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# The modularity of certain non-rigid Calabi–Yau threefolds

By

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**Abstract**

Let $X$ be a Calabi–Yau threefold fibred over $\mathbb{P}^1$ by non-constant semi-stable K3 surfaces and reaching the Arakelov–Yau bound. In [25], X. Sun, Sh.-L. Tan, and K. Zuo proved that $X$ is modular in a certain sense. In particular, the base curve is a modular curve. In their result they distinguish the rigid and the non-rigid cases. In [17] and [28] rigid examples were constructed. In this paper we construct explicit examples in non-rigid cases. Moreover, we prove for our threefolds that the "interesting" part of their $L$-series is attached to an automorphic form, and hence that they are modular in yet another sense.

## 1. Introduction

Let $X$ be an algebraic threefold and let $f : X \to \mathbb{P}^1$ be a non-isotrivial morphism whose fibers are semi-stable K3 surfaces. Let $S \subset \mathbb{P}^1$ be the finite set of points above which $f$ is non-smooth, and assume that the monodromy at each point of $S$ is non-trivial. Jost and Zuo [9] proved the Arakelov–Yau type inequality:

$$\deg f_*\omega_{X/\mathbb{P}^1} \leq \deg \Omega^1_{\mathbb{P}^1}(\log S).$$

Let $\Delta \subset X$ be the pull-back of $S$. Let $\omega_{X/\mathbb{P}^1}$ be the canonical sheaf. The Kodaira–Spencer maps $\theta^{2,0}$ and $\theta^{1,1}$ are maps of sheaves fitting into the following diagram:

$$\begin{aligned} f_*\Omega^2_{X/\mathbb{P}^1}(\log\Delta) &\overset{\theta^{2,0}}{\to} R^1 f_*\Omega^1_{X/\mathbb{P}^1}(\log\Delta)\otimes\Omega^1_{\mathbb{P}^1}(\log S) \\ &\overset{\theta^{1,1}}{\to} R^2 f_*\mathcal{O}_{X/\mathbb{P}^1}\otimes\Omega^1_{\mathbb{P}^1}(\log S)^{\otimes 2}. \end{aligned}$$

The iterated Kodaira–Spencer map of $f$ is defined to be the map $\theta^{1,1}\theta^{2,0}$.

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It is known (see [25]) that when the (iterated) Kodaira–Spencer map is 0, one actually has the stronger inequality

$$\deg f_*\omega_{X/\mathbb{P}^1} \le \frac{1}{2}\deg \Omega^1_{\mathbb{P}^1}(\log S).$$

Assume from now on that $X$ is a Calabi–Yau threefold. Then the triviality of the canonical bundle implies that $\deg f_*\omega_{X/\mathbb{P}^1} = 2$ (see (3.2) below).

Recently X. Sun, S-L. Tan and K. Zuo [25] considered Calabi–Yau threefolds for which the Arakelov–Yau inequality becomes equality. Thus $S$ consists of 6 points when the Kodaira–Spencer map is 0 and of 4 points otherwise.

As a consequence of the main theorem of [25], the following results were obtained.

**Theorem 1.1** ([25], Corollary 0.4)**.** (i) *If the iterated Kodaira–Spencer map $\theta^{1,1}\theta^{2,0}$ of $f$ is non-zero, then $X$ is rigid (i.e., $h^{2,1}=0$) and birational to the Nikulin–Kummer construction of a symmetric square of a family of elliptic curves $f : E \to \mathbb{P}^1$. After passing to a double cover $E' \to E$ (if necessary), the family $g' : E' \to \mathbb{P}^1$ is one of the six modular families of elliptic curves on the Beauville's list* ([2])*.*

(ii) *If the iterated Kodaira–Spencer map $\theta^{1,1}\theta^{2,0}$ of $f$ is zero, then $X$ is a non-rigid Calabi–Yau threefold (i.e., $h^{2,1}\neq 0$), the general fibers have Picard number at least* 18*, and $\mathbb{P}^1 \setminus S \simeq \mathfrak{H}/\Gamma$ where $\Gamma$ is a congruence subgroup of $PSL(2,\mathbb{Z})$ of index* 24*.*

**Remark.** The base curve $\mathbb{P}^1\setminus S$ is a modular variety of genus zero, i.e., $\mathfrak{H}/\Gamma$ where $\Gamma$ is a torsion-free genus zero congruence subgroup of $PSL(2,\mathbb{Z})$ of index 12 in case (i), and of index 24 in case (ii). In the paper of Sun, Tan and Zuo [25], the word "modularity" refers to this fact.

The third cohomology of each of the six rigid Calabi–Yau threefolds in Theorem 1.1 (i) arises from a weight 4 modular form. In the articles of Saito and Yui [17] and of Verrill in Yui [28], these forms were explicitly determined. Saito and Yui use geometric structures; while Verrill uses point counting method, to obtain the results.

More precisely, the following was proved for the natural models over $\mathbb{Q}$ of these six rigid threefolds:

**Theorem 1.2** (Saito and Yui [17] and Verrill in Yui [28])**.** *For each of the six rigid Calabi–Yau threefold over $\mathbb{Q}$, the L-series of the third cohomology coincides with the L-series arising from the cusp form of weight* 4 *of one variable on the modular group in the Beauville's list. Beauville's list and the corresponding cusp forms are given in Table* 1*.*

It might be helpful to recall the six rigid Calabi–Yau threefolds in Theorem 1.2. These six rigid Calabi–Yau threefolds are obtained (by Schoen [18]; see also Beauville [2]) as the self-fiber products of stable families of elliptic curves admitting exactly four singular fibers. The base curve is a rational modular

Table 1. Rigid Calabi–Yau threefolds and cusp forms

| Group $\Gamma$ | Number of components of singular fibers | Cusp forms of weight 4 |
|---|---|---|
| $\Gamma(3)$ | $3,3,3,3$ | $\eta(q^3)^8$ |
| $\Gamma_1(4)\cap\Gamma(2)$ | $4,4,2,2$ | $\eta(q^2)^4\eta(q^4)^4$ |
| $\Gamma_1(5)$ | $5,5,1,1$ | $\eta(q)^4\eta(q^5)^4$ |
| $\Gamma_1(6)$ | $6,3,2,1$ | $\eta(q)^2\eta(q^2)^2\eta(q^3)^2\eta(q^6)^2$ |
| $\Gamma_0(8)\cap\Gamma_1(4)$ | $8,2,1,1$ | $\eta(q^4)^{16}\eta(q^2)^{-4}\eta(q^8)^{-4}$ |
| $\Gamma_0(9)\cap\Gamma_1(3)$ | $9,1,1,1$ | $\eta(q^3)^8$ |

Here $\eta(q)$ denotes the Dedekind eta-function: $\eta(q) = q^{1/24}\prod_{n\geq 1}(1-q^n)$ with $q = e^{2\pi i\tau}$.

curve and correspond to the torsion-free genus zero congruence subgroups $\Gamma$ of $PSL(2,\mathbb{Z})$ in Table 1. Note that the 4-tuples of natural numbers appearing in the second column add up to 12, which is the index of the modular group $\Gamma$ in $PSL(2,\mathbb{Z})$.

In [25] the authors indicate one example for the non-rigid extremal case. It is related to $\Gamma(4)$, which is a torsion-free congruence subgroup of genus 0 and index 24 in $PSL(2,\mathbb{Z})$. The list of torsion-free congruence subgroups of genus 0 and index 24 in $PSL(2,\mathbb{Z})$ is known (see Sebbar [19], and Table 2). In this paper we will show that most of them give rise to a similar collection of examples. In each of these cases we will compute the interesting part of the $L$-series of the third cohomology of an appropriate natural model over $\mathbb{Q}$ in terms of automorphic forms.

This paper is organized as follows. In Section 2, we use work of Sebbar [19] to determine the groups $\Gamma$ corresponding to case (ii) of Theorem 1.2. These are subgroups of $PSL(2,\mathbb{Z})$, and associated to each $PSL(2,\mathbb{Z})$-conjugacy class there is a natural elliptic fibration over the base curve, defined over $\mathbb{Q}$. The total spaces of these fibrations are elliptic modular surfaces in the sense of Shioda [21]. Moreover each is an extremal K3 surface (namely their Picard number is 20, the maximum possible). We explain the relation between the motive of their transcendental cycles, and specific CM forms of weight 3 using a result of Livné on orthogonal rank 2 motives in [14].

Section 3 contains our main results: we construct our examples, verify the required properties, and analyze the interesting part of their cohomology. (See the final Remark 3(2) for the other parts.) Then in Section 4 we give explicit formulas for the weight 3 cusp forms and defining equations for the elliptic fibrations of Section 2.

The paper is supplemented by the article of Hulek and Verrill [7] where they treat Kummar varieties, one of which is the case associated to the modular group $\Gamma_1(7)$. This case differs from the examples considered in our paper with the main difference being the fact that the 2-torsion points do not decompose into four sections, leading to non-semi-stable fibrations. But it still gives rise to a Calabi–Yau threefold (Theorem 2.2 of Hulek and Verrill [7]), and the modularity question can still be considered, and this is exactly what Hulek and Verrill deals with in the supplement [7] to this article.

## 2. Extremal congruence K3 surfaces

The torsion-free genus zero congruence subgroups of $PSL(2,\mathbb{Z})$ of index 24 were classified in Sebbar [19]. There are precisely nine conjugacy classes of such congruence subgroups.

The second column in the following Table 2 gives the complete list of the torsion-free congruence subgroups of $PSL(2,\mathbb{Z})$ of index 24 up to conjugacy. Each has precisely 6 cusps. The third column in the table gives the widths of these cusps.

Table 2. Torsion-free congruence subgroups of index 24

| # | The group $\Gamma$ | Widths of the cusps |
|---|---|---|
| 1 | $\Gamma(4)$ | $4,4,4,4,4,4$ |
| 2 | $\Gamma_0(3)\cap\Gamma(2)$ | $6,6,6,2,2,2$ |
| 3 | $\Gamma_1(7)$ | $7,7,7,1,1,1$ |
| 4 | $\Gamma_1(8)$ | $8,8,4,2,1,1$ |
| 5 | $\Gamma_0(8)\cap\Gamma(2)$ | $8,8,2,2,2,2$ |
| 6 | $\Gamma_1(8;4,1,2)$ | $8,4,4,4,2,2$ |
| 7 | $\Gamma_0(12)$ | $12,4,3,3,1,1$ |
| 8 | $\Gamma_0(16)$ | $16,4,1,1,1,1$ |
| 9 | $\Gamma_1(16;16,2,2)$ | $16,2,2,2,1,1$ |

Here

$$\Gamma_1(8;4,1,2) := \left\{\pm\begin{pmatrix}1+4a & 2b\\ 4c & 1+4d\end{pmatrix},\ a\equiv c \pmod 2\right\}$$

and

$$\Gamma_1(16;16,2,2) := \left\{\pm\begin{pmatrix}1+4a & b\\ 8c & 1+4d\end{pmatrix},\ a\equiv c \pmod 2\right\}.$$

**Remark 1.** If we are interested in conjugacy as Fuchsian groups (in $PSL(2,\mathbb{R})$), Examples #1,#5, and #8 are conjugate (use the matrix $\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$), Examples #2 and #7 are conjugate (use the same matrix), and Examples #4, #6, and #9 are conjugate (use the same matrix as well as $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$).

**Proposition 2.1.** *Let $\Gamma$ be one of the congruence subgroups in Table 2. Then $\Gamma$ has an explicit congruence lift $\tilde{\Gamma}$ to $SL(2,\mathbb{Z})$ with the following properties*:

(1) $\tilde{\Gamma}$ *has no elliptic elements. In particular* $-\mathrm{Id}$ *is not in* $\tilde{\Gamma}$.
(2) $\tilde{\Gamma}$ *contains no element of trace* $-2$.

*Proof.* We let $\tilde{\Gamma}$ be the subgroup of $SL(2,\mathbb{Z})$ consisting of the elements above $\Gamma$. Indeed, the lifts $\tilde{\Gamma}$ are sometimes the same as the groups $\Gamma$ themselves. In fact, for the cases #1, #2, #3, #4 and #5, the lifts are the same and respectively given by: $\Gamma(4)$, $\Gamma_0(3)\cap\Gamma(2), \Gamma_1(7), \Gamma_1(8)$ and $\Gamma_0(8)\cap\Gamma(2)$. In the cases #6, #7, #8 and #9, the lifts are not unique, and we choose respectively the following lifts: $\Gamma_1'(8;4,1,2)$, $\Gamma_0(12)\cap\Gamma_1(3)$, $\Gamma_0(16)\cap\Gamma_1(4)$ and $\Gamma_1'(16;16,2,2)$. Here $\Gamma_1'(8;4,1,2)$ and $\Gamma_1'(16;16,2,2)$ are defined in the same way as $\Gamma_1(8;4,1,2)$ and $\Gamma_1(16;16,2,2)$ but without the $\pm$. Note that the widths of the cusps are not affected by taking a lift as $-\mathrm{Id}$ is the only difference.

(We should remark that the lifts are not unique; for instance, in the case #7, there are four lifts, but only one has no elements of trace $-2$, which is the one given above.) □

In fact, Proposition 2.1 has also been obtained by A. Sebbar in his unpublished note.

Proposition 2.1 will pave a way to the definition of elliptic modular surfaces, which we will discuss next.

**Elliptic Modular Surfaces**: In [21] Shioda has shown how to associate to any subgroup $G$ of $SL(2,\mathbb{Z})$ of finite index and not containing $-\mathrm{Id}$ an elliptic fibration $E(G)$, called the elliptic modular surface associated to $G$, over the modular curve $X(G)=\overline{G\backslash\mathfrak{H}}$.

(Shioda's construction requires that modular groups ought to be a subgroup of $SL(2,\mathbb{Z})$ (rather than a subgroup of $PSL(2,\mathbb{Z})$) that contains no element of order 2. This is a reason we consider a lift $\tilde{\Gamma}$ of $\Gamma$ in our discussion.)

**Remark 2.** It follows from Kodaira's theory that when $G$ contains no elliptic elements and no elements of trace $-2$, all the singular fibers are above the cusps and are of type $I_n$, where $n$ is the width of the cusp. On the other hand, elements of trace $-2$ give rise to $I_n^*$-fibers above the cusps.

For the $\tilde{\Gamma}$'s of Proposition 2.1 the modular curve $X(\tilde{\Gamma})$ has genus 0, and, since the sum of the widths of the cusps in Table 2 is always 24, each $E(\tilde{\Gamma})$ is an extremal K3 surface. The space $S_3(\tilde{\Gamma})$ of cusp forms of weight 3 for $\tilde{\Gamma}$ is therefore one-dimensional. Up to a square, the discriminant of the intersection form on the rank 2 motive of the transcendental cycles $T := T(E(\tilde{\Gamma})) = H^2(E(\tilde{\Gamma}),\mathbb{Q})/\mathrm{NS}(E(\tilde{\Gamma}))$ is given by

$$\delta=\delta_k=-1,-3,-7,-2,-1,-2,-3,-1,-2 \tag{2.1}$$

in cases $k=1,\dots,9$ respectively. To see this one computes the discriminant of the (known) lattice $NS(E(\tilde{\Gamma}))$ and passes to the orthogonal complement in $H^2(E(\tilde{\Gamma}),\mathbb{Q})$. For details, see e.g. Besser-Livné [3]. By [14] it follows that the normalized newform $g_{3,\Gamma}$ generating $S_3(\tilde{\Gamma})$ has CM by $\mathbb{Q}(\sqrt{\delta}\,)$.

To each of our 9 examples there is a naturally associated moduli problem of classifying (generalized) elliptic curves with a given level structure (Katz and Mazur [10]). Each of these moduli problems refines the respective moduli problem $Y_1(M)$, of classifying elliptic curves with a point of order $M$, where $M$ is as above. Since $M \geq 4$, these moduli problems are all represented by universal families $E(\tilde{\Gamma})/X(\tilde{\Gamma})/\mathbb{Z}[1/M]$ (see Katz–Mazur [10]). The geometric fibers are geometrically connected in all these examples, and their compactified fibers over $\mathbb{C}$ are the corresponding elliptic modular surfaces above.

We shall now compute the $L$-series $L(T, s)$ of the transcendental cycles. By the Eichler–Shimura Isomorphism, this is the parabolic cohomology

$$\tilde{H} := \tilde{H}^1(X(\tilde{\Gamma}) \times_{\mathbb{Z}[1/M]} \overline{\mathbb{Q}}, R^1(E(\tilde{\Gamma}) \to X(\tilde{\Gamma}))).$$

Moreover, Deligne proved ([5]) the Eichler–Shimura congruence relation

$$\mathrm{Frob}_p + \mathrm{Frob}'_p = T_p \qquad \text{for any } p \not| M,$$

where $T_p$ is the $p$-th Hecke operator on $S_3(\tilde{\Gamma})$. This is the same as the $p$-th Fourier coefficient of the normalized newform $g_{3,\Gamma}$. Summarizing, we proved

$$L(T(X(\tilde{\Gamma}), s) = L(g_{3,\Gamma}, s).$$

Explicit Weierstrass equations for the elliptic fibrations

$$E(\tilde{\Gamma})/X(\tilde{\Gamma})/\mathbb{Z}[1/M]$$

will be given in Section 4 below.

**Remarks.** (1) The list in Table 2 exhausts all of the families of semi-stable elliptic $K3$ surfaces with exactly six singular fibers, which correspond to torsion-free genus zero *congruence* subgroups of $PSL_2(\mathbb{Z})$ of index 24. The 6-tuples of natural numbers appearing in the third column add up to 24. Therefore, the number of such 6-tuples is a priori finite. That this list is complete was proved by Sebbar [19].

(2) Miranda and Persson [16] studied all possible configurations of $I_n$ fibers on elliptic $K3$ surfaces. In the case of exactly six singular fibers, they obtained 112 possible configurations including the above nine cases. All these $K3$ surfaces have the maximal possible Picard number 20. It should be emphasized that the exactly nine configurations correspond to genus zero congruence subgroups of $PSL_2(\mathbb{Z})$ of index 24.

(3) The theory of Miranda and Persson had been extended to prove the uniqueness (over $\mathbb{C}$) of K3 surfaces having each of these types of singular fibers by Artal-Bartolo, Tokunaga and Zhang [1]. Confer also the article of Shimada and Zhang [20] for a useful table of extremal elliptic $K3$ surfaces.

## 3. The non-rigid examples

Let $Y = E(\tilde{\Gamma})$ be one of the K3 surfaces of the previous Section, and let $g_Y = g_{3,\Gamma}$ denote the corresponding cusp form of level 3. If $\tilde{\Gamma}$ contains $\Gamma_1(M)$

(in Table 2 this happens in cases #3, #4, #7, #8, and #9), and $M = M_Y$ is the maximal possible, then $M$ is the *level* of $g_Y$, and the *newtype* of $g_Y$ is the Dirichlet character

$$\epsilon = \epsilon_Y, \tag{3.1}$$

of conductor $M$, so that $g_Y$ is in $S_3(\Gamma_0(M), \epsilon_Y)$. Notice that $\epsilon$ is odd (namely $\epsilon(-1) = -1$). Moreover, since the coefficients of $g_Y$ are integers, $\epsilon$ must be quadratic. (We will determine $\epsilon$ for our examples in Proposition 4.1.)

Let $E$ be an elliptic curve. We view the product $Y \times E$ as a family of abelian surfaces over $X(\tilde{\Gamma})$. The fiber $A_t = A_{\Gamma,t}$ over each point $t \in X(\tilde{\Gamma})$ is the product of the fiber $E_{\Gamma,t}$ of $E(\tilde{\Gamma})$ with $E$. Then we obtain the following results.

**Theorem 3.1.** (1) *The product $Y \times E$ has the Hodge numbers*

$$h^{0,3}(Y \times E) = 1,\ h^{1,0}(Y \times E) = 1 \text{ and } B_3(Y \times E) = 44$$

(*so that $Y \times E$ is not a Calabi–Yau threefold*).

(2) *The motive $T(Y \times E) = T(Y) \times H^1(E)$ is a submotive of $H^3(Y \times E)$. If $E$ and $Y$ are defined over $\mathbb{Q}$, this submotive is modular, in the sense that its $L$-series is associated to a cusp form $g_Y$ on $GL(2, \mathbb{Q}(\sqrt{\delta}\,))$.*
*Let $g_E$ be the cusp form of weight* 2 *associated to $E$ by Wiles et al.* ([27]). *Let $A(p)$* (*respectively $B(p)$*) *be the pth Fourier coefficient of $g_E$* (*respectively of $g_Y$*), *and let $\epsilon_Y$ be the newtype character of $g_Y$* (*see Section* 3). *Then for any good prime $p$, the local Euler factor $L_p(s)$ of the $L$-series $L(T(Y \times E), s) = L(g_E \otimes g_Y, s)$ is*

$$\begin{aligned} 1 - A(p)B(p)p^{-s} + (B(p)^2 + \epsilon_Y(p)pA(p)^2 - 2p^2\epsilon_Y(p))p^{1-2s} \\ -A(p)B(p)\epsilon_Y(p)p^{3-3s} + p^{6-4s}. \end{aligned}$$

*Proof.* The statements about the Hodge and Betti numbers follow from the Künneth formula. Since $T(Y)$ is a factor of $H^2(Y)$, it follows again from the Künneth formula that $T(Y) \times H^1(E)$ is a factor of $H^3(Y \times E)$.

For the second part, we know that $g_Y$ is a CM form. Hence it is induced from an algebraic Hecke character $\chi = \chi_Y$ of the imaginary quadratic field $F = K_i$. Let $\chi_G$ be the compatible system of 1-dimensional $\ell$-adic representations of $G_F = \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ corresponding to $\chi$. Then the 2-dimensional Galois representation associated to $T(Y)$ is $\mathrm{ind}_{G_F}^{G_{\mathbb{Q}}}\chi_G$. Hence we obtain the 4-dimensional Galois representation

$$\rho_E \otimes \mathrm{ind}_{G_F}^{G_{\mathbb{Q}}}\chi_G \simeq \mathrm{ind}_{G_F}^{G_{\mathbb{Q}}}(\chi_G \otimes \mathrm{Res}_{G_F}^{G_{\mathbb{Q}}}\rho_E),$$

where $\rho_E$ is the Galois representation associated to $H^1(E)$. The operation of restricting $\rho_E$ to $G_F$ and of twisting by characters have automorphic analogs. Let $\pi_E$ be the automorphic representation associated to $E$. Then $\pi' = \chi\otimes$

$\mathrm{Res}_F^{\mathbb{Q}}\pi_E$ makes sense as an automorphic cuspidal irreducible representation of $GL(2,F)$, and we have the characterizing relationship

$$L(\pi', s) = L(\mathrm{ind}_F^{\mathbb{Q}}\pi', s) = L(\pi_E \otimes \mathrm{ind}_F^{\mathbb{Q}}\chi, s) = L(g_E \otimes g_Y, s).$$

For the last part, write the $p$th Euler factors of $g_E$ and $g_Y$ respectively as

$$(1-\alpha_p p^{-s})(1-\alpha_p' p^{-s}) = 1 - A(p)p^{-s} + p^{1-2s}$$

and

$$(1-\beta_p p^{-s})(1-\beta_p' p^{-s}) = 1 - B(p)p^{-s} + \epsilon_Y(p)p^{2-2s}.$$

Then the Euler factor $L_p$ is defined as

$$L_p(s) = (1-\alpha_p\beta_p p^{-s})(1-\alpha_p'\beta_p p^{-s})(1-\alpha_p\beta_p' p^{-s})(1-\alpha_p'\beta_p' p^{-s}),$$

and the claim follows by a direct calculation. □

**Remark.** If a K3 surface has the Picard number 19 or 18, the modularity question for the product $Y \times E$ is still open. However, if the Picard number is 19, one knows at least that the rank 3 motive $T(Y)$ of the transcendental cycles is self dual orthogonal via the cup product. (For explicit examples of K3 surfaces with Picard number 19, see e.g. Besser and Livné [3].) Thus one can use a result of Tate to lift each $\ell$-adic representation to the associated spin cover, which is the multiplicative group of some quaternion algebra over $\mathbb{Q}$. If the spin representation is modular (which should always be the case), then it is associated to a cusp form $h$ of weight 2 on $GL(2)$, so that $\mathrm{Symm}^2 h$ realizes $T(Y)$. Let $g_E$ be again the weight 2 cusp form associated with $E$. It follows, by work of Gelbart and Jacquet, that $T(Y)$ is realized by an automorphic representation on $GL(3,\mathbb{Q})$. Hence, by the work of Kim and Shahidi ([11]), $T(Y)\times E$ is realized by an automorphic form on $GL(6,\mathbb{Q})$. In particular, $L(\mathrm{Symm}^2 h \otimes g_E, s)$ has the expected analytic properties.

To construct our promised examples, let $X = X_\Gamma \to X(\tilde{\Gamma})$ be the associated Kummer family, in which we divide each fiber $A_t$ of $Y(\tilde{\Gamma}) \times E$ by $\pm 1$ and then blow up the locus of points of order 2. We now have the following result.

**Theorem 3.2.** *In the Examples #1, #2, #5, and #6 of Table 2 the resulting $X$ is a smooth Calabi–Yau threefold. It is non-rigid, and the given fibration $f : X \to X(\tilde{\Gamma})$ is semi-stable, with vanishing (iterated) Kodaira–Spencer mapping. We have*

$$\deg f_*\omega_{X/\mathbb{P}^1} = 2 = \frac{1}{2}\, deg\,\Omega^1_{\mathbb{P}^1}(log\, S),$$

*In other words, $X$ reaches the (stronger) Arakelov–Yau bound.*

**Remark.** For the first case in Table 2 ($\tilde{\Gamma} = \Gamma(4)$) this example is indicated in [25].

*Proof.* The Examples we chose are those in which $\tilde{\Gamma}$ is a subgroup of $\Gamma(2)$. (This is because otherwise, the points of order 2of $X(\Gamma)$ coincide (over the cusps).) Thus the 2-torsion points (of $E_t$ and hence of $A_t$) are distinct for *all* $t \in X(\tilde{\Gamma})$. It follows that the locus $A[2]$ of 2-torsion points is smooth, and hence so is the blow-up $X$. We have $H^i(X) = H^i(Y(\tilde{\Gamma}) \times E)^{<\pm 1>}$. But $\pm 1$ acts as $\pm 1$ on both the non-trivial holomorphic 1-form $\omega_1$ of $E$ and on the non-trivial holomorphic 2-form $\omega_2$ of $Y(\tilde{\Gamma})$. Hence $\omega_1 \wedge \omega_2$ descends to a holomorphic 3-form $\omega_3$ on $X$. Its divisor can only be supported on the proper transform $\mathcal{F}$ of $A[2]$; however $\mathcal{F}$ intersects each fiber $f^{-1}(t)$ in sixteen $(-2)$-curves, which do not contribute to the canonical class, so that $\omega_3$ is indeed nowhere-vanishing. The Künneth formula gives that

$$H^1(X) = H^1(Y)^{<\pm 1>} = H^1(E)^{<\pm 1>} = 0,$$

and

$$H^{2,0}(X) = H^{2,0}(Y \times E)^{<\pm 1>} = H^2(Y)^{<\pm 1>} = 0.$$

Thus $X$ is indeed a smooth Calabi–Yau threefold. It is non-rigid, because $T(Y \times E)$ descends to $X$ and each of its Hodge pieces $H^{p,q}(T(Y \times E))$ is 1-dimensional.

To compute the monodromy around each singular fiber, we notice that for a generic fiber $X_t = f^{-1}(t)$ the Kummer structure gives a canonical decomposition

$$H^2(X_t, \mathbb{Q}) = (H^2(A_t, \mathbb{Q}) \oplus \mathbb{Q}A_t[2])^{<\pm 1>}.$$

Our examples were chosen so that the action of $\pm 1$ on $A_t[2] = E_{\Gamma,t}[2] \times E_t[2]$ is trivial. Moreover, in the Künneth decomposition

$$H^2(A_t) = H^2(E) \oplus (H^1(E) \otimes H^1(E_{\Gamma,t})) \oplus H^2(E_{\Gamma,t})$$

the $\pm 1$ action is trivial on the first and last factors, is trivial on $H^1(E)$ and is unipotent on $H^1(E_\Gamma, t)$ around each singular fiber of $f$ (namely the cusps of $\Gamma$). Thus the monodromy of the fibration $f$ is unipotent as well.

To compute the Kodaira–Spencer map $\Theta(f)$ for our $f$ we embed it into the Kodaira–Spencer map for $Y \times E \to X(\tilde{\Gamma})$. This map is the cup product with the Kodaira–Spencer class $\Theta$ which itself is $\Theta_{Y/X(\tilde{\Gamma})} \otimes \Theta_{X(\tilde{\Gamma}) \times E/X(\tilde{\Gamma})}$. Since the Kodaira–Spencer class of a trivial fibration vanishes, it follows that $\Theta(f) = 0$.

Our examples all have 6 singular fibers. Hence

$$\frac{1}{2} \deg \Omega^1_{\mathbb{P}^1}(\log S) = \frac{1}{2} \deg \mathcal{O}_{\mathbb{P}^1}(-2+6) = 2.$$

On the other hand, since $X$ is a Calabi–Yau variety we have

$$\omega_{X/\mathbb{P}^1} = \omega_X \otimes (f^* \omega_{\mathbb{P}^1})^{-1} = (f^* \omega_{\mathbb{P}^1})^{-1}.$$

Hence

$$f_* \omega_{X/\mathbb{P}^1} = f_* f^* (\omega_{\mathbb{P}^1})^{-1} = (f_* f^* \omega_{\mathbb{P}^1})^{-1} = \omega_{\mathbb{P}^1}^{-1}, \tag{3.2}$$

whose degree is 2 as well, concluding the proof of Theorem 3.2. □

**Remark 3.** (1) In the other cases in Table 2 the monodromy on the points of order 2 of $A_t[2]$ is non-trivial, and the calculation gives that the monodromy of $f$ around the cusps is not unipotent. We know by Remark 1, the groups #1, #5 and #8 are in the same $PSL(2,\mathbb{R})$-conjugacy class. However, this group theoretic property does not guarantee isomorphisms of the corresponding Calabi–Yau threefolds, since the fiber structures are not preserved. Similarly, the groups #4, #6 and #9 are $PSL(2,\mathbb{R})$-conjugate, but geometric structures are different (as the fibers over the cusps are different). The same applies to Examples #2 and #7. Therefore, Examples #4, #7, #8 and #9 are not covered by our examples. Also we do not know how to construct examples corresponding to Example #3 in Table 2, for which $\tilde{\Gamma} = \Gamma_1(7)$. We also do not know whether our examples are the only ones.

(2) It is an interesting exercise to compute the full $L$-series of our examples. The results are as follows: Let $N_+$ (respectively $N_-$) be the motive of algebraic cycles on $Y$ invariant (respectively anti-invariant) by $\pm 1$ acting on the elliptic fibrations of $Y$. Let $n_\pm$ be the rank of $N_\pm$. Then $n_+ + n_- = 20$, and if we let $\chi_{\delta'}$ denote the quadratic character cut by $\mathbb{Q}(\sqrt{\delta'}\,)$ (not necessarily the same quadratic field pre-determined by the modular group corresponding to the surface), then $N_+ = \mathbb{Z}(1)^{n'_+} \oplus \mathbb{Z}(\chi_{\delta'}(1))^{n''_+}$ and $N_- = \mathbb{Z}(1)^{n'_-} \oplus \mathbb{Z}(\chi'_\delta(1))^{n''_-}$. Here $n'_+$ (resp. $n''_+$) denotes the number of cycles defined over $\mathbb{Q}$ (resp. $\mathbb{Q}(\delta')$) and similarly for $n'_-$ (resp. $n''_-$). We have $n_\pm = (n' + n'')_\pm$. Then we have

$$\begin{aligned}
L(H^0,s) &= L(\mathbb{Z},s) = \zeta(s)\\
L(H^1,s) &= 1\\
L(H^2,s) &= L(H^2(\mathbb{P}^1\times\mathbb{P}^1),s)L(\mathbb{Z}(1),s)L(N_+,s)\\
&= \zeta(s-1)^{16}\zeta(s-1)^{1+n'_+}L(\mathbb{Q}\otimes\chi_{\delta'},s-1)^{n''_+}\\
L(H^3,s) &= L(T(Y)\otimes H^1(E),s)L(N_-\times H^1(E),s)\\
&= L(g_3\otimes g_2,s)L(E,s-1)^{n'_-}\prod_{\delta'} L(E\otimes\chi_{\delta'},s-1)^{n''_-}
\end{aligned}$$

(The higher cohomologies are determined by Poincaré duality.)

**Lemma 3.1.** *In cases #1, #2, #5, and #6 in Table 2, we have $n_+ = 14$ and $n_- = 6$ (so $n_+ - n_- = 2 + 6 = 8$). Furthermore, we have*

$$(n'_+, n''_+) = \begin{cases} (12,2) & \text{for } \#1\\ (14,0) & \text{for } \#2\\ (13,1) & \text{for } \#5,\ \#6\end{cases}$$

*and*

$$(n'_-, n''_-) = \begin{cases} (3,3) & \text{for } \#1\\ (6,0) & \text{for } \#2\\ (5,1) & \text{for } \#5, \#6\end{cases}$$

*In case #3, $n_+ = 11$ and $n_- = 9$. (Forthe last case, confer the article of Hulek and Verrill* [7] *for more detailed discussion.)*

For the computations of $n_+$ and $n_-$, confer Proposition 2.4 of Hulek and Verrill [7]. The Proof of Lemma 3.1 will be given at the end of Section 4.

## 4. Explicit formulas

We shall now give explicit formulas for the weight 3 forms $g_Y = g_{3,\Gamma}$ for the examples in Table 2. We will denote the weight 3 form in the $i$th case by $h_i$. By Remark 1 it suffices to compute $h_i$ for $i = 8, 7, 3, 4$, and then $h_8(\tau) = h_5(\tau/2) = h_1(\tau/4)$, $h_7(\tau) = h_2(\tau/2)$, and $h_6(\tau) = h_9(\tau/2) = h_4(\tau/2 - 1/2)$. Two kinds of formulas suggest themselves for the $h_i$'s: as a product of $\eta$-functions or as inverse Mellin transforms of the Dirichlet series attached to Hecke characters. The second method is always possible since the $g_Y$'s are CM forms. In [15] Martin determined which modular forms on $\Gamma_1(N)$ can be expressed as a product of $\eta$-functions. This applies to cases #8, #7, #3, and #4 in Table 2. Hence the same is also true for the #6 and #9 cases. For cases #3, #7, and #8 the corresponding spaces of cusp forms of weight 3 are 1-dimensional, hence the conditions in [15] are satisfied and Martin gives the corresponding forms as $h_3 = \eta(q)^3\eta(q^7)^3$ and $h_7 = \eta(q^2)^3\eta(q^6)^3$. The modular form in case #1 is classically known to be $h_1 = \eta(q)^6$, which implies $h_8 = \eta(q^4)^6$. Lastly $h_2 = \eta(q)^3\eta(q^3)^3$, and $h_5 = \eta(q^2)^6$. For #4, we have $h_4(q) = \eta(q)^2\eta(q^2)\eta(q^4)\eta(q^8)^2$. We will prove the following results.

**Proposition 4.1.** *Let $\chi_i$ be the Hecke character for which $L(h_i, s) = L(\chi_i, s)$ (so that the inverse Mellin transform of $L(\chi_i, s)$ is $h_i$). Let $a_p(h_i)$ be the pth Fourier coefficient of $h_i$, and let $K_i = \mathbb{Q}(\sqrt{\delta_i})$. Then we have the following*:

(1) *The infinite component of $\chi_i : \mathbb{A}_{K_i}^\times \to \mathbb{C}$ is given by $\chi_{i,\infty}(z) = z^{-2}$. Moreover $\chi_i$ is the unique such Hecke character of conductor $c_i\mathcal{O}_{K_i}$, where $c_i = 2, 2, 1, 1 \in \mathcal{O}_{K_i}$ for $i = 8, 7, 3, 4$ respectively.*

(2) *For each rational prime $p$ which is prime to the level of the corresponding $\Gamma$, we have $a_p(h_i) = 0$ if $p$ is inert in $K_i$. Otherwise, there are $a$, $b$, which are integers in case #8 and half integers in the three other cases, so that $p = a^2 + d_ib^2$, where $d_i = 4, 3, 7, 2$ for $i = 8, 7, 3, 4$ respectively. Then $a$ and $b$ are unique up to signs, and $a_p(h_i) = (a^2 - d_ib^2)/2$.*

(3) *The newtype of $h_i$ (see (3.1)) is the character defining $K_i$, namely $p \mapsto \left(\frac{\delta_i}{p}\right)$.*

*Proof.* See e.g. [14] for the generalities (in particular regarding the $\infty$-component of $\chi_i$), as well as the following formula: the conductors of $\chi_i$ and of $h_i$ are related by

$$\mathrm{cond}(h_i) = \mathrm{Nm}_{\mathbb{Q}}^{K_i}\mathrm{cond}(\chi_i)\mathrm{Disc}(K_i).$$

Since the level of $h_i$ is respectively $M = 16$, 12, 7, and 8 in cases #8, #7, #3, and #4 of Table 2, we get the asserted value of the $c_i$'s, and since all the fields $K_i$ involved have class number 1 we have

$$\mathbb{A}_{K_i}^\times = (K_i^\times \times U_i \times \mathbb{C}^\times)/\mu(K_i) \tag{4.1}$$

where $U_i$ is the maximal compact subgroup $\hat{\mathcal{O}}_{K_i}^\times$ of the finite idèles of $K_i$, and $\mu(K_i)$ is the group of roots of unity of $K_i$, acting diagonally (we view $\mathbb{C}$ as the infinite completion of $K_i$). The existence and the uniqueness of $\chi_i$ are then verified in each case by a straightforward calculation (compare [13]).

For the second part, the vanishing of $a_p(h_i)$ for $p$ inert in $K_i$ is a general property of CM forms. For a split $p$ (prime to $\mathrm{cond}(h_i)$), write $p = a^2 + d_i b^2 = \mathrm{Nm}_{\mathbb{Q}}^{K_i}\pi$. Here $\pi$ is a prime element of $\mathcal{O}_{K_i}$, so $a$ and $b$ are half integers. We verify that, up to multiplying $\pi$ by a unit, we can guarantee that $a$ and $b$ are integers for $i = 8$. In all cases, the $a$'s and the $b$'s are unique up to signs. Next one verifies that $\pi \equiv \pm 1 \pmod{c_i\mathcal{O}_{K_i}}$. Let $\wp$ be the ideal generated by $\pi$ (notice that changing the sign of $b$ replaces $\wp$ by its conjugate). Let tr denote the trace from $K_i$ to $\mathbb{Q}$. By the general theory, we have that

$$a_p(h_i) = \mathrm{tr}\,\chi_\wp(\pi) = \mathrm{tr}\,\chi_\infty(\pi)^{-1} = \mathrm{tr}\,\pi^2 = 2(a^2 - d_i b^2),$$

where the second equality holds since the finite components of $\chi$ other than $\pi$ are now 1.

For the third part we notice that the restriction of $\chi_i$ to $U_i$ in the decomposition (4.1) above gives a Dirichlet character $\epsilon_i'$ on $\mathcal{O}_{K_i}$ of conductor $c_i\mathcal{O}_{K_i}$. The newtype Dirichlet character $\epsilon_i$ (on $\mathbb{Z}$) is then the product $\chi_{K_i}$ by the restriction of $\epsilon_i'$ to $\mathbb{Z}$. However, the conductors $c_i\mathcal{O}_{K_i}$ of $\chi_i$ are all 1 or 2, and the only character of $\mathbb{Z}$ of conductor 1 or 2 is trivial. Hence the newtype character of $h_i$ is $K_i$, concluding the proof of Proposition 4.1. $\square$

**Defining equations for extremal K3 surfaces**: We now discuss how to determine defining equations for the extremal K3 surfaces in Theorem 3.2. This problem has been getting a considerable attention lately, for instance, Shioda [Sh3] and (independently and by a different method by Y. Iron [8]) have determined a defining equation for the semi-stable elliptic K3 surface with singular fibers of type $I_1, I_1, I_1, I_1, I_1, I_{19}$ whose existence was established in Miranda and Persson [16] (this is given as #1 in their list). As we shall see, our examples can be determined by a more classical method.

There are several cases where defining equations can be found in the literature, i.e., Example #1 in Table 2 is the classical Jacobi quartic corresponding to $\Gamma(4)$,

$$y^2 = (1 - \sigma^2 x^2)(1 - x^2\sigma^{-2}) \tag{4.2}$$

where $\sigma$ is a parameter for $X(4)$. A Legendre form is given by

$$Y^2 = X(X-1)(X-\lambda) \quad \text{with } \lambda = \frac{1}{4}(\sigma + \sigma^{-1})^2 \tag{4.3}$$

(see e.g. Shioda [22]). One checks that the singular fibers, all of type $I_4$, occur at the cusps $\sigma = 0, \infty, \pm 1, \pm\sqrt{-1}$. Moreover the $j$-invariant is given by

$$j = 2^4 \frac{(1 + 14\sigma^2 + \sigma^4)^3}{\sigma^4(\sigma^4 - 1)^4}.$$

For the remaining cases, we can find defining equations using a method due to Tate. Since we could not find Tate's method in the literature, we sketch it here. (Actually, we found out after completing the paper that there are several papers dealing with this exact issue, e.g., Kubert [12] and his arguments were reproduced in Howe–Leprévost–Poonen [6]. Also, the paper of Billing and Mahler [4] dealt with the same problem.)

**A method of Tate to calculate** $E_1(N)$ : Let $Y_1(N)$ be the modular curve, and let $E_1^0(N) \to Y_1(N)$, with $N \geq 4$, be the universal family of elliptic curves having a point (or section) $P = P_N$ of order $N$. Tate's method gives a defining equation for this family over $\mathbb{Z}[1/N]$. We start with the general Weierstrass equation:

$$E: \quad y^2 + a_1xy + a_3y = x^3 + a_1x^2 + a_4x + a_6.$$

Let $P = (x, y) \in E$ be a rational point and assume that $P, 2P, 3P \neq 0$. Changing coordinates, we may put $P$ at $(x, y) = (0, 0)$. So we may assume $a_6 = 0$. Since $P$ does not have order 2, the tangent line at $(0, 0)$ cannot be the $y$-axis (i.e., $x = 0$), which implies that $a_3$ cannot vanish. We can therefore change coordinates again to obtain $a_4 = 0$ and the equation takes the form: $y^2+a_1xy+a_3y = x^3+a_2x^2$. By making a dilation, we furthermore get $a_2 = a_3$. Therefore, $E$ has a Weierstrass equation of the form:

$$(*) \quad y^2 + axy + by = x^3 + bx^2 \quad \text{with } b \neq 0.$$

To get a defining equation for $E_1^0(N)$, we need to find the relations on $a$ and $b$ that arise if $P$ has order $N$. The coordinates of $P, -P, 2P, -2P$ are easily checked to be

$$P = (0, 0),\ -P = (-0, -b),\ 2P = (-b, (a - 1)b),\ -2P = (-b, 0).$$

We will also determine the coordinates of $3P$ and of $4P$. At $-2P$ the tangent line is:

$$y = \frac{b}{1-a}(x + b).$$

Substituting this to the equation (*) to get

$$4P = \left(\frac{b}{1-a} + \frac{b^2}{(1-a)^2},\quad \frac{b^2}{1-a}\left(1 + \frac{b}{(1-a)^2} + \frac{1}{1-a}\right)\right).$$

Likewise, the line $x + y + b = 0$ intersects $E$ at $-P$, $-2P$ and $3P$, giving $3P = (1 - a, a - 1 - b)$. We will give Weierstrass equations for $E_1(N)$ when $N = 4$, 6, 8, or 7:

$\boxed{E_1(4)}$ Here we get $a = 1$, giving the equation

$$y^2 + xy + ty = x^3 + tx^2$$

(we replaced $b$ by $t$). Here $X_1(4)$ is the $t$-line. By a direct calculation or from Shioda's result (see also Remark 5), we see that the singular fibers are over the three cusps, of types, $I_1^*$, $I_1$, and $I_4$.

$\boxed{E_1(6)}$ The equation $x(4P) = x(-2P)$ readily gives $b = -(a-1)(a-2)$, giving us the equation for $E_1(6)$:

$$E_1(6): \quad y^2 + axy - (a-1)(a-2)y = x^3 - (a-1)(a-2)x^2.$$

Here $a$ is a parameter (Hauptmodul) on $X_1(6)$. There are four cusps, and as before one gets from Remark 2 or by a direct computation that the types are $I_1$, $I_2$, $I_3$, and $I_6$, matching the widths of the cusps given in the third column of Table 2.

$\boxed{E_1(8)}$ The equation $y(4P) = y(-4P)$ is equivalent to $ax(4P) + b = -2y(4P)$. Expanding, cancelling $b$, and clearing denominators gives

$$ab(1-a) + (1-a)^2 = -2b\left((1-a)(2-a) + b\right).$$

Substituting $b = k(a-1)$ gives

$$(a,b) = \left(\frac{-2k^2+4k-1}{k}, \quad -2k^2+3k-1\right),$$

Thus $k$ is a parameter on $X_1(8)$ and $E_1(8)$ (Example #4) is given by

$$y^2 + \frac{-2k^2+4k-1}{k}xy + (-2k^2+3k-1)y = x^3 + (-2k^2+3k-1)x^2. \tag{4.4}$$

The fibers above the cusps are found as before to have types $I_1$, $I_1$, $I_2$, $I_4$, $I_8$, and $I_8$.

$\boxed{E_1(7)}$ In a similar way one gets the equation for $E_1(7)$ (Example #3); the result is (see for instance, Silverman [24, Example 13.4])

$$y^2 + (1+t-t^2)xy + (t^2-t^3)y = x^3 + (t^2-t^3)x^2.$$

Three singular fibers have type $I_1$, and three have type $I_7$.

$\boxed{E(6,2)}$ Returning to our cases, we now handle Example #2 in Table 2, corresponding to $\tilde{\Gamma} = \Gamma_0(3) \cap \Gamma(2)$, whose associated modular curve is

$$X(6,2) = X_1(6) \times_{X_1(2)} X(2)$$

by pulling $E_1(6)$ back to $X(6,2)$. To do this we cannot use Tate's method directly, since the moduli problems associated to $Y(2)$ and to $Y_1(2)$ are not representable. However $X(2)$ is the Legendre $\lambda$-line, and any elliptic curve with $\Gamma(2)$-level structure can always be written in Legendre form

$$y^2 = x(x-1)(x-\lambda) = x(x^2 + (-1-\lambda)x + \lambda).$$

Likewise, given an elliptic curve $E$ with a point $P$ of order 2 (over $\mathbb{Z}[\frac{1}{2}]$), write $E$ in Weierstrass form $y^2 = x(x^2 + cx + d)$ where $P = (0, 0)$. This form is unique up to homothety, and hence $c^2/d$ is a parameter on $X_0(2)$. The natural map $X(2) \to X_0(2)$ is therefore given by $u = \frac{(1+\lambda)^2}{\lambda}$. Hence the fibred product for $E(6, 2)$ above is given by equating

$$\frac{(1+\lambda)^2}{\lambda} = \frac{(4-3a^2)^2}{16(a-1)^3}.$$

A computation gives that

$$\xi = \frac{32(a-1)^3}{(4-3a^2)(a-2)^2}\left(\lambda + 1 - \frac{(4-3a^2)^2}{32(a-1)^3}\right)$$

is a parameter on $X(6, 2)$ such that the map to $X_1(6)$ is given by

$$a = \frac{2\xi^2 - 10}{\xi^2 - 9}.$$

Under a base change of ramification index $b$ an $I_a$ fiber pulls back to an $I_{ab}$ fiber (and an $I_a^*$ fiber pulls back to an $I_{ab}^*$ fiber if $a \geq 1$, and $b$ is odd). From this or again by Remark 5 the fiber types are as expected from Table 2, namely, three $I_6$ and three $I_2$ fibers.

**Remark.** Even though we do not need the following example here, we mention that a parameter for $X_0(12)$ can be computed in the same way via the natural map $X_0(12) \to X_0(6) = X_1(6)$. By pull-back this will give a family of elliptic curves over $X_0(12)$, which will turn out to be $\Gamma_0(12)$ case (Example #7) in Table 2. For this we let $t$ be the parameter for $X_0(4) = X_1(4)$ as before, and let $a$ be the parameter from before on $X_0(6)$. Then $X_0(12)$ is the fibred product

$$X_1(4) \times_{X_1(2)} X_1(6).$$

To compute the natural "forgetting" maps of $X_1(4)$ and of $X_1(6)$ to $X_1(2)$ we again bring both $E_1(4)$ and $E_1(6)$ to the form $y^2 = x(x^2 + ax + b)$:

$$E_1(4) : w^2 = v\left(v^2 + \left(\frac{1}{4} - 2b\right)v + b^2\right)$$

(here we completed the square and set $w = y + (x + b)/2$ and $v = x + b$).

$$E_1(6) : w^2 = v\left(v^2 + \frac{4-3a^2}{4}v + (a-1)^3\right).$$

By the above, the fibred product is given by equating

$$\frac{(\frac{1}{4} - 2b)^2}{b^2} = \frac{(4-3a^2)^2}{16(a-1)^3}.$$

Thus $a-1$ is a square, say $a = u^2+1$, where $u$ is a parameter on $X_0(12)$ and the pulled-back family is given by

$$y^2 + (u^2+1)xy - u^2(u^2-1)y = x^3 - u^2(u^2-1)x^2.$$

One again routinely verifies that the bad fibers are as expected. (Notice, however, that the pull-back of the universal family from $X_1(4)$ has $I_a^*$ fibers!)

$\boxed{E(8,2)}$ Next we handle Example #5 in Table 2, whose associated modular curve is $X(8,2)$. As was explained in Remark 1, we can take as a parameter the same $\sigma$ as for $X(4)$ above. However to get the right family, we will divide the universal elliptic curve by a section $s$ of order 2. This changes the type of the singular fibers $E(4)_c$ at a cusp $c$ from $I_4$ to $I_8$ if $s$ meets $E(4)_c$ at the same component as the identity section, and to type $I_2$ otherwise. The singular fibers obtained in this way are as expected from Table 2.

To get the new family, recall that if an elliptic curve is given in Weierstrass form

$$y^2 = x(x^2+ax+b),$$

then the quotient by the two-torsion point $(0,0)$ is given by the similar equation

$$y^2 = x(x^2 - 2ax + a^2 - 4b). \tag{4.5}$$

In particular, for a curve given in Legendre form $y^2 = x(x-1)(x-\lambda)$ the resulting quotient is $y^2 = x(x^2 + 2(1+\lambda)x + (1-\lambda)^2)$. Applying this to the Legendre form (4.3) of the Jacobi quartic gives the quotient family in the form

$$y^2 = x\left(x^2 + \left(2 + \frac{1}{2}(\sigma+\sigma^{-1})^2\right)x + \frac{1}{16}(\sigma-\sigma^{-1})^4\right).$$

As before one sees that the singular fibers have types $I_8$, $I_8$, $I_4$, $I_4$, $I_4$, and $I_2$.

$\boxed{E(8;4,1,2)}$ To handle Example #6 in Table 2 we proceed in the same way, dividing the family $E_1(8)$ by its section of order 2. The cusps of $\Gamma_1(8)$ are $N\backslash G/N$, where $N$ is the upper unipotent subgroup of G$=SL(2,\mathbb{Z}/8\mathbb{Z})/<\pm\,\mathrm{Id}>$. Explicitly the cusps are $\begin{bmatrix}0\\1\end{bmatrix}$, $\begin{bmatrix}0\\3\end{bmatrix}$, $\begin{bmatrix}1\\2\end{bmatrix}$, $\begin{bmatrix}1\\4\end{bmatrix}$, $\begin{bmatrix}1\\0\end{bmatrix}$, and $\begin{bmatrix}3\\0\end{bmatrix}$. The corresponding widths are 1,1,2,4,8, and 8 respectively. We identify the torsion sections of the universal family $E_1(8) \to X_1(8)$ with the subgroup $\begin{pmatrix}*\\0\end{pmatrix} \in (\mathbb{Z}/8\mathbb{Z})^2$. Let $s = \begin{pmatrix}4\\0\end{pmatrix}$ be the section of order 2 of this elliptic fibration. Then $s$ belongs to the connected component of the 0-section at a cusp $\begin{bmatrix}a\\b\end{bmatrix}$, if and only if it is in the subgroup generated by $\begin{pmatrix}a\\b\end{pmatrix}$. This happens at the first four cusps above but not at the last two.

The Kodaira type of the fibers of the quotient family $E_1(8)/(s) \to X_1(8)$ (which is our $E(\Gamma)$ of Example #6) are accordingly multiplied by 2 at the first four cusps, and divided by 2 at the last two cusps. This results in fiber types $I_8$, $I_4$, $I_4$, $I_4$, $I_2$, and $I_2$, in agreement with Table 2.

Starting with our equation (4.4) for $E_1(8)$ we set

$$Y = 8k^3\, y\left(y + \frac{-2k^2+4k-1}{2k}x + \frac{-2k^2+3k-1}{2}\right)$$

and

$$X = 4k^2(x - k + k^2).$$

A straightforward computation then gives for $E_1(8)$ the form

$$Y^2 = X(X^2 + (8k^4 - 16k^3 + 16k^2 - 8k + 1)X + (2k(k-1))^4).$$

By formula (4.5), the quotient family $E_1(8)/(s)$ is given by

$$y^2 = x(x^2 - 2(8k^4 - 16k^3 + 16k^2 - 8k + 1)x + (8k^2 - 8k + 1)(2k-1)^4).$$

*Proof of Lemma* 3.1. We will use the arguments due to Klaus Hulek and Matthias Schütt for the calculation of $n_+$ and $n_-$. The Galois action on $NS(Y)$ is computed as follows. Tensoring with $\mathbb{Q}$, $NS(Y) \otimes \mathbb{Q}$ has for basis the (classes of the) general fiber, the 0-section and those components of the singular fibers which do not meet the identity component (the section). The Galois action clearly preserves the fiber class and the 0-section. The action of $\pm 1$ on each fiber of type $I_n$ is given as follows. A fiber $I_n$ contributes $n-1$ to the cohomology. If we enumerate the components $e_1, e_2, \ldots, e_{n-1}$ cyclically, then $e_j$ will be sent to $e_{-j}$. If $n$ is even, $n/2$ cycles, $e_{n/2}$, $e_j + e_{-j}$ $(1 \le j < n/2)$ are fixed contributing to $n_+$; while $e_j - e_{-j}$ $(1 \le j < n/2)$ contributing to $n_-$. If $n$ is odd, $(n-1)/2$ cycles are fixed contributing to $n_+$, and equally $(n-1)/2$ cycles to $n_-$. Further, the fields of definition of the components $e_j$ will determine $n'_\pm$ and $n''_\pm$.

In Example #1, the cusps (singularities) are $t = 0, \pm 1, \infty$ and $\pm\sqrt{-1}$. Put $i = \sqrt{-1}$. Then $N_+$ is spanned by the zero-section, the fiber and the following divisors: $e_{t,2}$, $e_{t,1} + e_{t,3}$ where $t = 0, \pm 1, \pm i, \infty$. When $t \in \mathbb{Q}$ or $t = \infty$, these divisors are defined over $\mathbb{Q}$, giving 10 divisors out of 14. Over $t = \pm i$, we see that $e_{i,2} + e_{-i,2}$ and $(e_{i,1} + e_{i,3}) + (e_{-i,1} + e_{-i,3})$ are fixed by complex conjugation, so that these are defined over $\mathbb{Q}$ contributing to $n'_+$. Hence, as Galois representations, we get

$$N_+ = \mathbb{Z}(1)^{12} \oplus \mathbb{Z}(\chi_i(1))^2$$

so that $n'_+ = 12$, and $n''_+ = 2$.

On the other hand, the space $N_-$ is simply spanned by $e_{t,1} - e_{t,3}$ for the six cusps $t$. Over $t = \pm 1$, both are defined over $\mathbb{Q}$, contributing to $n'_-$. Over $t = 0, \infty$, $e_{t,1}$ and $e_{t,3}$ are conjugate, so the difference is not fixed under complex conjugation, so it contributes to $n''_-$. Over $t = \pm i$, we have two divisors

$(e_{i,1} - e_{i,3}) \pm (e_{-i,1} - e_{-i,3})$. One of these is fixed by complex conjugation, while the other is not. Thus, as Galois representation,

$$N_- = \mathbb{Z}(1)^3 \oplus \mathbb{Z}(\chi_i(1))^3$$

and reading off the ranks, we get $n'_- = 3$ and $n''_- = 3$.

In Example #2, the cusps are all defined over $\mathbb{Q}$, the torsion sections meet all the components of the fibers (this can be seen either from the moduli viewpoint or from the equations in the previous section). Then $N_+$ is spanned by the zero-section, the fiber and the divisors $e_{t,1} + e_{t,5}$, $e_{t,2} + e_{t,4}$ and $e_{t,3}$ for $I_6$ type singular fibers and $e_{t,1}$ for $I_2$ type singular fibers. Thus we compute that $n_+ = 3 + 3 + 3 + 1 + 1 + 1 + 1 + 1 = 14$. For the space $N_-$ is spanned by $e_{t,1} - e_{t,5}$, $e_{t,2} - e_{t,4}$ for $I_6$ type singular fibers. Thus we have $n_- = 2 + 2 + 2 + 0 + 0 + 0 = 6$. Since all divisors are defined over $\mathbb{Q}$, all these algebraic cycles are also defined over $\mathbb{Q}$, and we have

$$N_+ = \mathbb{Z}(1)^{14} \quad \text{and} \quad N_- = \mathbb{Z}(1)^6.$$

(In particular, this implies that $n''_\pm = 0$ in this case.)

For the other two cases, #5 and #6, we use the above argument to compute $n_\pm$. In fact, for Example #5 (resp. #6),

$$n_+ = 4+4+1+1+1+1+1+1+1 = 14 \quad (\text{resp.}\, 4+2+2+2+1+1+1+1 = 14),$$

and

$$n_- = 3 + 3 = 6 \quad \text{for both cases.}$$

Thus $n_+ = 14$ and $n_- = 6$. However, for these examples, not all algebraic cycles are defined over $\mathbb{Q}$. In fact, we use the fact that each of these $K3$ surfaces is realized as a quadratic base change of a rational modular elliptic surface (see Top and Yui [26] for detailed argument).

In the case of Example #5, this surface is obtained as a pull-back of a rational elliptic modular surface with 4 singular fibers of type $I_4$ over $infty$, $I_4$ over 0, $I_2$ over 1 and $I_2$ over $-1$. All cusps of the pull-back over $\infty$, 0 and 1 are defined over $\mathbb{Q}$. However, the two cusps of the pull-back above $-1$ are defined only over $\mathbb{Q}(\sqrt{-1})$. Put $\sqrt{-1} = i$. Then the divisor $e_{i,1} + e_{-i,1}$ is invariant under complex conjugation, while the divisor $e_{i,1} - e_{-i,1}$ is not. Thus, we get

$$N_+ = \mathbb{Z}(1)^{13} \oplus \mathbb{Z}(\chi_i(1))$$

so that $(n'_+, n''_+) = (13, 1)$. On the other hand, all algebraic cycles spanning $N_-$ are defined over $\mathbb{Q}$ so that $N_- = \mathbb{Z}(1)^6$ and $n_- = n'_- = 6$.

For Example #6, the cusps are $t = 0$, $\infty$, $\pm 1$ and $\pm\sqrt{2}$. But the pull-back of the components $e_{0,1}$ and $e_{0,3}$ are conjugate over $\mathbb{Q}(i)$. This gives

$$N_+ = \mathbb{Z}(1)^{13} \oplus \mathbb{Z}(\chi_2(1))$$

so that $n'_+ = 13$ and $n''_+ = 1$. While

$$N_- = \mathbb{Z}(1)^5 \oplus \mathbb{Z}(\chi_i(1))$$

so that $n'_- = 5$ and $n_-'' = 1$. □

**Remark.** For #3, the singular fibers are of type $I_7$ and $I_1$ (3 copies each). Hulek and Verrill [7]compute that $n_+ = 11$ and $n_- = 9$, and show that all cycles are defined over $\mathbb{Q}$. This example does not admit semi-stable fibrations, but still gives rise to a non-rigid Calabi–Yau threefold defined over $\mathbb{Q}$, and one can still look into the modularity question for the $L$-series associated to the third cohomology group. This is exactly what is done in the article of Hulek and Verrill [7] supplementing this paper.

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