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Weighted integral inequalities for differential forms

By

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Abstract

In this paper, we obtain some weighted integral inequalities for differential forms, which can be considered as generalizations of the Poincaré inequality, the Caccioppoli-type estimate, and the weak reverse Hölder inequality, respectively. These results can be used to study the integrability of differential forms and to estimate the integrals of differential forms. We also give some applications of the above results.

1. Introduction

Throughout this paper, we always assume that Ω is a connected open subset of $\mathbf{R^n}$ and write $\mathbf{R} = \mathbf{R^1}$. Balls are denoted by B and σB is the ball with the same center as B and with $diam\,(\sigma B) = \sigma diam(B)$. We do not distinguish balls from cubes throughout this paper. The n-dimensional Lebesgue measure of a set $E \subset \mathbf{R^n}$ is denoted by $|E|$. We call $w\,(x)$ a weight if $w \in L^1_{loc}\,(\mathbf{R^n})$ and $w > 0$ a.e.. For $0 < p < \infty$, we denote the weighted L^p-norm of a measurable function f over E by

$$\|f\|_{p.E.w^\alpha} = \left(\int_E |f(x)|^p w^\alpha dx\right)^{1/p},$$

where α is a real number.

A differential l-form ω on Ω is a Schwartz distribution on Ω with values in $\wedge^l\,(\mathbf{R^n})$. We denote the space of differential l-forms by $D'\left(\Omega, \wedge^l\right)$. We write $L^p(\Omega, \wedge^l)$ for the l-forms $\omega(x) = \sum_I \omega_I(x)dx_I = \sum \omega_{i_1 i_2 \cdots i_l}(x)dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$ with $\omega_I \in L^p(\Omega, \mathbf{R})$ for all ordered l-tuples I. Thus $L^p(\Omega, \wedge^l)$ is a Banach space with norm

$$\|\omega\|_{p,\Omega} = \left(\int_\Omega |\omega(x)|^p dx\right)^{1/p} = \left(\int_\Omega \left(\sum |\omega_I(x)|^2\right)^{p/2} dx\right)^{1/p}.$$

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Similarly, $W_p^1(\Omega, \wedge^l)$ are those differential l-forms on Ω whose coefficients are in $W_p^1(\Omega, \mathbf{R})$. The notations $W_{p,loc}^1(\Omega, \mathbf{R})$ and $W_{p,loc}^1(\Omega, \wedge^l)$ are self-explanatory. We denote the exterior derivative by

$$d : D^{'}(\Omega, \wedge^l) \to D^{'}(\Omega, \wedge^{l+1})$$

for $l = 0, 1, \ldots, n$. Its formal adjoint operator $d^\star : D^{'}(\Omega, \wedge^{l+1}) \to D^{'}(\Omega, \wedge^l)$ is given by $d^\star = (-1)^{nl+1} \star d \star$ on $D^{'}(\Omega, \wedge^{l+1})$, $l = 0, 1, \ldots, n$.

Differential forms ω is called an A-harmonic tensor if ω satisfies the A-harmonic equation

$$d^\star A(x, d\omega) = 0, \tag{1.1}$$

where $A : \Omega \times \wedge^l(\mathbf{R^n}) \to \wedge^l(\mathbf{R^n})$ satisfies the following conditions:

$$|A(x, \xi)| \le a|\xi|^{p-1} \tag{1.2}$$

and

$$\langle A(x, \xi), \xi \rangle \ge |\xi|^p \tag{1.3}$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbf{R^n})$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1.1). A solution to (1.1) is an element of the Sobolev space $W_{p,loc}^1\left(\Omega, \wedge^{l-1}\right)$ such that

$$\int_\Omega \langle A(x, d\omega), d\varphi \rangle = 0$$

for all $\varphi \in W_p^1\left(\Omega, \wedge^{l-1}\right)$ with compact support.

Iwaniec and Lutoborski prove the following result in [7]: Let $Q \subset \mathbf{R^n}$ be a cube or a ball. To each $y \in Q$ there corresponds a linear operator $K_y : C^\infty\left(Q, \wedge^l\right) \to C^\infty\left(Q, \wedge^{l-1}\right)$ defined by

$$(K_y\omega)(x; \xi_1, \ldots, \xi_l) = \int_0^1 t^{l-1}\omega(tx + y - ty; x - y, \xi_1, \ldots, \xi_{l-1})\, dt$$

and the decomposition

$$\omega = d(K_y\omega) + K_y(d\omega).$$

We define another linear operator $T_Q : C^\infty\left(Q, \wedge^l\right) \to C^\infty\left(Q, \wedge^{l-1}\right)$ by averaging K_y over all points y in Q

$$T_Q\omega = \int_Q \varphi(y) K_y \omega dy,$$

where $\varphi \in C_0^\infty(Q)$ is normalized by $\int_Q \varphi(y) = 1$. We define the l-form $\omega_Q \in D^{'}\left(Q, \wedge^l\right)$ by

$$\omega_Q = |Q|^{-1} \int_Q \omega(y), l = 0,$$

and

$$\omega_Q = d\left(T_Q\omega\right), l = 1, 2, \ldots, n$$

for all $\omega \in L^p\left(Q, \wedge^l\right)$, $1 \le p < \infty$.

2. $A_r(\lambda, \Omega)$-weighted Poincaré inequality

The following two $A_r(\lambda, \Omega)$-weights (or the two-weight) are introduced in [17]. And if we choose $w_1 = w_2$ in Definition 2.1, we obtain the usual $A_r(\lambda)$-weights introduced in [11]. See [11] for more properties of $A_r(\lambda)$-weights.

Definition 2.1. We say the weight $(w_1(x), w_2(x))$ satisfies the $A_r(\lambda, \Omega)$ condition for $r > 1$ and $0 < \lambda < \infty$, write $(w_1(x), w_2(x)) \in A_r(\lambda, \Omega)$, if $w_1(x) > 0, w_2(x) > 0$ a.e., and

$$\sup_B \left(\frac{1}{|B|}\int_B w_1^{\lambda} dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w_2}\right)^{1/(r-1)} dx\right)^{(r-1)} < C_{r,\lambda,w_1,w_2}$$

for any ball $B \subset\subset \Omega$.

By a direct computation, we get that

$$(w_1(x), w_2(x)) = (|x|^{\theta/\lambda}, |x|^{\theta})$$

is two $A_r(\lambda, \Omega)$-weights if and if only $-n < \theta < n(r-1)$. We will need the following generalized Hölder inequality.

Lemma 2.2. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$ and $s^{-1} = \alpha^{-1} + \beta^{-1}$. If f and g are measurable functions on $\mathbf{R^n}$, then*

$$\|fg\|_{s,\Omega} \le \|f\|_{\alpha,\Omega} \cdot \|g\|_{\beta,\Omega}$$

for any $\Omega \subset \mathbf{R^n}$.

The following weak reverse Hölder inequality appears in [9].

Lemma 2.3. *Let u be a differential form satisfying the A-harmonic equation* (1.1) *in Ω, $\sigma > 1$ and $0 < s, t < \infty$. Then there exists a constant C, depending only on s, t, a, p, n and σ, such that*

$$\|u\|_{s,B} \le C|B|^{(t-s)/st}\|u\|_{t,\sigma B}$$

for all balls or cubes B with $\sigma B \subset \Omega$.

Different versions of the Poincaré inequality have been established in the study of the Sobolev spaces of differential forms, (see [3], [7], [9]). The following version of the Poincaré inequality appears in [9].

Lemma 2.4. *Let $u \in D'(B, \wedge^l)$ and $du \in L^p(B, \wedge^{l+1})$. Then $u - u_B$ is in $W_p^1(B, \wedge^l)$ with $1 < p < \infty$ and*

$$\|u - u_B\|_{p,B} \le C(n,p)|B|^{1/n}\|du\|_{p,B}$$

for B a cube or a ball in $\mathbf{R^n}$, $l = 0, 1, \ldots, n$.

We now generalize Lemma 2.4 into the following two $A_r(\lambda, \Omega)$-weights Poincaré inequality for differential forms.

Theorem 2.5. *Let $u \in D'\left(\Omega, \wedge^l\right)$ be a differential form satisfying the A-harmonic equation (1.1) in a domain $\Omega \subset \mathbf{R^n}$ and $du \in L^s\left(\Omega, \wedge^{l+1}\right), l = 0, 1, \ldots, n$. Suppose that $(w_1(x), w_2(x)) \in A_r(\lambda, \Omega)$ for some $r > 1$ and $0 < \lambda < \infty$. If $0 < \alpha < 1, \sigma > 1$, and $s > \alpha(r-1)+1$, Then there exists a constant C, depending on $a, p, n, s, r, \sigma, \alpha, \lambda, w_1, w_2$, such that*

$$\left(\int_B |u - u_B|^s w_1^{\alpha\lambda} dx\right)^{1/s} \le C|B|^{1/n}\left(\int_{\sigma B} |du|^s w_2^{\alpha} dx\right)^{1/s} \tag{2.6}$$

for all balls B with $\sigma B \subset \Omega$. Here u_B is a closed form.

Note that (2.6) can be written as

$$\|u - u_B\|_{s,B,w_1^{\alpha\lambda}} \le C|B|^{1/n}\|du\|_{s,\sigma B,w_2^{\alpha}}. \tag{2.6$'$}$$

Proof. Let $t = s/(1-\alpha)$, then $1 < s < t$. Since $1/s = 1/t + (t-s)/st$, by Lemma 2.2, we have

$$\begin{aligned} \left(\int_B |u - u_B|^s w_1^{\alpha\lambda} dx\right)^{1/s} &= \left(\int_B \left(|u - u_B| w_1^{\alpha\lambda/s}\right)^s dx\right)^{1/s} \\ &\le \|u - u_B\|_{t,B}\left(\int_B w_1^{\alpha\lambda t/(t-s)} dx\right)^{(t-s)/st} \\ &= \|u - u_B\|_{t,B}\left(\int_B w_1^{\lambda} dx\right)^{\alpha/s}. \end{aligned} \tag{2.7}$$

Taking $m = s/(\alpha(r-1)+1)$, we find that $m > 1$ and $m < s < t$. Since u_B is a closed form, by Lemma 2.3 and Lemma 2.4, we find that

$$\begin{aligned} \|u - u_B\|_{t,B} &\le C_1(s, a, p, n, \sigma, \alpha, r)|B|^{(m-t)/mt}\|u - u_B\|_{m,\sigma B} \\ &\le C_2(s, a, p, n, \sigma, \alpha, r)|B|^{(m-t)/mt}|B|^{1/n}\|du\|_{m,\sigma B} \end{aligned} \tag{2.8}$$

for all balls B with $\sigma B \subset \Omega$. Now $1/m = 1/s + (s-m)/sm$. By Lemma 2.2

again, we obtain

$$
\begin{aligned}
\|du\|_{m,\sigma B} &= \left(\int_{\sigma B} |du|^m dx\right)^{1/m} \\
&= \left(\int_{\sigma B} \left(|du| w_2^{\alpha/s} w_2^{-\alpha/s}\right)^m dx\right)^{1/m} \\
&\le \left(\int_{\sigma B} |du|^s w_2^{\alpha} dx\right)^{1/s} \left(\int_{\sigma B} (1/w_2)^{\alpha m/(s-m)} dx\right)^{(s-m)/sm} \\
&= \left(\int_{\sigma B} |du|^s w_2^{\alpha} dx\right)^{1/s} \left(\int_{\sigma B} (1/w_2)^{1/(r-1)} dx\right)^{\alpha(r-1)/s} .
\end{aligned}
\tag{2.9}
$$

Combining (2.7), (2.8), and (2.9), we have

$$
\begin{aligned}
&\left(\int_B |u-u_B|^s w_1^{\alpha\lambda} dx\right)^{1/s} \\
&\quad \le C_2 |B|^{(m-t)/mt} |B|^{1/n} \|w_1\|_{\lambda,B}^{\alpha\lambda/s} \|1/w_2\|_{1/(r-1),\sigma B}^{\alpha/s} \\
&\qquad \times \left(\int_{\sigma B} |du|^s w_2^{\alpha} dx\right)^{1/s} .
\end{aligned}
\tag{2.10}
$$

Since $(w_1(x), w_2(x)) \in A_r(\lambda, \Omega)$, then

$$
\begin{aligned}
&\left(\int_B w_1^{\lambda} dx\right)^{\alpha/s} \left(\int_{\sigma B} (1/w_2)^{1/(r-1)} dx\right)^{\alpha(r-1)/s} \\
&\le \left(\left(\int_{\sigma B} w_1^{\lambda} dx\right)\left(\int_{\sigma B} (1/w_2)^{1/(r-1)} dx\right)^{r-1}\right)^{\alpha/s} \\
&= \left(|\sigma B|^r \left(\frac{1}{|\sigma B|}\int_{\sigma B} w_1^{\lambda} dx\right)\left(\frac{1}{|\sigma B|}\int_{\sigma B} (1/w_2)^{1/(r-1)} dx\right)^{r-1}\right)^{\alpha/s} \\
&\le C_3(r,\lambda,w_1,w_2)|\sigma B|^{\alpha r/s} \\
&\le C_4(r,\lambda,w_1,w_2,\sigma,\alpha,s)|B|^{\alpha r/s}.
\end{aligned}
\tag{2.11}
$$

Substituting (2.11) in (2.10) and noting $(m-t)/mt = -\alpha r/s$, we obtain

$$
\left(\int_B |u-u_B|^s w_1^{\alpha\lambda} dx\right)^{1/s} \le C_5 |B|^{1/n} \left(\int_{\sigma B} |du|^s w_2^{\alpha} dx\right)^{1/s} .
$$

Where C_5 depends on $a, p, n, s, r, \sigma, \alpha, \lambda, w_1, w_2$. We have completed the proof of Theorem 2.5. □

3. $A_{r,\lambda}(\Omega)$-weighted Caccioppoli-type inequality

The following two $A_{r,\lambda}(\Omega)$-weights (or the two-weight), which can be considered as an extension of the usual A_r-weights [5], appear in [13]. Also, see [14], [15] for more applications of two $A_{r,\lambda}(\Omega)$-weights.

Definition 3.1. We say a pair of weights $(w_1(x), w_2(x))$ satisfy the $A_{r,\lambda}(\Omega)$-condition , write $(w_1(x), w_2(x)) \in A_{r,\lambda}(\Omega)$, for some $0 < \lambda < \infty$ and $1 < r < \infty$ with $1/r + 1/r' = 1$ in a domain $\Omega \subset \mathbf{R^n}$ if $w_1(x) > 0, w_2(x) > 0$ a.e. and

$$\sup_B \left(\frac{1}{|B|} \int_B w_1^\lambda dx \right)^{1/\lambda r} \left(\frac{1}{|B|} \int_B \left(\frac{1}{w_2} \right)^{\lambda r'/r} dx \right)^{1/\lambda r'} < C_{\lambda,r,w_1,w_2}$$

for any ball $B \subset\subset \Omega$.

As two special examples of $A_{r,\lambda}(\Omega)$-weights we know that the weights $(w_1(x), w_2(x)) = (|x|^\delta, |x|^\delta)$ are in $A_{r,\lambda}(\Omega)$ if and if only $-n/\lambda < \delta < n(r-1)/\lambda$ and the weights $(w_1(x), w_2(x)) = (|x|^{\delta_1}, |x|^{\delta_2})$ are in $A_{r,\lambda}(\Omega)$ if Ω is a bounded domain and $\delta_1 > -n/\lambda, \delta_2 < \min(\delta_1, n(r-1)/\lambda)$. These are easily verified by a direct computation.

we will need the following local Caccioppoli-type estimate for differential forms appearing in [9].

Lemma 3.2. *Let $u \in D'(\Omega, \wedge^l)$ be a differential form satisfying the A-harmonic equation* (1.1) *in a domain $\Omega \in \mathbf{R^n}$, $l = 0, 1, \ldots, n$, and $\sigma > 1$. Let $1 < s < \infty$ is a fixed exponent associated with the A-harmonic equation* (1.1). *Then there exists a constant C, depending only on a, s, n, such that*

$$\|du\|_{s,B} \leq C diam(B)^{-1} \|u - c\|_{s,\sigma B}$$

for all balls or cubes B with $\sigma B \subset \Omega$ and all closed forms c.

We now prove the following two $A_{r,\lambda}(\Omega)$-weights Caccioppoli-type inequality for differential forms.

Theorem 3.3. *Let $u \in D'(\Omega, \wedge^l)$ be a differential form satisfying the A-harmonic equation* (1.1) *in a domain $\Omega \in \mathbf{R^n}$, $l = 0, 1, \ldots, n$,. Assume that $\rho > 1$ and $(w_1(x), w_2(x)) \in A_{r,\lambda}(\Omega)$ for some $1 < r < \infty$ and $0 < \lambda < \infty$ with $1/r + 1/r' = 1$. If $1 < s < \infty$ is a fixed exponent associated with the A-harmonic equation* (1.1). *Then there exists a constant C, depending on $a, s, n, r, \lambda, w_1, w_2, \beta, \rho$, but independent of u, such that*

$$\left(\int_B |du|^s w_1^\beta dx \right)^{1/s} \leq C diam(B)^{-1} \left(\int_{\rho B} |u - c|^s w_2^\beta dx \right)^{1/s} \tag{3.4}$$

for all balls B with $\rho B \subset \Omega$ and all closed forms c and any real number β with $0 < \beta < \lambda$.

Proof. Choose $t = \lambda s/(\lambda - \beta)$, then $1 < s < t$. Since $1/s = 1/t + (t-s)/st$, by the Hölder inequality and Lemma 3.2, we have

(3.5)
$$\begin{aligned}\left(\int_B |du|^s w_1^\beta dx\right)^{1/s} &= \left(\int_B \left(|du| w_1^{\beta/s}\right)^s dx\right)^{1/s} \\ &\le \|du\|_{t,B}\left(\int_B w_1^{\beta t/(t-s)} dx\right)^{(t-s)/st} \\ &\le C_1(a,s,n) diam(B)^{-1}\|u-c\|_{t,\sigma B}\left(\int_B w_1^\lambda dx\right)^{\beta/\lambda s}\end{aligned}$$

for all balls B with $\sigma B \subset \Omega$ and all closed form c. Since c is a closed form. Then, taking $m = \lambda s/(\lambda + \beta(r-1))$, we find that $m < s < t$. Applying Lemma 2.3 yields

$$\begin{aligned}\|u-c\|_{t,\sigma B} &\le C_2(a,n,s,\lambda,r,\beta,\rho)|B|^{(m-t)/mt}\|u-c\|_{m,\sigma^2 B} \\ &= C_2(a,n,s,\lambda,r,\beta,\rho)|B|^{(m-t)/mt}\|u-c\|_{m,\rho B},\end{aligned} \tag{3.6}$$

where $\rho = \sigma^2$. Substituting (3.6) in (3.5), we have

$$\begin{aligned}\left(\int_B |du|^s w_1^\beta dx\right)^{1/s} &\le C_3 diam(B)^{-1}|B|^{(m-t)/mt}\|u-c\|_{m,\rho B} \\ &\quad \times \left(\int_B w_1^\lambda dx\right)^{\beta/\lambda s}.\end{aligned} \tag{3.7}$$

Where the constant C_3 depends on $a, n, s, \lambda, r, \beta, \rho$. Now $1/m = 1/s + (s-m)/sm$, by the Hölder inequality again, we obtain

$$\begin{aligned}\|u-c\|_{m,\rho B} &= \left(\int_{\rho B} |u-c|^m dx\right)^{1/m} \\ &= \left(\int_{\rho B} \left(|u-c| w_2^{\beta/s} w_2^{-\beta/s}\right)^m dx\right)^{1/m} \\ &\le \left(\int_{\rho B} |u-c|^s w_2^\beta dx\right)^{1/s}\left(\int_{\rho B}\left(\frac{1}{w_2}\right)^{\beta m/(s-m)} dx\right)^{(s-m)/sm} \\ &= \left(\int_{\rho B} |u-c|^s w_2^\beta dx\right)^{1/s}\left(\int_{\rho B}\left(\frac{1}{w_2}\right)^{\lambda/(r-1)} dx\right)^{\beta(r-1)/\lambda s}\end{aligned} \tag{3.8}$$

for all balls B with $\rho B \subset \Omega$ and all closed forms c. Combining (3.7) and (3.8), we obtain

(3.9)
$$\begin{aligned}\left(\int_B |du|^s w_1^\beta dx\right)^{1/s} &\le C_3 diam(B)^{-1}|B|^{(m-t)/mt}\left(\int_{\rho B} |u-c|^s w_2^\beta dx\right)^{1/s} \\ &\quad \times \left(\int_B w_1^\lambda dx\right)^{\beta/\lambda s}\left(\int_{\rho B}\left(\frac{1}{w_2}\right)^{\lambda/(r-1)} dx\right)^{\beta(r-1)/\lambda s}.\end{aligned}$$

Since $(w_1(x), w_2(x)) \in A_{r,\lambda}(\Omega)$, then

(3.10)
$$\begin{aligned}
&\left(\int_B w_1^{\lambda} dx\right)^{\beta/\lambda s} \left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{\lambda/(r-1)} dx\right)^{\beta(r-1)/\lambda s} \\
&\le \left(\left(\int_{\rho B} w_1^{\lambda} dx\right) \left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{\lambda/(r-1)} dx\right)^{(r-1)}\right)^{\beta/\lambda s} \\
&= \left(|\rho B|^r \left(\frac{1}{|\rho B|}\int_{\rho B} w_1^{\lambda} dx\right) \left(\frac{1}{|\rho B|}\int_{\rho B} \left(\frac{1}{w_2}\right)^{\lambda/(r-1)} dx\right)^{r-1}\right)^{\beta/\lambda s} \\
&\le C_4(r,\lambda,w_1,w_2)|\rho B|^{\beta r/\lambda s} \\
&\le C_5(r,\lambda,w_1,w_2,s,\beta,\rho)|B|^{\beta r/\lambda s}.
\end{aligned}$$

Substituting (3.10) in (3.9) and noting $(m-t)/mt = -\beta r/\lambda s$, we obtain

$$\left(\int_B |du|^s w_1^{\beta} dx\right)^{1/s} \le C_6 diam(B)^{-1} \left(\int_{\rho B} |u-c|^s w_2^{\beta} dx\right)^{1/s}.$$

Where C_6 depends on $s, a, n, r, \lambda, w_1, w_2, \rho, \beta$. We have completed the proof of Theorem 3.3. □

4. $A_{r,\lambda}(\Omega)$-weighted weak reverse Hölder inequality

Using similar methods, we can prove the following two-weight weak reverse Hölder inequality.

Theorem 4.1. *Let $u \in D'(\Omega, \wedge^l)$ be a differential form satisfying the A-harmonic equation (1.1) in a domain $\Omega \in \mathbf{R^n}$, $l = 0, 1, \ldots, n,$. Suppose that $0 < s, t < \infty$, $\sigma > 1$ and $(w_1(x), w_2(x)) \in A_{r,\lambda}(\Omega)$ for some $1 < r < \infty$ and $0 < \lambda < \infty$ with $1/r + 1/r' = 1$. Then there exists a constant C, depending on $a, p, n, s, t, r, \lambda, \beta, \sigma, w_1, w_2$, but independent of u, such that*

$$\left(\int_B |u|^s w_1^{\beta} dx\right)^{1/s} \le C|B|^{(t-s)/st} \left(\int_{\sigma B} |u|^t w_2^{\beta t/s} dx\right)^{1/t} \tag{4.2}$$

for all balls B with $\sigma B \subset \Omega$ and any real number β with $0 < \beta < \lambda$.

Note that (4.2) can be written as the symmetric version

$$\left(\frac{1}{|B|}\int_B |u|^s w_1^{\beta} dx\right)^{1/s} \le C \left(\frac{1}{|B|}\int_{\sigma B} |u|^t w_2^{\beta t/s} dx\right)^{1/t}. \tag{4.3}$$

Proof. Choose $k = \lambda s/(\lambda - \beta)$, then $s < k$. Since $1/s = 1/k + (k-s)/ks$, applying the Hölder inequality yields

$$
\begin{aligned}
\left(\int_B |u|^s w_1^\beta dx\right)^{1/s} &= \left(\int_B \left(|u| w_1^{\beta/s}\right)^s dx\right)^{1/s} \\
&\leq \|u\|_{k,B} \left(\int_B w_1^{\beta k/(k-s)} dx\right)^{(k-s)/sk} \\
&= \|u\|_{k,B} \left(\int_B w_1^\lambda dx\right)^{\beta/\lambda s}
\end{aligned} \tag{4.4}
$$

for all balls B with $\sigma B \subset \Omega$. Next, choose $m = \lambda st/\left(\lambda s + \beta t(r-1)\right)$, then $m < t$. Using Lemma 2.3, we have

$$
\|u\|_{k,B} \leq C_1 |B|^{(m-k)/mk} \|u\|_{m,\sigma B}. \tag{4.5}
$$

Where C_1 depending on $a, p, n, s, t, r, \lambda, \beta, \sigma$. Since $1/m = 1/t + (t-m)/tm$, by the Hölder inequality again, we obtain

$$
\begin{aligned}
\|u\|_{m,\sigma B} = \left(\int_{\sigma B} |u|^m dx\right)^{1/m} &= \left(\int_{\sigma B} \left(|u| w_2^{\beta/s} w_2^{-\beta/s}\right)^m dx\right)^{1/m} \\
&\leq \left(\int_{\sigma B} |u|^t w_2^{\beta t/s} dx\right)^{1/t} \left(\int_{\sigma B} \left(\frac{1}{w_2}\right)^{\beta mt/(t-m)s} dx\right)^{(t-m)/mt} \\
&= \left(\int_{\sigma B} |u|^t w_2^{\beta t/s} dx\right)^{1/t} \left(\int_{\sigma B} \left(\frac{1}{w_2}\right)^{\lambda/(r-1)} dx\right)^{\beta(r-1)/\lambda s}
\end{aligned} \tag{4.6}
$$

From (4.4), (4.5),and (4.6), we find that

$$
\begin{aligned}
\left(\int_B |u|^s w_1^\beta dx\right)^{1/s} &\leq C_1 |B|^{(m-k)/mk} \left(\int_B w_1^\lambda dx\right)^{\beta/\lambda s} \\
&\times \left(\int_{\sigma B} (\frac{1}{w_2})^{\lambda/(r-1)} dx\right)^{\beta(r-1)/\lambda s} \left(\int_{\sigma B} |u|^t w_2^{\beta t/s} dx\right)^{1/t}.
\end{aligned} \tag{4.7}
$$

Since $(w_1(x), w_2(x)) \in A_{r,\lambda}(\Omega)$, then

$$
\begin{aligned}
&\left(\int_B w_1^\lambda dx\right)^{\beta/\lambda s} \left(\int_{\sigma B} \left(\frac{1}{w_2}\right)^{\lambda/(r-1)} dx\right)^{\beta(r-1)/\lambda s} \\
&\leq \left(\left(\int_{\sigma B} w_1^\lambda dx\right) \left(\int_{\sigma B} \left(\frac{1}{w_2}\right)^{\lambda/(r-1)} dx\right)^{(r-1)}\right)^{\beta/\lambda s} \\
&= \left(|\sigma B|^r \left(\frac{1}{|\sigma B|} \int_{\sigma B} w_1^\lambda dx\right) \left(\frac{1}{|\sigma B|} \int_{\sigma B} \left(\frac{1}{w_2}\right)^{\lambda/(r-1)} dx\right)^{r-1}\right)^{\beta/\lambda s} \\
&\leq C_2(r, \lambda, w_1, w_2) |\sigma B|^{\beta r/\lambda s} \\
&\leq C_3(r, \lambda, w_1, w_2, s, \sigma, \beta) |B|^{\beta r/\lambda s}.
\end{aligned} \tag{4.8}
$$

Finally substituting (4.8) into (4.7) and using $(m-k)/km = 1/k - 1/m = 1/s - 1/t - \beta r/\lambda s$, we obtain

$$\left(\int_B |u|^s w_1^\beta dx\right)^{1/s} \le C_4 |B|^{(t-s)/st} \left(\int_{\sigma B} |u|^t w_2^{\beta t/s} dx\right)^{1/t}.$$

Where C_4 depends on $a, n, p, r, s, t, \lambda, \beta, \sigma, w_1, w_2$. The proof of Theorem 4.1 is completed. □

5. Applications of the above results

As applications of our main theorems obtained in this paper, we give three examples as follow.

Example 5.1. Let $u \in D'(\Omega, \wedge^l)$, $l = 0, 1, \ldots, n$, be a differential form satisfying the equation

$$d^\star(|du|^{p-2}du) = 0. \tag{5.2}$$

Then u satisfies (2.6), (3.4), and (4.2), respectively.

Proof. Let $A : \Omega \times \wedge^l(\mathbf{R^n}) \to \wedge^{\mathbf{l}}(\mathbf{R^n})$ be an operator defined by

$$A(x, \xi) = |\xi|^{p-2}\xi.$$

Then (1.1) reduces to (5.2) and A satisfies the conditions:

$$|A(x, \xi)| \le |\xi|^{p-1} \quad \text{and} \quad \langle A(x, \xi), \xi\rangle \ge |\xi|^p$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbf{R^n})$. By Theorem 2.5, Theorem 3.3, and Theorem 4.1, we find that u satisfies (2.6), (3.4), and(4.2), respectively. □

In particular, we consider the equation (5.2) in $\mathbf{R^3}$. Clearly, $u = (a_2x_3 - a_3x_2)dx_1 + (a_3x_1 - a_1x_3)dx_2 + (a_1x_2 - a_2x_1)dx_3$ is a 1-form in $\mathbf{R^3}$. Here a_i is a constant for $i = 1, 2, 3$. By simple calculation, we know u satisfies the equation (5.2) when $1 < p < \infty$, and $n = 3$. Then u satisfies the inequalities (2.6),(3.4), and (4.2), respectively.

Example 5.3 ([16])**.** Suppose that $f : \Omega \to R^n - \{0\}$ is a K-quasiregular mapping, i.e., $f \in W^{1,n}_{loc}(\Omega, R^n)$ and

$$|Df(x)|^n \le KJ_f(x), \text{for almost every } x \in \Omega \tag{5.4}$$

We define the matrix-valued function

$$G^{-1}(x) = \begin{cases} \frac{D^t f(x) Df(x)}{J_f(x)^{2/n}}, & \text{if } Df(x) \text{ exists and } J_f(x) \ne 0 \\ I, & \text{otherwise} \end{cases}$$

By (5.4) we see that $G^{-1}(x)$ is defined everywhere in Ω as a symmetric positive definite $n \times n$ matrix such that $\det G^{-1}(x) \equiv 1$ and

$$|\xi|^2 \le \langle G^{-1}(x)\xi, \xi\rangle \le K^{2/n}|\xi|^2$$

Hence the inverse matrix, denoted by $G(x)$, satisfies

$$K^{-2/n}|\xi|^2 \le \langle G(x)\xi, \xi\rangle \le |\xi|^2 \tag{5.5}$$

It is known that for any K-quasiregular mapping $f : \Omega \to R^n - \{0\}$ the function $u = -\log|f|$ is the weak solution of the equation

$$\mathrm{div} A(x, \bigtriangledown u(x)) = 0 \tag{5.6}$$

where $A(x,\xi) = \frac{n}{2}\langle G(x)\xi,\xi\rangle^{\frac{n-2}{2}} G(x)\xi$ and $G(x)$ satisfies (5.5). Then it can be easily derived by using (5.5) that A satisfies the following conditions

$$|A(x,\xi)| \le \frac{n}{2}|\xi|^{n-1}, \text{ and } \langle A(x,\xi),\xi\rangle \ge \frac{n}{2}K^{-1}|\xi|^n$$

Since equation (5.6) is a special case of (1.1), then by Theorems 2.5, 3.3, 4.1, we find that $u = -\log|f|$ satisfies (2.6), (3.4), and (4.2), respectively.

Example 5.7. It is known that if $f = \big(f^1, f^2, \ldots, f^n\big)$ be K-quasiregular in $\mathbf{R^n}$, then

$$u = f^l df^1 \wedge df^2 \wedge \cdots \wedge df^{l-1}$$

$l = 1, 2, \ldots, n$, is a differential form satisfying A-harmonic equation (1.1), where A is some operator satisfying (1.2) and (1.3) (see [6]). Then by Theorem 2.5, we obtain the following local weighted integral inequality for quasiregular mappings.

$$\begin{aligned}\Bigg(\int_B |f^l df^1 \wedge df^2 \wedge \cdots \wedge df^{l-1} \\ - (f^l df^1 \wedge df^2 \wedge \cdots \wedge df^{l-1})_B|^s w_1^{\alpha\lambda} dx\Bigg)^{1/s} \\ \le C|B|^{1/n}\left(\int_{\sigma B} |df^1 \wedge df^2 \wedge \cdots \wedge df^l|^s w_2^{\alpha} dx\right)^{1/s},\end{aligned}$$

where $0 < \alpha < 1$ is any real number.

By Theorem 3.3 and Theorem 4.1, we obtain the following two local weighted integral inequalities for quasiregular mappings, respectively.

$$\begin{aligned}&\left(\int_B |df^1 \wedge df^2 \wedge \cdots \wedge df^l|^s w_1^{\beta} dx\right)^{1/s} \\ &\qquad \le C diam(B)^{-1}\left(\int_{\rho B} |f^l df^1 \wedge df^2 \wedge \cdots \wedge df^{l-1} - c|^s w_2^{\beta} dx\right)^{1/s},\end{aligned}$$

where c is any closed form and β is any real number with $0 < \beta < \lambda$.

$$\begin{aligned}&\left(\int_B |f^l df^1 \wedge df^2 \wedge \cdots \wedge df^{l-1}|^s w_1^{\beta} dx\right)^{1/s} \\ &\qquad \le C|B|^{(t-s)/st}\left(\int_{\sigma B} |f^l df^1 \wedge df^2 \wedge \cdots \wedge df^{l-1}|^t w_2^{\beta t/s} dx\right)^{1/t},\end{aligned}$$

where β is any real number with $0 < \beta < \lambda$.

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