a}_j(t) = (\mathfrak{b}, x_j^t)$ with respect to rev. lex. and $x_j > x_i$ as smalles linear terms, for $t \ge \max\{\delta(b);\ b \in B_{ij} \cup M_{ij}\}$ as specified in Lemma 2.6. (Note: the order depends on i and j, so Gröbner bases also depend on i and j.) Under an additional assumption, the condition in the previous theorem can be checked as follows:

Proposition 3.4. *Let $\mathfrak{b}$ be a binomial ideal such that $\mathfrak{b} : x_i = \mathfrak{b}$ for all $0 \le i \le d$. Assume that the ideal $\mathfrak{a}_j(t) := (\mathfrak{b}, x_j^t)$ stabilizes at (T, k) with respect to x_i for all $0 \le i \ne j \le d$ for some T, k. For a fixed h, $0 \le h \ne i \le d$, the following are equivalent*:

(i) $\mathfrak{b}(i-1,T) : x_i^k \subseteq \mathfrak{b}(i-1,T) : x_h^k$.

(ii) $G_{ij}(T)|_{x_i=1} \subseteq (\mathfrak{b}(i-1,T) : x_h^k),\ 0 \le j \le i-1$. *Here $G_{ij}(T)|_{x_i=1}$ means the variable x_i is replaced by 1 in all monomials of $G_{ij}(T)$.*

Proof. We start by proving

Claim. $\bar{\mathcal{M}}_j(T)|_{x_i=1} \subseteq \mathfrak{b}(i-1,T) : x_h^k,\ 0 \le j \le i-1$ iff $G_{ij}(T)|_{x_i=1} \subseteq \mathfrak{b}(i-1,T) : x_h^k,\ 0 \le j \le i-1$, where $\bar{\mathcal{M}}_j(T)$ denotes the ideal generated by all monomials in $(\mathfrak{b}, x_j^T)$.

$\Rightarrow$. This follows since $G_{ij}(T)$ is contained in $\bar{\mathcal{M}}_j(t)$.

$\Leftarrow$. Let $m \in \bar{\mathcal{M}}_j(T)$. If $\delta_{x_i}(m) = 0$, then

$$m|_{x_i=1} = m \in (\mathfrak{b}, x_j^T) \subseteq \mathfrak{b}(i-1,T) \subseteq \mathfrak{b}(i-1,T) : x_h^k,$$

(since $j \le i-1$). Assume $\delta_{x_i}(m) > 0$, and $m \overset{B_{ij}}{\to} \tilde{m}$, $\tilde{m}$ irr. mod B_{ij}. Then there exists $\bar{m} \in G_{ij}(T) \cup M_{ij}$ such that $\bar{m} | \tilde{m}$ (since $\tilde{m} \in \mathfrak{a}_j(T)$ and $\{B_{ij} \cup M_{ij}, G_{ij}(t)\}$ is a reduced Gröbner basis of this ideal). By Lemma 2.9, $\delta_{x_i}(m) \le \delta_{x_i}(\tilde{m})$. If $\tilde{m} = q\bar{m}$, then

$$x_i^{\delta_{x_i}(m)}(m|_{x_i=1}) - x_i^{\delta_{x_i}(\tilde{m})}[(q|_{x_i=1})(\bar{m}|_{x_i=1})] = m - \tilde{m} \in (B_{ij}) \subseteq \mathfrak{b}.$$

Since $\mathfrak{b} : x_i = \mathfrak{b}$, we have

$$m|_{x_i=1} - x_i^{\delta_{x_i}(\bar{m})-\delta_{x_i}(m)}[(q|_{x_i=1})(\bar{m}|_{x_i=1})] \in \mathfrak{b}.$$

If $x_i^{\delta_{x_i}(\bar{m})}(\bar{m}|_{x_i=1}) = \bar{m} \in M_{ij} \subseteq \mathfrak{b}$, then again $\bar{m}|_{x_i=1} \in \mathfrak{b} \subseteq \mathfrak{b}(i-1,T)$. Otherwise $\bar{m} \in G_{ij}(T)$, and by induction assumption $x_h^k(\bar{m}|_{x_i=1}) \in \mathfrak{b}(i-1,T)$. Therefore, in both cases, $x_h^k(m|_{x_i=1}) \in \mathfrak{b}(i-1,T)$, as required.

Proof of Proposition 3.4. Since $\sum_{j=0}^{i-1} \bar{\mathcal{M}}_j(T)$ is the ideal of all monomials in $\mathfrak{b}(i-1,T)$ and $\mathfrak{b} : x_i = \mathfrak{b}$, we have:

$$\mathfrak{b}(i-1,T) : x_i^k \subseteq \mathfrak{b}(i-1,T) : x_h^k \tag{2}$$

iff

$$\begin{aligned}\mathfrak{b}(i-1,T) &\supseteq x_h^k(\mathfrak{b}(i-1,T) : x_i^k) \\ &= x_h^k\left[\left(\mathfrak{b}, \sum_{j=0}^{i-1} \bar{\mathcal{M}}_j(T)\right) : x_i^k\right] \\ &= x_h^k\left[\mathfrak{b} + \sum_{j=0}^{i-1}(\bar{\mathcal{M}}_j(T) : x_i^k)\right] \qquad \text{(by Lemma 3.1 and [KR, Satz 5])}\end{aligned}$$

iff

$$x_h^k(\bar{\mathcal{M}}_j(T) : x_i^k) \subseteq \mathfrak{b}(i-1,T) \qquad \text{for} \quad \text{all } 0 \le j \le i-1 \tag{3}$$

(since $x_h^k\mathfrak{b} \in \mathfrak{b}(i-1,T)$). By stabilization, we may also replace $\bar{\mathcal{M}}_j(T) : x_i^k$ in (3) by $\bar{\mathcal{M}}_j(T) : x_i^\infty = \bar{\mathcal{M}}_j(T)|_{x_i=1}$. Hence, by the initial claim, (2) is equivalent to $G_{ij}(T)|_{x_i=1} \subseteq \mathfrak{b}(i-1,T) : x_h^k$ for all $0 \le j \le i-1$. □

4. Local cohomology and stabilization

The rest of this paper is devoted to the termination of Algorithm A. For short we also use R to denote R_{r+1}. For a homogeneous ideal $\mathfrak{c} \subseteq R$, denote

$$a_i(R/\mathfrak{c}) = \begin{cases} \max\{t;\ [H^i_{\mathfrak{m}}(R/\mathfrak{c})]_t \neq 0\} & \text{if} \quad H^i_{\mathfrak{m}}(R/\mathfrak{c}) \neq 0, \\ -\infty & \text{if} \quad H^i_{\mathfrak{m}}(R/\mathfrak{c}) = 0, \end{cases}$$

where $[\cdot]_t$ denotes the t-th graded part. As usual, $\dim \mathfrak{c} = d+1 \ge 2$ and $\mathfrak{c} \neq 0$.

Lemma 4.1. *There exist $z_0, \dots, z_d \in [R]_1$ such that $\alpha_0 z_0 + \cdots + \alpha_d z_d$ is a parameter element for $\mathfrak{c}$ for all $(\alpha_0, \dots, \alpha_d) \in K^{d+1} \setminus \{(0,\dots,0)\}$.*

Proof. The vector space $[R]_1 = Kx_0 \oplus \cdots \oplus Kx_r$ has dimension $r+1 > d+1$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be all highest dimension associated prime ideals of $R/\mathfrak{c}$. Then $\mathfrak{p}_1 \cap [R]_1, \dots, \mathfrak{p}_s \cap [R]_1$ are proper linear subspaces of $[R]_1$ of dimension at most $r-d$. Since K is infinite, one can find a subspace $H \subset [R]_1$ of dimension $d+1$ such that $H \cap \mathfrak{p}_1 = \cdots = H \cap \mathfrak{p}_s = 0$. Any basis $z_0, \dots, z_d$ of H will satisfy the conclusion of the lemma. $\square$

A special case (when $d = 1$) of the following result is Proposition 2.8 in [M], which was proved by a different method.

Proposition 4.2. *Assume that $R/\mathfrak{c}$ is a generalized Cohen-Macaulay ideal of dimension $d+1 \geq 2$ and $n \leq 0$. Then*

$$\dim_K [H^1_{\mathfrak{m}}(R/\mathfrak{c})]_{n-1} \leq \max\{0, \dim_K [H^1_{\mathfrak{m}}(R/\mathfrak{c})]_n - d\}.$$

Proof. Choose a s.o.p. $z_0, \dots, z_d \in [R]_1$ of $R/\mathfrak{c}$ as in Lemma 4.1. Let $(\alpha_0, \dots, \alpha_d) \in K^{d+1} \setminus \{(0, \dots, 0)\}$ and $z = \alpha_0 z_0 + \cdots + \alpha_d z_d$. Then z is a parameter element for $\mathfrak{c}$, and by Definition 1.2, $\dim_K(0 :_{R/\mathfrak{c}} z) < \infty$. Since $[H^0_{\mathfrak{m}}(R/(\mathfrak{c}, z))]_n = 0$ for $n \leq 0$, from the exact sequence

$$0 \to R/(\mathfrak{c} : z)(-1) \xrightarrow{\cdot z} R/\mathfrak{c} \to R/(\mathfrak{c}, z) \to 0,$$

we get an injective map

$$0 \to [H^1_{\mathfrak{m}}(R/\mathfrak{c})]_{n-1} \xrightarrow{\cdot z} [H^1_{\mathfrak{m}}(R/\mathfrak{c})]_n$$

for all $n \leq 0$ and all z. By [Br1, Lemma 3.1], we get

$$\dim_K [H^1_{\mathfrak{m}}(R/\mathfrak{c})]_{n-1} \leq \max\{0, \dim_K [H^1_{\mathfrak{m}}(R/\mathfrak{c})]_n - d\}.$$

$\square$

Corollary 4.3. *Under the assumptions of Proposition 4.2,*

$$[H^1_{\mathfrak{m}}(R/\mathfrak{c})]_n = 0 \qquad \text{for} \quad \text{all } n \leq -\frac{\dim_K [H^1_{\mathfrak{m}}(R/\mathfrak{c})]_0}{d}.$$

We would like to mention that Brodmann already gave in [Br2], Theorem 5.6 a priori lower bound for the vanishing of $[H^1_{\mathfrak{m}}(R/\mathfrak{c})]_n$ in negative degrees. His bound, which works under a much weaker assumption, is worse than the above bound.

Lemma 4.4. *Let α be an arbitrary integer such that*

$$\alpha \geq \max\{a_0(R/\mathfrak{c}), a_1(R/\mathfrak{c})\} + 1,$$

and $z \in [R]_\alpha$ be a parameter element of $R/\mathfrak{c}$ such that $\dim_K(0 :_{R/\mathfrak{c}} z) < \infty$. Then

$$\dim_K [H^1_{\mathfrak{m}}(R/\mathfrak{c})]_0 = \dim_K \left[\frac{(\mathfrak{c}, z) : \mathfrak{m}^\infty}{(\mathfrak{c}, z)} \right]_\alpha.$$

Proof. Note that $[H^0_{\mathfrak{m}}(R/\mathfrak{c})]_\alpha = [H^1_{\mathfrak{m}}(R/\mathfrak{c})]_\alpha = 0$. Hence, from the exact sequence

$$0 \to R/(\mathfrak{c}:z)(-\alpha) \xrightarrow{\cdot z} R/\mathfrak{c} \to R/(\mathfrak{c},z) \to 0,$$

we get an isomorphism

$$[H^0_{\mathfrak{m}}(R/(\mathfrak{c},z))]_\alpha \cong [H^1_{\mathfrak{m}}(R/\mathfrak{c})]_0.$$

By definition

$$[H^0_{\mathfrak{m}}(R/(\mathfrak{c},z))]_\alpha = \left[\frac{(\mathfrak{c},z):\mathfrak{m}^\infty}{(\mathfrak{c},z)}\right]_\alpha,$$

which completes the proof. □

Recall that the Castelnuovo-Mumford regularity of $R/\mathfrak{c}$ is the number

$$\operatorname{reg}(R/\mathfrak{c}) = \max\{a_i(R/\mathfrak{c}) + i;\ 0 \le i \le d+1\}.$$

This invariant can be computed using a minimal free resolution of $R/\mathfrak{c}$, thus via a Gröbner basis calculation (see [E, Chapter 20]). For the next theorem, the initial ideal is taken with respect to a rev. lex. term order with x_0 the smallest linear term. For an integer a, let $\beta(a)$ be the smallest integer such that

$$\beta(a) \ge \frac{1}{d}\dim_K\left[\frac{(\operatorname{in}(\mathfrak{b})|_{x_0=1})}{(\operatorname{in}(\mathfrak{b}) + x_1^a(\operatorname{in}(\mathfrak{b})|_{x_0=1} : x_1^a))}\right]_a.$$

Theorem 4.5. *Let $\mathfrak{b}$ be a binomial ideal as in Section 2. Assume that $\mathfrak{b}$ is a generalized Cohen-Macaulay ideal. Let*

$$k = \beta(\operatorname{reg}(R/\mathfrak{b}) + 1) + 2\operatorname{reg}(R/\mathfrak{b}).$$

Then

(i) $\mathfrak{a}_j(t) = (\mathfrak{b}, x_j^t)$ *stabilizes at* $(1,k)$ *w.r.t.* x_i *for all* $0 \le i \ne j \le d$, *and*

(ii) $\mathfrak{b}$ *is a k-Buchsbaum ideal.*

The same conclusions remain true for $k' = \beta((r+1)(D-1)+1) + 2(r+1)(D-1)$, where D is the maximal degree of a reduced Gröbner base of $\mathfrak{b}$ (w.r.t. any term order).

Proof. For short, let $a_i = a_i(R/\mathfrak{b})$, $i = 0,1$. Then $a_0 \le \operatorname{reg}(R/\mathfrak{b})$ and $a_1 + 1 \le \operatorname{reg}(R/\mathfrak{b})$. By [HT, Corollary 1.3], we also have $\operatorname{reg}(R/\mathfrak{b}) \le (r+1)(D-1)$. On the other hand, let $a \ge 1$ be any integer. Since $\mathfrak{b}$ is a generalized Cohen-Macaulay ideal, $\dim_K(0 :_{R/\mathfrak{b}} x_1^a) < \infty$, and $(\mathfrak{b}, x_1^a) : \mathfrak{m}^\infty = (\mathfrak{b}, x_1^a) : x_0^\infty$. Recall that x_0 is the smallest term. By [E, Proposition 15.12], we have

$$\begin{aligned}\dim_K\left[\frac{(\mathfrak{b},x_1^a):\mathfrak{m}^\infty}{(\mathfrak{b},x_1^a)}\right]_a &= \dim_K\left[\frac{(\mathfrak{b},x_1^a):x_0^\infty}{(\mathfrak{b},x_1^a)}\right]_a \\ &= \dim_K\left[\frac{(\operatorname{in}(\mathfrak{b}):x_0^\infty, x_1^a)}{(\operatorname{in}(\mathfrak{b}),x_1^a)}\right]_a \\ &= \dim_K\left[\frac{(\operatorname{in}(\mathfrak{b})|_{x_0=1})}{(\operatorname{in}(\mathfrak{b}) + x_1^a(\operatorname{in}(\mathfrak{b})|_{x_0=1} : x_1^a))}\right]_a.\end{aligned}$$

Hence by Lemma 4.4,

$$\beta(\operatorname{reg}(R/\mathfrak{b})+1) = \beta((r+1)(D-1)+1) \geq \frac{1}{d}\dim_K[H^1_{\mathfrak{m}}(R/\mathfrak{b})]_0.$$

In particular $k' \geq k$ and we only have to show (i) and (ii) for k.

By Lemma 4.3 we get $[H^1_{\mathfrak{m}}(R/\mathfrak{b})]_n = 0$ for all $n \leq -\beta(\operatorname{reg}(R/\mathfrak{b})+1)$. By definition of a_1, $[H^1_{\mathfrak{m}}(R/\mathfrak{b})]_n = 0$ for all $n \geq a_1+1$. Hence

$$\mathfrak{m}^{\beta(\operatorname{reg}(R/\mathfrak{b})+1)+a_1+1} H^1_{\mathfrak{m}}(R/\mathfrak{b}) = 0. \tag{4}$$

Since $[H^0_{\mathfrak{m}}(R/\mathfrak{b})]_n = 0$ for all $n \leq 0$,

$$\mathfrak{m}^{a_0} H^0_{\mathfrak{m}}(R/\mathfrak{b}) = 0. \tag{5}$$

Let $t \geq 1$ and $0 \leq j \leq d$. From the exact sequence

$$0 \to R/(\mathfrak{b} : x_j^t) \xrightarrow{\cdot x_j^t} R/\mathfrak{b} \to R/(\mathfrak{b}, x_j^t) \to 0,$$

we get an exact sequence

$$H^0_{\mathfrak{m}}(R/\mathfrak{b}) \to H^0_{\mathfrak{m}}(R/(\mathfrak{b}, x_j^t)) \to H^1_{\mathfrak{m}}(R/\mathfrak{b}). \tag{6}$$

Since $\beta(\operatorname{reg}(R/\mathfrak{b})+1)+a_1+1+a_0 \leq k$, (4), (5) and (6) imply $\mathfrak{m}^k H^0_{\mathfrak{m}}(R/(\mathfrak{b}, x_j^t)) = 0$. Therefore

$$(\mathfrak{b}, x_j^t) : \mathfrak{m}^k = (\mathfrak{b}, x_j^t) : \mathfrak{m}^\infty. \tag{7}$$

Note that $(\mathfrak{b}, x_j^t)$ is also a generalized Cohen-Macaulay ideal. By Definition 1.2 and (7) for any $0 \leq i \neq j \leq d$ the following holds:

$$(\mathfrak{b}, x_j^t) : \mathfrak{m}^k \subseteq (\mathfrak{b}, x_j^t) : x_i^k \subseteq (\mathfrak{b}, x_j^t) : x_i^\infty \subseteq (\mathfrak{b}, x_j^t) : \mathfrak{m}^\infty \subseteq (\mathfrak{b}, x_j^t) : \mathfrak{m}^k. \tag{8}$$

Hence $(\mathfrak{b}, x_j^t) : x_i^k = (\mathfrak{b}, x_j^t) : x_i^\infty$, i.e. $\mathfrak{a}_j(t) = (\mathfrak{b}, x_j^t)$ stabilizes at $(1,k)$. Thus (i) is proven.

Finally let $t = 2k$. In the proof of Proposition 3.2 we have shown that

$$\mathfrak{b}(i-1,t) : x_i^t = \sum_{j=0}^{i-1} (\mathfrak{b}, x_j^t) : x_i^t \subseteq \sum_{j=0}^{i-1} (\mathfrak{b}, x_j^t) : x_i^k.$$

By (8) we can conclude that $\mathfrak{m}^k(\mathfrak{b}(i-1,t) : x_i^t) \subseteq \mathfrak{b}(i-1,t)$ for all $0 \leq i \leq d$. Using Theorem 1.3, we then get (ii). □

Note that in the above theorem all parameters can be computed via Gröbner bases. Under an additional assumption we get the following nice result.

Theorem 4.6. *Let $\mathfrak{b}$ be as in the above theorem. Moreover assume that it is reduced. Then*

(i) $\mathfrak{a}_j(t) = (\mathfrak{b}, x_j^t)$ *stabilizes at* $(1, \mathrm{reg}(R/\mathfrak{b}))$ *w.r.t.* x_i *for all* $0 \le i \ne j \le d$, *and*

(ii) $\mathfrak{b}$ *is a* $\mathrm{reg}(R/\mathfrak{b})$*-Buchsbaum ideal.*

Proof. $\mathfrak{b}$ reduced implies $H^0_{\mathfrak{m}}(R/\mathfrak{b}) = 0$. Moreover, by [HSV, Lemma 1 (ii) (a)], $[H^1_{\mathfrak{m}}(R/\mathfrak{b})]_n = 0$ for all $n < 0$ and $n \ge \mathrm{reg}(R/\mathfrak{b})) \ge a_1 + 1$. Thus

$$\mathfrak{m}^{\mathrm{reg}(R/\mathfrak{b})} H^1_{\mathfrak{m}}(R/\mathfrak{b}) = 0.$$

The exact sequence (6) even gives an injection:

$$0 \to H^0_{\mathfrak{m}}(R/(\mathfrak{b}, x_j^t)) \to H^1_{\mathfrak{m}}(R/\mathfrak{b}).$$

From this

$$(\mathfrak{b}, x_j^t) : \mathfrak{m}^{\mathrm{reg}(R/\mathfrak{b})} = (\mathfrak{b}, x_j^t) : \mathfrak{m}^{\infty},$$

and we can repeat the last part of the above proof. □

In spite of this theorem, it would be nice to have a similar result for arbitrary generalized Cohen-Macaulay homogeneous ideals.

Now we can state the main two theorems of this paper.

Theorem 4.7. *If Algorithm A applied to any* $\mathfrak{a}_j(t)$, $0 \le j \le d$, *does not stop after* $\beta(\mathrm{reg}(R/\mathfrak{b}) + 1) + 2\,\mathrm{reg}(R/\mathfrak{b}) + 1$ (*or* $\beta((r+1)(D-1)+1) + 2(r+1)(D-1)+1$) *steps, then* $\mathfrak{b}$ *is not a* l*-Buchsbaum ideal for any* l. *Moreover, if* $\mathfrak{b}$ *is reduced, only* $\mathrm{reg}(R/\mathfrak{b}) + 1$ (*or* $2(r+1)(D-1)+1$) *steps are required.*

Proof. Fix an index j, $0 \le j \le d$. For simplicity we use the same notation as in Section 2 for $\mathfrak{a}_j(t)$, namely $\mathfrak{a}(t) = \mathfrak{a}_j(t)$. If $\mathfrak{b}$ is a l-Buchsbaum ideal for some l, then by Theorem 4.5, $\mathfrak{a}_j(t)$ stabilizes at $(1, k)$, where

$$k = \beta(\mathrm{reg}(R/\mathfrak{b}) + 1) + 2\,\mathrm{reg}(R/\mathfrak{b}).$$

By Theorems 2.14 and Lemma 2.10, $\delta_y(m) \ge \Delta_y$ for all $m \in G(\Delta + \Delta_y(k+1))$. Thus Algorithm A must stop not later than the $(k+1)$-st step.

Similarly, using Theorems 4.5 and 4.6, one can get other statements. □

This theorem together with Theorem 3.3 implies

Theorem 4.8. *Assume* $\mathfrak{b}$ *be as in Section* 2. *There exists an algorithm to determine if* $\mathfrak{b}$ *is* ℓ*-Buchsbaum for some* ℓ. *In this case the algorithm also gives the smallest value of such* ℓ.

Following Algorithm A we have to do many Gröbner bases calculation, if $\mathfrak{a}_j$ does not stabilize for some j. One can avoid it by using the following result. However here also the calculations could become too large.

Proposition 4.9. *Assume the notation of Theorem* 4.5. *Let* e *denote the multiplicity of* $R/\mathfrak{b}$. *Then* $\mathfrak{b}$ *is a generalized Cohen-Macaulay ideal iff*

$$(9) \qquad \ell(R/(\mathfrak{b}, x_0^{4k}, \ldots, x_d^{4k})) - \ell(R/(\mathfrak{b}, x_0^{2k}, \ldots, x_d^{2k})) = (2k)^{d+1}(2^{d+1} - 1)e.$$

Proof. If (9) holds, then $x_0^{2k}, \ldots, x_d^{2k}$ is a so-called standard s.o.p. of $R/\mathfrak{b}$, and thus $\mathfrak{b}$ is generalized Cohen-Macaulay (see Theorem and Definition 17 in the Appendix of [SV]). Conversely, if $\mathfrak{b}$ is generalized Cohen-Macaulay, by Theorem 4.5 it is k-Buchsbaum. From Proposition 13 and Theorem and Definition 17 in the Appendix of [SV] it follows that $x_0^{2k}, \ldots, x_d^{2k}$ is a standard s.o.p., and therefore (9) is satisfied. □

Example 4.10. a) Theorem 4.8 is applicable to all simplicial semigroups. For this assume the prime ideal $\mathfrak{p} \subseteq R = K[x_0, \ldots, x_r]$ has generic zero as follows:

$$x_0 = t_0^{\alpha_{00}} \cdots t_d^{\alpha_{0d}}, \ldots, x_i = t_0^{\alpha_{i0}} \cdots t_d^{\alpha_{id}}, \ldots, x_r = t_0^{\alpha_{r0}} \cdots t_d^{\alpha_{rd}},$$

such that
(i) $r > d$,
(ii) $\sum_{j=0}^{d} \alpha_{ij} = D,\ 0 \leq i \leq r$,
(iii) There are exactly $d+1$ variables of the form $x_{j_h} = t_h^D,\ 0 \leq h \leq d$.

Then by [CLO] a generating set for $\mathfrak{p}$ is algorithmically defined, thus $\mathfrak{p}$ satisfies the conditions of Theorem 4.8. However in this case the theoretical part of Theorem 4.6 is not new, because we already know from the proof of Lemma 4.11 in [TH] that the local cohomology modules $H^i_{\mathfrak{m}}(R/\mathfrak{p}) = 0$ for all $i \neq 1, d+1$ iff $R/\mathfrak{p}$ is a generalized Cohen-Macaulay ideal, and $H^1_{\mathfrak{m}}(R/\mathfrak{p})$ may have only positive degrees. Moreover one can derive from the proof of Lemma 4.11 in [TH] a simple combinatorial characterization for $R/\mathfrak{p}$ to be a generalized Cohen-Macaulay ideal. Note that for this class of ideals there is a good bound on the Castelnuovo-Mumford regularity. Namely it was recently shown in [HS] that

$$\operatorname{reg}(R/\mathfrak{p}) \leq \dim(R/\mathfrak{p})(\operatorname{degree}(R/\mathfrak{p}) - \operatorname{codim}(R/\mathfrak{p}) - 2) + 3.$$

On the other hand if we modify $\mathfrak{p}$ slightly by adding some monomials or binomials not containing variables specialized in (iii) above, then one cannot apply the theory of affine semigroup rings, but Theorem 4.8 remains valid. In this case it is not clear how the Castelnuovo-Mumford regularity could be bounded.

b) Example 1.7 is an example of the following binomial ideals: $\mathfrak{b} = (B)$, B a binomial generating set. Assume $V_1 = \{x_{i_1}, \ldots, x_{i_k}\}$ and $V_2 = \{x_{j_{k+1}}, \ldots, x_{i_{j_{r+1}}}\}$ are disjoint sets of variables and B decomposes into two disjoint sets of binomials B_1 and B_2 such that:

1. $B_1 = \{b;\ b = x_{i_h}^{\alpha_{i_h}} - m_{i_h},\ 1 \leq h \leq k,\ x_{j_t} | m_{i_h}$ for some $j_t,\ k+1 \leq t \leq r+1\}$ is the set of binomials with a pure power term.
2. $b \in B_2 = B \setminus B_1$ implies $x_{j_t} \not| b,\ k+1 \leq t \leq r+1$.
3. Let $<_{j_t}$ be the term order rev. lex. with $x_{j_l} <_{j_t} x_{i_h},\ 1 \leq h \leq k,\ k+1 \leq l \leq r+1$, and x_{j_t} as smallest linear term. For every such $<_{j_t}$ and $b_1 = m_{11} - m_{12},\ b_2 = m_{21} - m_{22}$ in B either $\operatorname{in}(b_1)$ and $\operatorname{in}(b_2)$ are relatively prime or their g.c.d. d divides m_{12} and m_{22}, thus by 2. d is a term in $K[x_{i_1}, \ldots, x_{i_k}]$.

Then 1. implies V_1 is a s.o.p. for $\mathfrak{b}$, 3. implies B is a Gröbner basis since s-polynomials reduce to 0 and therefore by 2. $\mathfrak{b} : x_{j_t} = \mathfrak{b},\ k+1 \leq t \leq r+1$. So Proposition 3.4 could be applied.

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