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Asymptotic stability of small solitary waves to 1D nonlinear Schrödinger equations with potential

By

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Abstract

We consider asymptotic stability of a small solitary wave to supercritical 1-dimensional nonlinear Schrödinger equations

$$iu_t + u_{xx} = Vu \pm |u|^{p-1}u \quad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R},$$

in the energy class. This problem was studied by Gustafson-Nakanishi-Tsai [18] in the 3-dimensional case using the endpoint Strichartz estimate.

To prove asymptotic stability of solitary waves, we need to show that a dispersive part $v(t,x)$ of a solution belongs to $L_t^2(0,\infty;X)$ for some space X. In the 1-dimensional case, this property does not follow from the Strichartz estimate alone.

In this paper, we prove that a local smoothing estimate of Kato type holds globally in time and combine the estimate with the Strichartz estimate to show $\|(1+x^2)^{-3/4}v\|_{L_x^\infty L_t^2} < \infty$, which implies the asymptotic stability of a solitary wave.

1. Introduction

In this paper, we consider asymptotic stability of solitary wave solutions to

$$\begin{cases} iu_t + u_{xx} = Vu + f(u) & \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}, \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $V(x)$ is a real potential, $f(u) = \alpha|u|^{p-1}u$ with $\alpha = \pm 1$.

Let

$$H(u) = \int_{\mathbb{R}} \left(|u_x|^2 + V(x)|u|^2 + \frac{2\alpha}{p+1}|u|^{p+1} \right) dx,$$
$$N(u) = \int_{\mathbb{R}} |u|^2 dx.$$

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Then a solution to (1.1) satisfies

$$H(u(t)) = H(u_0), \quad N(u(t)) = N(u_0) \tag{1.2}$$

during the time interval of existence. Stability of solitary waves to NLS was first studied by Cazenave and Lions [9], Grillakis-Shatah-Strauss [16] and Weinstein [45] (see also Rose-Weinstein [34], Oh [30] and Shatah-Strauss [38]). In the case of integrable equations such as cubic NLS and KdV, the inverse scattering theory tells us that if the initial data decays rapidly as $x \to \pm\infty$, a solution decomposes into a sum of solitary waves and a radiation part as $t \to \infty$ (see [37]). Soffer and Weinstein [41], [42] considered NLS with potential

$$iu_t + \Delta u = Vu \pm |u|^{p-1}u \quad \text{for } x \in \mathbb{R}^n \text{ and } t > 0, \tag{1.3}$$

where $n \geq 2$ and $1 < p < (n+2)/(n-2)$. They proved that if $-\Delta + V$ has exactly one negative eigenvalue and initial data is well localized and close to a nonlinear bound state, a solution tends to a sum of a nonlinear bound state nearby and a radiation part which disperses to 0 as $t \to \infty$. This result was extended by Yau and Tsai [46], [47], [48] and Soffer-Weinstein [43] to the case where $-\Delta + V$ have two bound states. In the 1-dimensional case, Buslaev and Perelman [6], [7] and Buslaev and Sulem [8] studied the asymptotic stability of (1.1) with $V \equiv 0$. Using the Jost functions, they built a local energy decay estimate of solutions to the linearized equation and prove asymptotic stability of solitary waves for super critical nonlinearities (see also [17]). Their results are extended to the higher dimensional case by Cuccagna [12] (see also Perelman [33] and Rodnianski-Schlag-Soffer [35] which study asymptotic stability of multi-solitons).

However, all these results assume that initial data is well localized so that a solution decays like $t^{-3/2}$. Martel and Merle [24], [25] proved the asymptotic stability of solitary waves to generalized KdV equations using the monotonicity of L^2-mass, which is a variant of the local smoothing effect proved by Kato [19]. They elegantly use the fact that a dispersive remainder part of a solution $v(t, x)$ satisfies

$$\int_0^\infty \|v(t, \cdot)\|_{H^1_{loc}}^2 \, dt < \infty \tag{1.4}$$

to prove the asymptotic stability of solitary waves in H^1 (see also El-Dika [14] and Mizumachi [27] for BBM equation and Pego-Weinstein [32] and Mizumachi [26] for KdV with localized initial data). Recently, Gustafson-Nakanishi-Tsai [18] has proved asymptotic stability of a small solitary wave of (1.3) in the energy class with $n = 3$. Their idea is to use the endpoint Strichartz estimate instead of (1.4), which tells us that $\|v\|_{L^2_t W^{1,6}_x}$ remains small globally in time for super critical nonlinearity. However, the Strichartz estimate is not sufficient in the lower dimensional case to obtain some estimate like (1.4) because a dispersive wave decays more slowly than the 3-dimensional case. To overcome this difficulty, we prove

$$\|\langle x\rangle^{-3/2}e^{it(-\partial_x^2+V)}Qf\|_{L_x^\infty L_t^2}\le C\|f\|_{L^2}, \tag{1.5}$$

$$\|\partial_x e^{it(-\partial_x^2+V)}Qf\|_{L_x^\infty L_t^2}\le C\|f\|_{H^{1/2}}, \tag{1.6}$$

where Q is a spectral projection associated to the continuous spectrum of $-\partial_x^2+V$. The local smoothing estimate of $1/2$ gain derivative has been studied by many authors (see e.g. Constantin and Saut [11], Kato and Yajima [20] and Kenig-Ponce-Vega [21], [22]) to show the local well-posedness of semilinear equations with derivative terms. Most of them are without potential ([11], [39]) or local in time (see [36]).

Ben-Artzi and Klainerman [3] proved a time global local smoothing estimate for the n-dimensional case with $n\ge 3$. See also Barceló-Ruiz-Vega [2] who use a Morawetz type inequality to obtain the result. Recently, Burq and Planchon [5] has proved local smoothing estimates including an estimate similar to (1.6) for $Lu=-\partial_x(a(x)\partial_x u)$ (they use $\dot B^{1/2}_{2,\infty}$ instead of L^∞). In the present paper, we show (1.5) and (1.6) assuming the non-resonance condition for $L=-\Delta+V$. Another difference between [5] is that L may have negative eigenvalues.

Our proof given in this paper for the 1-dimensional case is different from [3], [5], [2]. We use the Born series (see Artbazar-Yajima [1] and Goldberg-Schlag [15]) for the high frequency part and a theory of Jost functions for the low frequency part.

Finally, we introduce several notations. For complex valued functions $f(x)$ and $g(x)$, we denote $\langle f,g\rangle=\int_{\mathbb R} f(x)\overline{g(x)}dx$. Let

$$\|u\|_{L_t^qL_x^p}=\left(\int_{\mathbb R}\left(\int_{\mathbb R}|u(t,x)|^p dx\right)^{q/p}dt\right)^{1/q},$$

$$\|u\|_{L_x^sL_t^r}=\left(\int_{\mathbb R}\left(\int_{\mathbb R}|u(t,x)|^r dt\right)^{s/r}dx\right)^{1/s},$$

and let $H^{1,k}(\mathbb R)$ be the Hilbert space equipped with the norm

$$\|u\|_{H^{1,k}}=\left(\sum_{i=0,1}\int_{\mathbb R}(1+x^2)^k|\partial_x^i u(x)|^2dx\right)^{1/2}.$$

For any Banach spaces X, Y, we denote by $B(X,Y)$ the space of bounded linear operators from X to Y. We abbreviate $B(X,X)$ as $B(X)$.

We define the Fourier and transform of $f(x)$ as

$$\mathcal F_x f(\xi)=\hat f(\xi)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb R} f(x)e^{-ix\xi}dx,$$

and the inverse Fourier transform of $g(\xi)$ as

$$\mathcal F_\xi^{-1}g(x)=\hat g(-x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb R} g(\xi)e^{ix\xi}d\xi.$$

We define $\mathcal{S}_{\otimes}(\mathbb{R}^2)$ as a set of functions written as $f(t,x) = \sum_{i=1}^N f_i(t)g_i(x)$ with f_i, $g_i \in \mathcal{S}(\mathbb{R})$ $(1 \le i \le N)$.

For an interval $I \subset \mathbb{R}$, let $\chi_I(x)$ be a characteristic functions satisfying $\chi_I(x) = 1$ for $x \in I$ and $\chi_I(x) = 0$ for $x \notin I$. We denote $\sqrt{1+|x|^2}$ by $\langle x \rangle$.

2. The Main result and Preliminaries

In the present paper, we assume that the linear potential $V(x)$ is a continuous function on $\mathbb{R}$ and satisfies the following.

(V1) $(1+x^2)V(x) \in L^1(\mathbb{R})$.

(V2) $L = -\partial_x^2 + V$ has exactly one negative eigenvalue E_*, and 0 is neither a resonance nor an eigenvalue of L.

Let $E \in \mathbb{R}$ and $e^{-iEt}\phi_E(x)$ be a solitary wave solution of (1.1). Then $\phi_E(x)$ is a solution to

$$\begin{cases} \phi_E'' + E\phi_E = V\phi_E + \alpha|\phi_E|^{p-1}\phi_E & \text{for } x \in \mathbb{R}, \\ \lim_{x\to\pm\infty} \phi_E(x) = 0. \end{cases} \tag{2.1}$$

Using the bifurcation theory, we have the following.

Proposition 2.1. *Assume* (V1) *and* (V2). *Then there exist a $\delta > 0$, $E = E(s) \in C^4((-\delta,\delta))$ and $h(s)$ satisfying the following*:

1. *$\phi_E = s\phi_* + h(s)$ is a positive solution of* (2.1), *where ϕ_* is an normalized eigenfunction of H (satisfying $\|\phi_*\|_{L^2} = 1$) belonging to E_*,*
2. *for every $k \in \mathbb{N}$, the function $h(s) \in H^{1,k} \cap {}^{\perp}\operatorname{span}\{\phi_*\}$ are C^4 in $s \in (-\delta,\delta)$,*
3. *$E(s) = E_* + \alpha\|\phi_*\|_{L^{p+1}}^{p+1}(|s|^{p-1} + o(1))$ and $\|h(s)\|_{H^{1,k}} = O(s^p)$ as $s \to 0$.*

Proposition 2.1 follows from a rather standard argument. See for example [29] and [41, pp.123–124].

Now, we introduce our main result.

Theorem 2.1. *Assume* (V1) *and* (V2). *Let $p \ge 5$ and ε_0 be a sufficiently small positive number. Suppose $\|u_0\|_{H^1} < \varepsilon_0$. Then there exist an $E_+ < 0$, a real-valued function $\theta(t)$ and $v_+ \in H^1(\mathbb{R})$ such that*

$$|E_+ - E_*|^{1/(p-1)} + \|v_+\|_{H^1} = O(\|u_0\|_{H^1}),$$
$$\lim_{t\to\infty} \|u(t) - e^{i\theta(t)}\phi_{E_+} - We^{it\partial_x^2}v_+\|_{H^1(\mathbb{R})} = 0,$$

where $W = \lim_{t\to\infty} e^{-itL}e^{-it\partial_x^2}$.

Since $\theta(t)$ could be discontinuous at a time $E(t) = E_*$, we introduce following parameters as [18]: $a = (a_1,a_2) = s(\cos\theta, -\sin\theta)$ and

$$\phi(a) = \phi_{E(s)}e^{-i\theta} = \begin{cases} (a_1 + ia_2)\left(\phi_* + \dfrac{h(|a|)}{|a|}\right) & \text{if } a \ne 0, \\ 0 \quad \text{if } a = 0. \end{cases}$$

Remark 1. By Proposition 2.1,

$$\|\partial_{a_1}\phi - \phi_*\|_{H^{1,k}(\mathbb{R})} + \|\partial_{a_2}\phi - i\phi_*\|_{H^{1,k}(\mathbb{R})} \lesssim |E - E_*| = O(|a(t)|^{p-1}).$$

Remark 2. Let us decompose a solution to (1.1) into a solitary wave part and a radiation part:

$$u(t,x) = \phi(a(t))(x) + v(t,x). \tag{2.2}$$

If we take initial data in the energy class, the dispersive part of the solutions decays more slowly than it does for well localized initial data. So, being different from Soffer-Weinstein [41, 42] or Buslaev-Perelman [6], we cannot expect that $\int_t^\infty \dot{E}(s)ds$ is integrable. Thus in general, we need dispersive estimates for a time-dependent linearized equations to prove asymptotic stability of solitary waves in $H^1(\mathbb{R})$. To avoid this difficulty, we assume the smallness of solitary waves so that a generalized kernel of the linearized operator is well approximated by a 1-dimensional subspace $\{\beta\phi_* \mid \beta \in \mathbb{C}\}$.

Substituting (2.2) into (1.1), we obtain

$$iv_t = Lv + g_1 + g_2 + g_3, \tag{2.3}$$

where

$$g_1(t) = -i\dot{a}(t)\cdot\nabla_a\phi(a(t)) + E\phi(a(t)),$$
$$g_2(t) = f(\phi(a(t)) + v(t)) - f(\phi(a(t))) - \partial_\varepsilon f(\phi(a(t)) + \varepsilon v(t))|_{\varepsilon=0},$$
$$g_3(t) = \partial_\varepsilon f(\phi(a(t)) + \varepsilon v(t))|_{\varepsilon=0}.$$

By the gauge covariance of $\phi(a)$, we have

$$i\phi(a) = a_1\partial_{a_2}\phi - a_2\partial_{a_1}\phi \tag{2.4}$$

and g_1 is rewritten as $g_1 = -i(\dot{a}_1 - Ea_2)\partial_{a_1}\phi - i(\dot{a}_2 + Ea_1)\partial_{a_2}\phi$. To fix the decomposition (2.2), we assume a secular term condition

$$\Im\langle v(t), \partial_{a_1}\phi(a(t))\rangle = \Im\langle v(t), \partial_{a_2}\phi(a(t))\rangle = 0, \tag{2.5}$$

which is equivalent to $\Re\langle e^{i\theta(t)}v(t), \phi_{E(t)}\rangle = \Im\langle e^{i\theta(t)}v(t), \partial_E\phi_{E(t)}\rangle = 0$ if $a(t) \neq 0$. By Proposition 2.1, we have

$$|a(0)| + \|v(0)\|_{H^1} \lesssim \|u_0\|_{H^1}. \tag{2.6}$$

Since $u \in C(\mathbb{R}; H^1(\mathbb{R}))$, it follows from the implicit function theorem that there exist a $T > 0$ and $a(t) \in C^1([-T,T]; \mathbb{R}^2)$ such that (2.5) holds for $t \in [-T,T]$. See [18] for the proof.

Differentiating (2.5) with respect to t and substituting (2.3) into the resulting equation, we have

$$\Re\langle Lv + g_1 + g_2 + g_3, \partial_{a_i}\phi(a)\rangle = \sum_{j=1,2}\Im\langle v, \dot{a}_j\partial^2_{a_ia_j}\phi(a)\rangle \quad \text{for } i = 1,\ 2.$$

Using (2.1), (2.4), (2.5) and the fact that E is a real-valued C^1-function of $|a|^2$, we have

$$\begin{aligned}\Re\langle Lv+g_3,\partial_{a_i}\phi\rangle &= \Re\left\langle v,\left(L+\frac{p+1}{2}|\phi|^{p-1}\right)\partial_{a_i}\phi+\frac{p-1}{2}|\phi|^{p-3}\phi^2\overline{\partial_{a_i}\phi}\right\rangle\\ &= \Re\langle v,E\partial_{a_i}\phi+(\partial_{a_i}E)\phi\rangle\\ &= E\Im\langle v,a_2\partial^2_{a_ia_1}\phi-a_1\partial^2_{a_ia_2}\phi\rangle.\end{aligned}$$

Thus we have

$$\mathcal{A}\begin{pmatrix}\dot a_1-Ea_2\\ \dot a_2+Ea_1\end{pmatrix}=\begin{pmatrix}\Re\langle g_2,\partial_{a_2}\phi\rangle\\ \Re\langle g_2,\partial_{a_1}\phi\rangle\end{pmatrix}, \tag{2.7}$$

where

$$\begin{aligned}\mathcal{A}&=-\Im\begin{pmatrix}\langle\partial_{a_1}\phi,\partial_{a_2}\phi\rangle-\langle v,\partial^2_{a_2}\phi\rangle & \langle\partial_{a_2}\phi,\partial_{a_2}\phi\rangle-\langle v,\partial^2_{a_1a_2}\phi\rangle\\ \langle\partial_{a_1}\phi,\partial_{a_1}\phi\rangle-\langle v,\partial^2_{a_1a_2}\phi\rangle & \langle\partial_{a_2}\phi,\partial_{a_1}\phi\rangle-\langle v,\partial^2_{a_1}\phi\rangle\end{pmatrix}\\ &=\begin{pmatrix}1&0\\0&-1\end{pmatrix}+O(\|v\|_{L^2}+|a|^{p-1}).\end{aligned}$$

To prove our main result, we will use the Strichartz estimate and a time global estimate of Kato type. The Strichartz estimate along with $L^\infty-L^1$-estimate for 1-dimensional Schrödinger equations with linear potential was obtained by Goldberg and Schlag [15].

Let $Pu=\langle u,\phi_*\rangle\,\phi_*$, $Qu=(I-P)u$. Then we have the following.

Lemma 2.1 (Strichartz estimate ([15], [23])). *Assume* (V1) *and* (V2).
(a) *There exists a positive number C such that for any $f\in L^2(\mathbb{R})$,*

$$\|e^{-itL}Qf\|_{L^4_tL^\infty_x\cap L^\infty_tL^2_x}\le C\|f\|_{L^2}.$$

(b) *There exists a positive number C such that for any $g(t,x)\in\mathcal{S}(\mathbb{R}^2)$,*

$$\left\|\int_0^t e^{-i(t-s)L}Qg(s,\cdot)ds\right\|_{L^4_tL^\infty_x\cap L^\infty_tL^2_x}\le C\|g\|_{L^{4/3}_tL^1_x+L^1_tL^2_x}.$$

To estimate the the quadratic term of v in g_2, we need the following lemma.

Lemma 2.2. *Assume* (V1) *and* (V2).
(a) *There exists a positive constant C such that for any $f\in\mathcal{S}(\mathbb{R})$,*

$$\|\langle x\rangle^{-3/2}e^{-itL}Qf\|_{L^\infty_xL^2_t}\le C\|f\|_{L^2}, \tag{2.8}$$

$$\|\partial_xe^{-itL}Qf\|_{L^\infty_xL^2_t}\le C\|f\|_{H^{1/2}}. \tag{2.9}$$

(b) *There exists a positive constant C such that for any $g(t,x)\in\mathcal{S}(\mathbb{R}^2)$,*

$$\left\|\int_{\mathbb{R}}e^{isL}Qg(s,\cdot)ds\right\|_{L^2_x}\le C\|\langle x\rangle^{3/2}g\|_{L^1_xL^2_t}, \tag{2.10}$$

Lemma 2.3. *There exists a positive constant C such that for any $g(t,x) \in \mathcal{S}(\mathbb{R}^2)$ and $t \in \mathbb{R}$,*

$$\sum_{j=0,1} \left\| \langle x\rangle^{-1}\partial_x^j \int_0^t e^{-i(t-s)L} Qg(s,\cdot)ds \right\|_{L_x^\infty L_t^2} \le C\|\langle x\rangle g\|_{L_x^1 L_t^2}. \tag{2.11}$$

Furthermore, if $\sup_{x\in\mathbb{R}} e^{\alpha|x|}|V(x)| < \infty$ *holds for an* $\alpha > 0$*, there exists a positive number C such that*

$$\left\| \int_0^t \partial_x e^{-i(t-s)L} Qg(s,\cdot)ds \right\|_{L_x^\infty L_t^2} \le C\|g\|_{L_x^1 L_t^2}. \tag{2.12}$$

Lemma 2.1 is not applicable to a linear term g_3 in (2.3) because we do not have $g_3 \in L_t^{4/3}L_x^1 + L_t^1 L_x^2$. To deal with g_3, we use a lemma by Christ and Kiselev [10] to combine Lemmas 2.1 and 2.2.

Lemma 2.4. *Assume* (V1) *and* (V2)*. Then there exists a positive constant C such that for any $g(t,x) \in \mathcal{S}(\mathbb{R}^2)$ and $t \in \mathbb{R}$,*

$$\left\| \int_0^t e^{-i(t-s)L} Qg(s,\cdot)ds \right\|_{L_t^4 L_x^\infty \cap L_t^\infty L_x^2} \le C\|g\|_{L_t^2 L_x^2(\mathbb{R};\langle x\rangle^5 dx)}.$$

The proof of Lemmas 2.2–2.4 will be given in Section 4.

3. Proof of Theorem 2.1

In this section, we will prove Theorem 2.1. Let us translate (2.3) into an integral equation

$$v(t) = e^{-itL}v(0) - i\sum_{1\le j\le 3}\int_0^t e^{-i(t-s)L} g_j(s)ds. \tag{3.1}$$

All nonlinear terms in (3.1) can be estimated in terms of the following.

$$\begin{aligned}
&\mathbb{M}_1(T) = \sup_{0\le t\le T}|E(t)-E_*|, \quad \mathbb{M}_2(T) = \|\langle x\rangle^{-3/2}Qw\|_{L_x^\infty L^2(0,T)},\\
&\mathbb{M}_3(T) = \|Pv\|_{L_x^\infty L^2(0,T)} + \|\partial_x Pv\|_{L_x^\infty L^2(0,T)},\\
&\mathbb{M}_4(T) = \|Qv\|_{L^q(0,T;W_x^{1,2p})\cap L^\infty(0,T;H_x^1)} + \|Qv\|_{L^4(0,T;L_x^\infty)},\\
&\mathbb{M}_5(T) = \|Pv\|_{L^4(0,T;W_x^{1,\infty})\cap L^\infty(0,T;H_x^1)}, \quad \mathbb{M}_6(T) = \|\partial_x Qv\|_{L_x^\infty L^2(0,T)}.
\end{aligned}$$

where $4/q = 1 - 1/p$.

Proof of Theorem 2.1. By (2.7), we have

$$\begin{aligned}
|\dot a_1 - Ea_2| + |\dot a_2 + Ea_1| &\lesssim \sum_{i=1,2}|\langle g_2, \partial_{a_i}\phi\rangle| \\
&\lesssim \sum_{i=1,2}\left(\|\partial_{a_i}\phi v^2\|_{L^1} + \|\partial_{a_i}\phi f(v)\|_{L^1}\right).
\end{aligned} \tag{3.2}$$

Suppose that the decomposition (2.2) with (2.5) persists for $0 \le t \le T$ and that $\mathbb{M}_i(T)$ $(1 \le i \le 8)$ are bounded. Eq. (3.2) implies that

$$
\begin{aligned}
&\|\dot a_1 - Ea_2\|_{L^1(0,T)} + \|\dot a_2 + Ea_1\|_{L^1(0,T)} \\
&\le C(\mathbb{M})(\|\partial_{a_1}\phi v^2\|_{L^1(0,T;L^1_x)} + \|\partial_{a_2}\phi v^2\|_{L^1(0,T;L^1_x)}) \\
&\le C(\mathbb{M})\left(\sum_{i=1,2}\left\|\langle x\rangle^3\partial_{a_i}\phi\right\|_{L^1_xL^\infty(0,T)}\right)\|\langle x\rangle^{-3/2}v\|^2_{L^\infty_x L^2(0,T)} \\
&\le C(\mathbb{M})\left(\sum_{i=1,2}\left\|\langle x\rangle^5\partial_{a_i}\phi\right\|_{L^\infty(0,T;L^\infty_x)}\right)\|\langle x\rangle^{-3/2}v\|^2_{L^\infty_x L^2(0,T)} \\
&\le C(\mathbb{M})(\mathbb{M}_2(T)+\mathbb{M}_3(T))^2,
\end{aligned}
\tag{3.3}
$$

and

$$
\begin{aligned}
&\|\dot a_1 - Ea_2\|_{L^\infty(0,T)} + \|\dot a_2 + Ea_1\|_{L^\infty(0,T)} \\
&\lesssim \sup_{0\le t\le T}(\|v\|^2_{H^1} + \|v\|^p_{H^1}) \le C(\mathbb{M})(\mathbb{M}_4(T)+\mathbb{M}_5(T))^2.
\end{aligned}
\tag{3.4}
$$

Hereafter we denote by $C(\mathbb{M})$ various functions of $\mathbb{M}_1, \dots, \mathbb{M}_4$ that are bounded in a finite neighborhood of 0. By (3.3) and (2.6),

$$
\mathbb{M}_1(T) \lesssim \|u_0\|_{H^1} + C(\mathbb{M})(\mathbb{M}_2+\mathbb{M}_3)^2. \tag{3.5}
$$

By Remark 1 and (2.5), we have

$$
\begin{aligned}
&|\langle v(t),\phi_*\rangle| \\
&\le |\Re\langle v(t),\phi_*\rangle| + |\Im\langle v(t),\phi_*\rangle| \\
&\le |\Im\langle v(t),\partial_{a_2}\phi - i\phi_*\rangle| + |\Im\langle v(t),\partial_{a_1}\phi - \phi_*\rangle| \\
&\le \|\langle x\rangle^{-5/2}v\|_{L^2_x(\mathbb{R})}\left(\|\langle x\rangle^{5/2}(\partial_{a_1}\phi-\phi_*)\|_{L^2(\mathbb{R})} + \|\langle x\rangle^{5/2}(\partial_{a_2}\phi - i\phi_*)\|_{L^2(\mathbb{R})}\right) \\
&\lesssim |a|^{p-1}\|\langle x\rangle^{-5/2}v\|_{L^2_x(\mathbb{R})},
\end{aligned}
$$

and that

$$
\begin{aligned}
\mathbb{M}_3(T) &\le \sup_{t\in[0,T]}|a(t)|^{p-1}\|\langle x\rangle^{-5/2}w\|_{L^2(0,T;L^2_x)} \\
&\le C(\mathbb{M})\mathbb{M}_1(T)(\mathbb{M}_2(T)+\mathbb{M}_3(T)).
\end{aligned}
\tag{3.6}
$$

Similarly, we have

$$
\mathbb{M}_5(T) \le C(\mathbb{M})\mathbb{M}_1(T)(\mathbb{M}_4(T)+\mathbb{M}_5(T)). \tag{3.7}
$$

Next, we will estimate $\mathbb{M}_2(T)$. By (3.1),

$$
\mathbb{M}_2(T) \le I_1 + I_2 + I_3 + I_4,
$$

where

$$I_1 = \|\langle x\rangle^{-3/2} e^{-itL} Qw(0)\|_{L^\infty_x L^2(0,T)} \lesssim \|w(0)\|_{L^2},$$

and

$$I_j = \left\| \langle x\rangle^{-3/2} \int_0^t e^{-i(t-s)L} Qg_{j-1}(s)ds \right\|_{L^\infty_x L^2(0,T)} \quad \text{for } 2 \le j \le 4.$$

By Lemma 2.3, Remark 1, (3.3) and (3.4),

$$\begin{aligned}
I_2 &\lesssim \|\langle x\rangle Qg_1\|_{L^1_x L^2(0,T)} \\
&\le \|\langle x\rangle Q\partial_{a_1}\phi(a(t))\|_{L^1_x L^\infty(0,T)} \|\dot a_1 - Ea_2\|_{L^2(0,T)} \\
&\quad + \|\langle x\rangle Q\partial_{a_2}\phi(a(t))\|_{L^1_x L^\infty(0,T)} \|\dot a_2 + Ea_1\|_{L^2(0,T)} \\
&\le C(\mathbb{M})(\mathbb{M}_2(T) + \mathbb{M}_3(T) + \mathbb{M}_4(T) + \mathbb{M}_5(T))^2.
\end{aligned}$$

Lemmas 2.2 and 2.3 yield

$$\begin{aligned}
I_3 &\lesssim \|\langle x\rangle |\phi(a(t))|^{p-2} v^2\|_{L^1_x L^2(0,T)} + \int_0^T \||v|^p\|_{L^2_x} ds \\
&\lesssim \left\| \langle x\rangle^{5/2} \sup_{t\in[0,T]} |\phi(a(t))|^{p-2} \right\|_{L^1_x} \|\langle x\rangle^{-3/2} v\|_{L^\infty_x L^2(0,T)} \|v\|_{L^\infty(0,T;L^\infty_x)} \\
&\quad + \|v(t)\|^{p-q}_{L^\infty(0,T;L^\infty_x)} \|v(t)\|^q_{L^q(0,T;L^{2p}_x)} \\
&\le C(\mathbb{M}) \sum_{2\le i\le 5} \mathbb{M}_i(T)^2.
\end{aligned}$$

where $4/q = 1 - 1/p$. By Proposition 2.1 and Lemma 2.3,

$$\begin{aligned}
I_4 \lesssim \|\langle x\rangle g_3\|_{L^1_x L^2(0,T)} &\lesssim \left\| \langle x\rangle^{5/2} \sup_{t\in[0,T]} |\phi(a(t))|^{p-1} \right\|_{L^1_x} \|\langle x\rangle^{-3/2} v\|_{L^\infty_x L^2(0,T)} \\
&\le C(\mathbb{M})\mathbb{M}_1(T)(\mathbb{M}_2(T) + \mathbb{M}_3(T)).
\end{aligned}$$

Combining the above, we see that

$$\mathbb{M}_2(T) \le \|u_0\|_{H^1} + C(\mathbb{M}) \sum_{1\le i\le 5} \mathbb{M}_i(T)^2. \tag{3.8}$$

Likewise, we have

$$\mathbb{M}_6(T) \le \|u_0\|_{H^1} + C(\mathbb{M}) \sum_{1\le i\le 5} \mathbb{M}_i(T)^2. \tag{3.9}$$

Finally, we will estimate $\mathbb{M}_4(T)$. In view of (3.1),

$$\mathbb{M}_4(T) \le J_1 + J_2 + J_3 + J_4,$$

where

$$J_1 = \left\|e^{-itL}Qv(0)\right\|_{L^\infty(0,T;H^1_x)\cap L^q(0,T;W^{1,2p}_x)\cap L^4(0,T;L^\infty_x)}$$
$$J_j = \left\|\int_0^t e^{-i(t-s)L}Qg_{j-1}(s)ds\right\|_{L^\infty(0,T;H^1_x)\cap L^q(0,T;W^{1,2p}_x)\cap L^4(0,T;L^\infty_x)}$$

for $2 \le j \le 4$, where $4/q = 1 - 1/p$. By Lemma 2.1 and (3.3),

$$J_1 \lesssim \|v(0)\|_{H^1},$$

and

$$\begin{aligned} J_2 &\lesssim \|Qg_1(s)\|_{L^1(0,T;H^1_x)} \\ &\lesssim \|\dot{a}_1 - Ea_2\|_{L^1(0,T)} \sup_{t\in[0,T]} \|Q\partial_{a_1}\phi(a(t))\|_{H^1_x} \\ &\quad + \|\dot{a}_2 + Ea_2\|_{L^1(0,T)} \sup_{t\in[0,T]} \|Q\partial_{a_2}\phi(a(t))\|_{H^1_x} \\ &\le C(\mathbb{M})(\mathbb{M}_2(T)^2 + \mathbb{M}_3(T)^2). \end{aligned}$$

Using Minkowski's inequality and Lemma 2.1, we have

$$\begin{aligned} &J_3 \lesssim \||\phi(a(t))|^{p-2}v^2\|_{L^{6/5}(0,T;W^{1,6/5}_x)} + \|f(v)\|_{L^1(0,T;H^1_x)} \\ &\lesssim \|\langle x\rangle^2 \sup_{t\in[0,T]} |\phi(a(t))|^{p-2}\|_{W^{1,6/5}_x} \|v\|^{2/3}_{L^4(0,T;L^\infty_x)} \left(\sum_{i=0,1} \|\langle x\rangle^{-3/2}\partial_x^i v\|_{L^\infty_x L^2(0,T)}\right)^{4/3} \\ &\quad + \|v(t)\|^{p-q}_{L^\infty(0,T;H^1_x)} \|v(t)\|^q_{L^q(0,T;W^{1,2p}_x)} \\ &\le C(\mathbb{M}) \sum_{2\le i\le 6} \mathbb{M}_i(T)^2, \end{aligned}$$

where $4/q = 1 - 1/p$. By Proposition 2.1 and Lemma 2.4,

$$\begin{aligned} J_4 &\lesssim \|\langle x\rangle^{5/2} g_3\|_{L^2(0,T;L^2_x)} \\ &\lesssim \|\sup_{t\in[0,T]} \langle x\rangle^4 |\phi(a(t))|^{p-1}\|_{L^2_x} (\|\langle x\rangle^{-3/2}v\|_{L^\infty_x L^2(0,T)} + \|\partial_x v\|_{L^\infty_x L^2(0,T)}) \\ &\lesssim \mathbb{M}_1(T)(\mathbb{M}_2(T) + \mathbb{M}_3(T) + \mathbb{M}_6(T)). \end{aligned}$$

Combining the above, we have

$$\mathbb{M}_4(T) \le \|u_0\|_{H^1} + C(\mathbb{M}) \sum_{1\le i\le 6} \mathbb{M}_i(T)^2. \tag{3.10}$$

It follows from (3.5)–(3.10) that if ε_0 is sufficiently small,

$$\sum_{1\le i\le 6} \mathbb{M}_i(T) \lesssim \|u_0\|_{H^1}. \tag{3.11}$$

Thus by continuation argument, we may let $T \to \infty$.

By (3.3) and the fact that

$$d|a(t)|^2/dt = 2a_1(\dot{a}_1 - Ea_2) + 2a_2(\dot{a}_2 + Ea_1) \in L^1(0,\infty),$$

there exists an $E_+ < 0$ satisfying $\lim_{t\to\infty} E(t) = E_+$ and $|E_+ - E_*|^{1/(p-1)} \lesssim \|u_0\|_{H^1}$. In view of (3.11), we have

$$\begin{aligned} v_1 :=& -i \sum_{1\le j\le 3} \int_0^\infty e^{isL} Qg_j(s)ds \in H^1(\mathbb{R}),\\ \|v_1\|_{H^1} \lesssim& \|g_1(s)\|_{L_t^1 H_x^1} + \|g_2\|_{L_t^{6/5} W_x^{1,6/5} + L_t^1 H_x^1} + \|\langle x\rangle^{5/2} g_3\|_{L_t^2 H_x^1}\\ \lesssim& \|u_0\|_{H^1}, \end{aligned}$$

and

$$\lim_{t\to\infty} \|Qv(t) - e^{-itL}(Qv(0) + v_1)\|_{H^1} = 0.$$

By [15], we have $\|e^{-itL}Qf\|_{L^\infty} \lesssim t^{-1/2}\|f\|_{L^1}$. Since $L^1(\mathbb{R})$ is dense in $H^1(\mathbb{R})$, it follows that $\|e^{-itL}(Qv(0)+v_1)\|_{L^\infty} \to 0$ as $t \to \infty$, and that

$$\begin{aligned} &\|Qv(t)\|_{L^\infty}\\ \le& \|Qv(t) - e^{-itL}(Qv(0)+v_1)\|_{H^1} + \|Qe^{-itL}(Qv(0)+v_1)\|_{L^\infty}\\ \to& 0 \quad \text{as } t \to \infty. \end{aligned} \tag{3.12}$$

Analogously to (3.6), we have

$$\|Pv(t)\|_{L^\infty} \lesssim |a(t)|^{p-1}\|Qv(t)\|_{L^\infty}. \tag{3.13}$$

Combining (3.12) and (3.13), we have $\lim_{t\to\infty} \|Pv(t)\|_{H^1} = 0$. Thus by (2.2),

$$\lim_{t\to\infty} \left\|u(t) - \phi(a(t)) - e^{-itL}Q(v(0)+v_1)\right\|_{H^1} = 0.$$

From [1] and [44], we see that there exists a $v_+ \in H^1$ such that

$$\lim_{t\to\infty} \|e^{-itL}Q(v(0)+v_1) - We^{it\partial_x^2}v_+\|_{H^1} = 0,$$

where $W = \lim_{t\to\infty} e^{-itL}e^{-it\partial_x^2}$. Thus we complete the proof of Theorem 2.1. □

4. Linear estimates

Let $R(\lambda) = (\lambda - L)^{-1}$ and let $dE_{ac}(\lambda)$ be the absolute continuous part of the spectral measure of L. We have $R(\lambda - i0) = R(\lambda + i0)$ for $\lambda < 0$ and it follows from the spectral decomposition theorem that

$$\begin{aligned} Qe^{-itL}f =& \int_{-\infty}^\infty e^{-it\lambda} dE_{ac}(\lambda)f\\ =& \frac{1}{2\pi i}\int_{-\infty}^\infty e^{-it\lambda} Q(R(\lambda - i0) - R(\lambda + i0))fd\lambda. \end{aligned}$$

To prove Lemmas 2.2–2.4, we will apply Plancherel's theorem to the above formula.

4.1. High energy estimate

To begin with, we will estimate the high frequency part of the resolvent operators $R(\lambda \pm i0)$. Let $\chi(x)$ be a smooth function satisfying $0 \le \chi(x) \le 1$ for $x \in \mathbb{R}$ and

$$\chi(x) = \begin{cases} 1 & \text{if } x \ge 2, \\ 0 & \text{if } x \le 1, \end{cases}$$

and let $\chi_M(x)$ be an even function satisfying $\chi_M(x) = \chi(x - M)$ for $x \ge 0$.

Lemma 4.1. *Assume* (V1) *and* (V2). *Then there exist positive numbers M and C such that*

$$\sup_x \|\chi_M(\sqrt{\lambda}) R(\lambda \pm i0) u\|_{L^2_\lambda(0,\infty)} \le C\|u\|_{L^2(\mathbb{R})},$$
$$\sup_x \|\chi_M(\sqrt{\lambda}) \partial_x (R(\lambda - i0) - R(\lambda + i0)) u\|_{L^2_\lambda(0,\infty)} \le C\|u\|_{H^{1/2}(\mathbb{R})}$$

for every $u \in \mathcal{S}(\mathbb{R})$.

Lemma 4.2. *Assume* (V1) *and* (V2). *Then there exist positive numbers M and C such that*

$$\sum_{i=0,1} \|\chi_M(\sqrt{\lambda}) \partial_x^i R(\lambda \pm i0) u\|_{L^\infty_{x,\lambda}(\mathbb{R}^2)} \le C\|u\|_{L^1_x(\mathbb{R})}$$

for every $\lambda \in \mathbb{R}$ and $u \in \mathcal{S}(\mathbb{R})$.

Proof of Lemmas 4.1 *and* 4.2. Let $R_0(\lambda) = (\lambda + \partial_x^2)^{-1}$ and

$$G_1(x,k) = \frac{e^{ik|x|}}{2ik}, \quad G_2(x,k) = \frac{e^{-k|x|}}{-2k}.$$

We remark that $R_0(\lambda \mp i0)\delta = G_1(x, \pm k)$ for $\lambda = k^2$ with $k \ge 0$ and $R_0(\lambda)\delta = G_2(x,k)$ for $\lambda = -k^2$ with $k > 0$. If M is sufficiently large, we have

$$R(\lambda \pm i0)u = \sum_{j=0}^\infty R_0(\lambda \pm i0)(VR_0(\lambda \pm i0))^j u \tag{4.1}$$

for $\lambda \in \mathbb{R}$ with $|\lambda| \ge M$ and $u \in \mathcal{S}(\mathbb{R})$ since

$$\|\langle\cdot\rangle^{-1} R_0(\lambda \pm i0) \langle\cdot\rangle^{-1}\|_{B(L^2(\mathbb{R}))} \lesssim \langle\lambda\rangle^{-1/2}.$$

By the definition of $G_1(x,k)$ and Plancherel's theorem,

$$\begin{aligned} &\sup_x \int_{\mathbb{R}} dk \langle k\rangle \left|\chi_M(k)(G_1(\cdot,k) * u)(x)\right|^2 \\ \lesssim &\sup_x \int_{\mathbb{R}} dk \langle k\rangle^{-1} \left(\left|\int_x^\infty u(y)e^{-iky}dy\right|^2 + \left|\int_{-\infty}^x u(y)e^{iky}dy\right|^2 \right) dk \\ \lesssim &\|u\|_{L^2_x}^2. \end{aligned} \tag{4.2}$$

For $k=\sqrt{\lambda}\ge 0$, it holds that

$$
\begin{aligned}
&F_{1,n}(x,\pm k) := R_0(\lambda\mp i0)(VR_0(\lambda\mp i0))^n u(x)\\
&=\int_{\mathbb{R}^{n+1}} G_1(x-x_1,\pm k)\prod_{j=1}^{n}\left(V(x_j)G_1(x_j-x_{j+1},\pm k)\right)u(x_{n+1})dx_1\cdots dx_{n+1}.
\end{aligned}
\tag{4.3}
$$

Combining Minkowski's inequality with (4.2), we have for $n\ge 1$,

$$
\begin{aligned}
&\|\chi_M(\sqrt{\lambda})F_{1,n}(x,\pm\sqrt{\lambda})\|_{L^2_\lambda(0,\infty)}\\
&\quad\lesssim \int_{\mathbb{R}^{n+1}} dx_1\cdots dx_n\prod_{j=0}^{n-1}\{|V(x_{j+1})|\sup_{|k|\ge M}(|kG_1(x_j-x_{j+1},k)|)\}\\
&\quad\quad\times\left\{\int_{\mathbb{R}} dk\chi_M(k)^2\langle k\rangle^{-2n+1}\left|(G_1(\cdot,k)*u)(x_n)\right|^2\right\}^{1/2}\\
&\quad\lesssim \|V\|_{L^1}^n\sup_{x_n}\left(\int_{\mathbb{R}} dk k^{-2n+1}\chi_M(k)^2\left|(G_1(\cdot,k)*u)(x_n)\right|^2\right)^{1/2}\\
&\quad\lesssim M^{-n+1/2}\|V\|_{L^1}^n\|u\|_{L^2},
\end{aligned}
\tag{4.4}
$$

where $x_0=x$. Similarly, we have

$$
\|\chi_M(\sqrt{\lambda})\partial_x F_{1,n}(x,\pm\sqrt{\lambda})\|_{L^2_\lambda(0,\infty)}\lesssim M^{-n+3/2}\|V\|_{L^1}^n\|u\|_{L^2}.
\tag{4.5}
$$

Since

$$
\begin{aligned}
\partial_x(R_0(\lambda-i0)-R_0(\lambda+i0))u &= \frac{1}{2ik}\partial_x\int_{\mathbb{R}} dyu(y)(e^{-ik(x-y)}+e^{ik(x-y)})dy\\
&=\sqrt{\frac{\pi}{2}}\left(e^{ikx}\hat{u}(k)-e^{-ikx}\hat{u}(-k)\right),
\end{aligned}
$$

it follows from Plancherel's identity

$$
\begin{aligned}
\|\partial_x(R_0(\lambda-i0)-R_0(\lambda+i0))u\|_{L^2_\lambda} &\lesssim \left(\int_{\mathbb{R}} dk\langle k\rangle|\hat{u}(k)|^2dk\right)^{1/2}\\
&\lesssim \|u\|_{H^{1/2}}.
\end{aligned}
\tag{4.6}
$$

Combining (4.1), (4.4)–(4.6), we obtain Lemma 4.1.

Next, we will prove Lemma 4.2. In view of (4.3), we have

$$
\begin{aligned}
&\sup_{x,k}\left(|\chi_M(k)F_{1,n}(x,k)|+|\chi_M(k)\partial_x F_{1,n}(x,k)|\right)\\
&\quad\lesssim \sup_{x\in\mathbb{R},|k|\ge M}\langle k\rangle^{-n}\int_{\mathbb{R}^{n+1}}\prod_{j=1}^{n}|V(x_j)||u(x_{n+1})|dx_1\cdots dx_{n+1}\\
&\quad\lesssim M^{-n}\|V\|_{L^1(\mathbb{R})}^n\|u\|_{L^1(\mathbb{R})}.
\end{aligned}
\tag{4.7}
$$

For $\lambda = -k^2$ with $k > 0$, we have

$$\begin{aligned} F_{2,n}(x,k) &:= R_0(\lambda)VR_0(\lambda)^n u(x) \\ &= \int_{\mathbb{R}^{n+1}} G_2(x-x_1,k)\prod_{j=1}^{n}\left(V(x_j)G_2(x_j-x_{j+1},k)\right)u(x_{n+1})dx_1\cdots dx_{n+1}, \end{aligned}$$

and it follows that

$$\begin{aligned} &\sup_{x,k}\left(|\chi_M(k)F_{2,n}(x,k)| + |\chi_M(k)\partial_x F_{2,n}(x,k)|\right) \\ &\quad \lesssim M^{-n}\|V\|^n_{L^1(\mathbb{R})}\|u\|_{L^1(\mathbb{R})}. \end{aligned} \tag{4.8}$$

Combining (4.7) and (4.8), we obtain Lemma 4.2. □

4.2. Low energy estimate

Next, we will estimate the low frequency part of $R(\lambda \pm i0)$. Let $\widetilde{\chi}_M(x) = 1 - \chi_M(x)$.

Lemma 4.3. *Assume* (V1) *and* (V2). *Let M be a positive number given in Lemma* 4.1. *Then there exists a positive number C such that for every $u \in \mathcal{S}(\mathbb{R})$,*

$$\begin{gathered} \sup_x \|\langle x\rangle^{-3/2}\widetilde{\chi}_M(\sqrt{\lambda})R(\lambda \pm i0)u\|_{L^2_\lambda(0,\infty)} \le C\|u\|_{L^2(\mathbb{R})}, \\ \sup_x \|\widetilde{\chi}_M(\sqrt{\lambda})\partial_x R(\lambda \pm i0)u\|_{L^2_\lambda(0,\infty)} \le C\|u\|_{L^2(\mathbb{R})}. \end{gathered}$$

Lemma 4.4. *Assume* (V1) *and* (V2). *Let M be a positive number given in Lemma* 4.2. *Then there exists a positive number C such that*

$$\sum_{j=0,1}\sup_{\lambda\in[-M,M]}\left\|\langle x\rangle^{-1}\partial_x^j R(\lambda \pm i0)u\right\|_{L^\infty_x} \le C\|\langle x\rangle u\|_{L^1_x(\mathbb{R})}$$

for every $\lambda \in \mathbb{R}$ and $u \in \mathcal{S}(\mathbb{R})$.

Furthermore, if $\sup_{x\in\mathbb{R}} e^{\alpha|x|}|V(x)| < \infty$ holds for an $\alpha > 0$, there exists a positive number C such that

$$\sup_{\lambda\in[-M,M]}\|\partial_x R(\lambda \pm i0)u\|_{L^\infty_x} \le C\|u\|_{L^1_x(\mathbb{R})}$$

for every $\lambda \in \mathbb{R}$ and $u \in \mathcal{S}(\mathbb{R})$.

Before we start to prove Lemmas 4.3 and 4.4, we recall some properties of the Jost functions. We refer the readers to Deift-Trubowitz [13] for the details. Let $f_1(x,k)$ and $f_2(x,k)$ be the solutions to $Lu = k^2u$ satisfying

$$\lim_{x\to\infty}|e^{-ikx}f_1(x,k) - 1| = 0, \quad \lim_{x\to-\infty}|e^{ikx}f_2(x,k) - 1| = 0,$$

and let $m_1(x,k) = e^{-ikx}f_1(x,k)$ and $m_2(x,k) = e^{ikx}f_2(x,k)$. For each x, $m_1(x,k)$ and $m_2(x,k)$ are analytic in k with $\Im k > 0$, continuous in k with $\Im k \geq 0$, and satisfy

$$m_1(x,k) = 1 + \int_x^\infty \frac{e^{2ik(y-x)}-1}{2ik} V(y) m_1(y,k)dy,$$
$$m_2(x,k) = 1 + \int_{-\infty}^x \frac{e^{2ik(x-y)}-1}{2ik} V(y) m_2(y,k)dy.$$

Deift-Trubowitz [13] tells us that for $x \in \mathbb{R}$ and $k \in \mathbb{C}$ with $\Im k \geq 0$,

$$|m_1(x,k)-1| \lesssim \langle k\rangle^{-1}(1+\max(-x,0)) \int_x^\infty dy \langle y\rangle |V(y)|, \tag{4.9}$$

$$|m_2(x,k)-1| \lesssim \langle k\rangle^{-1}(1+\max(x,0)) \int_{-\infty}^x dy \langle y\rangle |V(y)|, \tag{4.10}$$

$$|\partial_x m_1(x,k)| \lesssim \langle k\rangle^{-1} \int_x^\infty dy \langle y\rangle |V(y)|, \tag{4.11}$$

$$|\partial_x m_2(x,k)| \lesssim \langle k\rangle^{-1} \int_{-\infty}^x dy \langle y\rangle |V(y)|. \tag{4.12}$$

For every $\delta > 0$, there exists a $C_\delta > 0$ such that for $x \in \mathbb{R}$ and $k \in \mathbb{C}$ with $\Im k \geq 0$ and $|k| \geq \delta$,

$$|m_1(x,k)-1| \leq C_\delta \int_x^\infty dy |V(y)|, \tag{4.13}$$

$$|m_2(x,k)-1| \leq C_\delta \int_{-\infty}^x dy |V(y)|. \tag{4.14}$$

There exist continuous functions $T(k)$, $R_1(k)$ and $R_2(k)$ on $\mathbb{R}$ satisfying

$$f_2(x,k) = \frac{R_1(k)}{T(k)} f_1(x,k) + \frac{1}{T(k)} f_1(x,-k), \tag{4.15}$$

$$f_1(x,k) = \frac{R_2(k)}{T(k)} f_2(x,k) + \frac{1}{T(k)} f_2(x,-k) \tag{4.16}$$

for $k \in \mathbb{R}$ with $k \neq 0$. Let $[f(x), g(x)] = f'(x)g(x) - f(x)g'(x)$ and let $W(k) = [f_1(x,k), f_2(x,k)]$. As is well known, the Wronskian $W(k)$ does not depend on x and $W(k) = 2ik/T(k) \neq 0$ for $k \neq 0$. Moreover, the assumption (V2) implies $W(0) \neq 0$.

Proof of Lemma 4.3. For $\lambda = k^2$ with $k \geq 0$, the resolvent operator $R(\lambda \pm i0)$ has the kernel

$$K_\pm(x,y,k) = \begin{cases} -\dfrac{f_1(x,\pm k)f_2(y,\pm k)}{W(\pm k)} & \text{for } x > y., \\ -\dfrac{f_2(x,\pm k)f_1(y,\pm k)}{W(\pm k)} & \text{for } x < y. \end{cases} \tag{4.17}$$

That is,

$$\begin{aligned} R(\lambda \pm i0)u = & -\frac{f_1(x,\pm k)}{W(\pm k)} \int_{-\infty}^{x} dy f_2(y,\pm k)u(y) \\ & -\frac{f_2(x,\pm k)}{W(\pm k)} \int_{x}^{\infty} dy f_1(y,\pm k)u(y) \\ =: & \, I(\pm k) + II(\pm k). \end{aligned} \tag{4.18}$$

We will estimate L^2_λ-norm of the right-hand side of (4.18). We may assume $x > 0$. Let

$$I = -\frac{f_1(x,k)}{W(k)}(I_1 + I_2 + I_3),$$

where

$$\begin{aligned} I_1 &= \int_0^x dy f_2(y,k)u(y), \\ I_2 &= \int_{-\infty}^0 dy e^{-iky}u(y) = \sqrt{2\pi}\mathcal{F}_y(\chi_{(-\infty,0]}u)(k), \\ I_3 &= \int_{-\infty}^0 dy e^{-iky}(m_2(y,k)-1)u(y). \end{aligned}$$

By (4.9) and (4.10), we see

$$\sup_{x>0} \big(|f_1(x,k)| + \langle x\rangle^{-1}|f_2(x,k)|\big) < \infty, \tag{4.19}$$

$$|I_1| \lesssim \int_0^x dy \langle y\rangle |u(y)| \lesssim \langle x\rangle^{3/2}\|u\|_{L^2}, \tag{4.20}$$

and

$$\sup_{x>0} |I_3| \lesssim \left\| \int_{-\infty}^x dy V(y) \right\|_{L^2_x(-\infty,0)} \|u\|_{L^2} \lesssim \|u\|_{L^2}. \tag{4.21}$$

Similarly, we have $II = -\frac{f_2(x,k)}{W(k)}(II_1 + II_2)$ with

$$\begin{aligned} II_1 &= \int_x^\infty dy e^{iky}u(y) = \sqrt{2\pi}\mathcal{F}^{-1}(\chi_{[x,\infty)}u)(k), \\ II_2 &= \int_x^\infty dy e^{iky}(m_1(y,k)-1)u(y), \end{aligned}$$

and

$$\sup_x |II_2| \lesssim \|u\|_{L^2}. \tag{4.22}$$

Obviously,

$$\sup_{x>0}(\|I_2\|_{L^2_k} + \|II_1\|_{L^2_k}) \lesssim \|u\|_{L^2}. \tag{4.23}$$

Since $W(k) \neq 0$ for every $k \in \mathbb{R}$ and $\widetilde{\chi}_M(k)$ is compactly supported, it follows from (4.19)–(4.23) that

$$\sup_x \int_{\mathbb{R}} dk|k| \left| \widetilde{\chi}_M(k) \int_{\mathbb{R}} dy K_\pm(x,y,k)u(y) \right|^2 \lesssim \langle x\rangle^3 \|u\|_{L^2}^2.$$

By (4.17), we have

$$\begin{aligned} \partial_x R(\lambda \pm i0)u &= -\frac{\partial_x f_1(x,\pm k)}{W(\pm k)} \int_{-\infty}^x dy f_2(y,\pm k)u(y) \\ &\quad - \frac{\partial_x f_2(x,\pm k)}{W(\pm k)} \int_x^\infty dy f_1(y,\pm k)u(y) \\ &= III(\pm k) + IV(\pm k). \end{aligned}$$

By symmetry, it suffices to consider the case where $x > 0$. Let us rewrite III as

$$III = -\frac{e^{ikx}}{W(k)}(III_1 + III_2 + III_3),$$

where

$$\begin{aligned} III_1 &= (ikm_1(x,k) + \partial_x m_1(x,k)) \int_{-\infty}^0 dy f_2(y,k)u(y), \\ III_2 &= ikm_1(x,k) \int_0^x dy f_2(y,k)u(y), \\ III_3 &= \partial_x m_1(x,k) \int_0^x dy f_2(y,k)u(y). \end{aligned}$$

By (4.9) and (4.11),

$$\sup_{k\in[-M,M]} \sup_{x\geq 0} (|km_1(x,k)| + |\partial_x m_1(x,k)|) < \infty.$$

Thus we have

$$\|III_1\|_{L_k^2(-M,M)+L_k^\infty(-M,M)} \lesssim \|u\|_{L^2} \tag{4.24}$$

in the same way as (4.21) and (4.23). Under the assumption (V1) and (V2), we have $1/T(k) \simeq k^{-1}$ as $k \to 0$ and $R_1(k)$ and $R_2(k)$ are continuous in $k \in \mathbb{R}$. Hence by using (4.15), we see that

$$\|III_2\|_{L_k^2(-M,M)+L_k^\infty(-M,M)} \lesssim \|u\|_{L^2} \tag{4.25}$$

follows in the same way as (4.21) and (4.23). By (V1), (4.11) and Schwarz's inequality, we have

$$\begin{aligned} |III_3| &\lesssim \left(\int_x^\infty dy|V(y)|\right) \int_0^x dy\langle y\rangle u(y) \\ &\lesssim \langle x\rangle^{-1/2} \|\langle x\rangle^2 V\|_{L^1} \|u\|_{L^2}. \end{aligned} \tag{4.26}$$

Similarly, we have

$$\sup_{x>0} \|IV\|_{L^2_k(-M,M)+L^\infty_k(-M,M)} \lesssim \|u\|_{L^2}. \tag{4.27}$$

Combining (4.25)–(4.27), we obtain

$$\sup_{x>0}\int_{\mathbb{R}} dk|k|\left|\widetilde{\chi}_M(k)\int_{\mathbb{R}} dy\partial_x K_\pm(x,y,k)u(y)\right|^2 \lesssim \|u\|^2_{L^2}.$$

Thus we complete the proof of Lemma 4.3. □

Proof of Lemma 4.4. Since $W(k)$ is continuous and $W(k) \neq 0$ on $\mathbb{R}$, it follows from (4.9)–(4.12) and (4.17) that

$$\sup_{k\in[-M,M]}\sup_{x,y\in\mathbb{R}} \langle x\rangle^{-1}|\partial_x^j K_\pm(x,y,k)|\langle y\rangle^{-1} < \infty \quad \text{for } j=0,\,1. \tag{4.28}$$

Thus we have

$$\sup_{\lambda\geq 0}\|\langle x\rangle^{-1}\widetilde{\chi}_M(\sqrt{\lambda})\partial_x^j R(\lambda\pm i0)u\|_{L^\infty_x} \leq C\|\langle x\rangle u\|_{L^1_x(\mathbb{R})} \quad \text{for } j=0,\,1. \tag{4.29}$$

For $\lambda<0$, the resolvent operator $R(\lambda)$ has the kernel

$$K(x,y,\lambda) = \begin{cases} -\dfrac{f_1(x,ik)f_2(y,ik)}{W(ik)} & \text{for } x>y, \\ -\dfrac{f_2(x,ik)f_1(y,ik)}{W(ik)} & \text{for } x<y, \end{cases} \tag{4.30}$$

where $k=\sqrt{-\lambda}$. We have $W(ik)=0$ for a $k>0$ if and only if $\lambda=-k^2$ is an eigenvalue of L. Thus the assumption (V2) yields that $W(ik)$ has a simple pole at $k=\sqrt{|E_*|}$ and $W(ik)\neq 0$ for $k\in[0,\sqrt{|E_*|})\cup(\sqrt{|E_*|},\infty)$. Thus by (4.9)–(4.12), we have

$$\sup_{\lambda<0}\left(\left\|\langle x\rangle^{-1}R(\lambda)Qu\right\|_{L^\infty_x} + \left\|\langle x\rangle^{-1}\partial_x R(\lambda)Qu\right\|_{L^\infty_x}\right) \lesssim \|\langle x\rangle u\|_{L^1_x}.$$

Combining the above, we obtain the former part of Lemma 4.4.

Next, we will estimate $\partial_x K_\pm(x,y,k)$ and $\partial_x K(x,y,\lambda)$ assuming that $V(x)$ decays like $e^{-\alpha|x|}$. In view of (4.9)–(4.12), we have

$$\sup_{k\in[-M,M]}\left(\sup_{x>0>y}|\partial_x K_\pm(x,y,k)| + \sup_{x<0<y}|\partial_x K_\pm(x,y,k)|\right) < \infty. \tag{4.31}$$

Suppose x and y has the same sign. By symmetry, we may assume $x>y>0$. By (4.9)–(4.11), we have $\sup_{k\in[-M,M]}|\partial_x m_2(y,k)| \lesssim \langle y\rangle \lesssim \langle x\rangle$, and

$$|\partial_x m_1(x,k)m_2(y,k)| \lesssim \langle x\rangle\int_x^\infty dy\langle y\rangle|V(y)| \lesssim \langle x\rangle e^{-\alpha|x|}.$$

As in the proof of Lemma 4.3, it follows from (4.15) that

$$\sup_{k\in[-M,M]}\sup_{y>0}|km_2(y,k)|<\infty.$$

Combining the above, we see that

$$\partial_x K_\pm(x,y,k)=\frac{-e^{\pm ik(x-y)}}{W(\pm k)}\{\pm ikm_1(x,\pm k)+\partial_x m_1(x,\pm k)\}m_2(y,\pm k).$$

are uniformly bounded with respect to $x>y>0$ and $k\in[-M,M]$.

By (4.9)–(4.14), we have

$$\sup_{\lambda\le-\alpha^2/16}\sup_{x,y\in\mathbb{R}}|(\lambda-E_*)\partial_x K(x,y,\lambda)|<\infty,$$
$$\sup_{\lambda<0}|\lambda-E_*|\left(\sup_{x<0<y}|\partial_x K(x,y,\lambda)|+\sup_{y<0<x}|\partial_x K(x,y,\lambda)|\right)<\infty.$$

Now we will prove the remaining case by using (4.15) and (4.16). We may assume $x>y>0$ by symmetry. Under the assumption $\sup_{x\in\mathbb{R}}e^{\alpha|x|}|V(x)|<\infty$, $m_1(x,k)$ and $m_2(x,k)$ are analytic in k with $\Im k>-\alpha/2$ and there exists a $C_a>0$ for every $a>0$ such that

$$|m_1(x,k)-1|\le C_a\int_x^\infty dy\langle y\rangle e^{-2\Im ky}|V(y)| \tag{4.32}$$

for $x>-a$ and $-\alpha/2<\Im k<0$, and

$$|m_2(x,k)-1|\le C_a\int_{-\infty}^x dy\langle y\rangle e^{2\Im ky}|V(y)| \tag{4.33}$$

for $x<a$ and $-\alpha/2<\Im k<0$. Furthermore, we see that

$$\begin{aligned}\frac{1}{T(k)}&=\frac{1}{2ik}[f_1(x,k),f_2(x,k)],\\ \frac{R_1(k)}{T(k)}&=\frac{1}{2ik}[f_2(x,k),f_1(x,-k)],\quad \frac{R_2(k)}{T(k)}=\frac{1}{2ik}[f_2(x,-k),f_1(x,k)],\end{aligned}$$

are meromorphic in k with $|\Im k|<\alpha/2$ and have a pole of order 1 at the origin. Hence it follows from (4.15), (4.9) and (4.32) that

$$\sup_{0<k\le\alpha/4}\sup_{y>0}k|m_2(y,ik)|<\infty.$$

By (4.10) and (4.11),

$$\sup_{0<k\le\alpha/4}|m_2(y,ik)|\lesssim\langle y\rangle\lesssim\langle x\rangle,$$
$$|\partial_x m_1(x,ik)|\lesssim\int_x^\infty dy\langle y\rangle|V(y)|\lesssim\langle x\rangle e^{-\alpha x}.$$

Hence, we have

$$\begin{aligned}&\sup_{0<k\le\alpha/4}\sup_{x>y>0}|\partial_x f_1(x,ik)f_2(y,ik)|\\ =&\sup_{0<k\le\alpha/4}\sup_{x>y>0}e^{-kx}\left|(-km_1(x,ik)+\partial_x m_1(x,ik))m_2(y,ik)\right|<\infty.\end{aligned}$$

Combining the above, we obtain $\sup_{\lambda<0}\|\partial_x R(\lambda)Q\|_{B(L^1,L^\infty)}<\infty$. Thus we prove the latter part of Lemma 4.4. □

4.3. Proof of Lemmas 2.2–2.4

Now, we are in position to prove Lemmas 2.2–2.4.

Proof of Lemma 2.2. By the spectral decomposition theorem, we have

$$Qe^{-itL}f=e^{-itL}\chi_M(L)f+Qe^{-itL}\widetilde{\chi}_M(L)f,$$

and

$$\chi_M(L)e^{-itL}f=\frac{1}{2\pi i}\int_{-\infty}^{\infty}e^{-it\lambda}\chi_M(\lambda)(R(\lambda-i0)-R(\lambda+i0))fd\lambda, \tag{4.34}$$

$$Qe^{-itL}\widetilde{\chi}_M(L)f=\frac{1}{2\pi i}\int_{-\infty}^{\infty}e^{-it\lambda}\widetilde{\chi}_M(\lambda)Q(R(\lambda-i0)-R(\lambda+i0))fd\lambda. \tag{4.35}$$

Integrating (4.34) by part, we see that

$$\begin{aligned}&\chi_M(L)e^{-itL}f\\ &\quad=\frac{(it)^{-j}}{2\pi i}\int_{-\infty}^{\infty}d\lambda e^{-it\lambda}\partial_\lambda^j\{(R(\lambda+i0)-R(\lambda-i0))\chi_M(\lambda)\}f\quad\text{in } \mathcal{S}'_x(\mathbb{R})\end{aligned}$$

for any $t\neq0$ and $f\in\mathcal{S}_x(\mathbb{R}^2)$. Since

$$\|\partial_\lambda^j QR(\lambda\pm i0)\|_{B(L^{2,(j+1)/2+0},L^{2,-(j+1)/2-0})}\lesssim\langle\lambda\rangle^{-(j+1)/2},$$

the above integral absolutely converges in $L_x^{2,-(j+1)/2-0}$ for $j\ge2$.

Suppose $g(t,x)=g_1(t)g_2(x)$, $g_1\in C_0^\infty(\mathbb{R}\setminus\{0\})$ and $g_2\in\mathcal{S}(\mathbb{R})$. We define $\langle\cdot,\cdot\rangle_x$ and $\langle\cdot,\cdot\rangle_{t,x}$ as

$$\langle u_1,u_2\rangle_x:=\int_{-\infty}^{\infty}u_1(x)u_2(x)dx,\quad\langle v_1,v_2\rangle_{t,x}:=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}v_1(t,x)v_2(t,x)dxdt.$$

Making use of Fubini's theorem and integration by parts, we have for $j\ge2$,

$$\begin{aligned}&\langle\chi_M(L)e^{-itL}f,g\rangle_{t,x}\\ &=\frac{1}{2\pi i}\int_{-\infty}^{\infty}dt(it)^{-j}g_1(t)\int_{-\infty}^{\infty}d\lambda e^{-it\lambda}\partial_\lambda^j\langle\chi_M(\lambda)(R(\lambda+i0)-R(\lambda-i0))f,g_2\rangle_x\\ &=\frac{1}{2\pi i}\int_{-\infty}^{\infty}d\lambda\partial_\lambda^j\langle\chi_M(\lambda)(R(\lambda+i0)-R(\lambda-i0))f,g_2\rangle_x\int_{-\infty}^{\infty}dt(it)^{-j}g_1(t)e^{-it\lambda}\\ &=\frac{1}{\sqrt{2\pi}i}\int_{-\infty}^{\infty}d\lambda(\mathcal{F}_tg_1)(\lambda)\langle\chi_M(\lambda)(R(\lambda+i0)-R(\lambda-i0))f,g_2\rangle_x.\end{aligned}$$

Hence it follows from the above and Fubini's theorem that

$$\begin{aligned}&\left\langle \chi_M(L)e^{-itL}f, g\right\rangle_{t,x}\\&= \frac{1}{\sqrt{2\pi}i}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}d\lambda\,(\chi_M(\lambda)(R(\lambda+i0)-R(\lambda-i0))f)\,\mathcal{F}_tg(\lambda,x)\end{aligned}$$

for every $g\in C_0^\infty(\mathbb{R}_t\setminus\{0\})\otimes\mathcal{S}(\mathbb{R}_x)$. Using Plancherel's theorem, we have

$$\begin{aligned}&\left|\langle \chi_M(L)e^{-itL}f, g\rangle_{t,x}\right|\\&\le (2\pi)^{-1/2}\|\chi_M(\lambda)(R(\lambda+i0)-R(\lambda-i0))f\|_{L_x^\infty L_\lambda^2}\|\mathcal{F}_tg(\lambda,\cdot)\|_{L_x^1L_\lambda^2}\\&=(2\pi)^{-1/2}\|\chi_M(\lambda)(R(\lambda+i0)-R(\lambda-i0))f\|_{L_x^\infty L_\lambda^2(0,\infty)}\|g\|_{L_x^1L_t^2}.\end{aligned}\tag{4.36}$$

Similarly, we have

$$\begin{aligned}&|\langle Qe^{-itL}\widetilde{\chi}_M(L)f\rangle_{t,x}|\\&\le(2\pi)^{-1/2}\|\langle x\rangle^{-3/2}\widetilde{\chi}_M(\lambda)Q(R(\lambda-i0)-R(\lambda+i0))f\|_{L_x^\infty L_\lambda^2}\|\langle x\rangle^{3/2}g\|_{L_x^1L_t^2},\end{aligned}\tag{4.37}$$

and

$$\begin{aligned}&|\langle\partial_xe^{-itL}Qf,g\rangle_{t,x}|\\&\le(2\pi)^{-1/2}\big(\|\chi_M(\lambda)\partial_x(R(\lambda-i0)-R(\lambda+i0))f\|_{L_x^\infty L_\lambda^2}\\&\qquad+\|\widetilde{\chi}_M(\lambda)\partial_x(R(\lambda-i0)-R(\lambda+i0))Qf\|_{L_x^\infty L_\lambda^2}\big)\|g\|_{L_x^\infty L_t^2}.\end{aligned}\tag{4.38}$$

Since $C_0^\infty(\mathbb{R}_t\setminus\{0\})\otimes\mathcal{S}(\mathbb{R}_x)$ is dense in $L_x^1L_t^2$, Eqs. (2.8) and (2.9) follow from (4.36)–(4.38) and Lemmas 4.1 and 4.3. By using the duality argument, we see that (2.10) follows from (2.8). Thus we complete the proof of Lemma 2.2. □

To prove Lemma 2.3, we need the following.

Lemma 4.5. *Assume* (V1) *and* (V2). *Let $g(t,x)\in\mathcal{S}_\otimes(\mathbb{R}^2)$ and*

$$U(t,x)=\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\lambda e^{-it\lambda}\{R(\lambda-i0)+R(\lambda+i0)\}Q(\mathcal{F}_t^{-1}g)(\lambda,\cdot).$$

Then,

$$\begin{aligned}U(t,x)=2\int_0^t dse^{-i(t-s)L}Qg(s,\cdot)+\int_{-\infty}^0 dse^{-i(t-s)L}Qg(s,\cdot)\\-\int_0^\infty dse^{-i(t-s)L}Qg(s,\cdot).\end{aligned}$$

Proof. We may assume that $g(t,x)$ is written as $g(t,x)=g_1(t)g_2(x)$ with g_1, $g_2\in\mathcal{S}(\mathbb{R})$. Let $h\in\mathcal{S}(\mathbb{R})$ and

$$\begin{aligned}f(\lambda)&=\langle Q\{R(\lambda-i0)+R(\lambda+i0)\}g_2,h\rangle,\\f_\varepsilon(\lambda)&=\langle Q\{R(\lambda-i\varepsilon)+R(\lambda+i\varepsilon)\}g_2,h\rangle.\end{aligned}$$

Then $f(\lambda)$ and $f_\varepsilon(\lambda)$ are smooth functions satisfying

$$\sup_{\lambda\in\mathbb{R},\varepsilon>0}\langle\lambda\rangle^{k+1/2}\left(\left|\partial_\lambda^k f(\lambda)\right|+\left|\partial_\lambda^k f_\varepsilon(\lambda)\right|\right)<\infty$$

for every $k\in\mathbb{N}\cup\{0\}$ (see e.g [28]), and

$$\begin{aligned}\int_{\mathbb{R}} U(t,x)\overline{h(x)}dx &= \frac{i}{\sqrt{2\pi}}\int_{\mathbb{R}} d\lambda e^{-it\lambda}f(\lambda)(\mathcal{F}^{-1}g_1)(\lambda)\\ &= \frac{i}{\sqrt{2\pi}}\int_{\mathbb{R}} ds\hat{f}(t-s)g_1(s).\end{aligned}\tag{4.39}$$

By the spectral decomposition theorem,

$$f_\varepsilon(\lambda)=\int_{\mathbb{R}}\frac{2(\lambda-\mu)}{(\lambda-\mu)^2+\varepsilon^2}d\langle E_{ac}(\mu)g_2,h\rangle.\tag{4.40}$$

Taking the Fourier transform of (4.40) and using Fubini's theorem, we have

$$\begin{aligned}\hat{f}_\varepsilon(t) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} d\langle E_{ac}(\mu)g_2,h\rangle\int_{\mathbb{R}} d\lambda e^{-it\lambda}\frac{2(\lambda-\mu)}{(\lambda-\mu)^2+\varepsilon^2}\\ &= \sqrt{2\pi}i\int_{\mathbb{R}} d\langle E_{ac}(\mu)g_2,h\rangle e^{-it\mu-\varepsilon|t|}\operatorname{sgn} t.\end{aligned}$$

Hence it follows that

$$\hat{f}(t)=\lim_{\varepsilon\downarrow 0}\hat{f}_\varepsilon(t)=\sqrt{2\pi}i\operatorname{sgn} t\langle Qe^{-itL}g_2,h\rangle.\tag{4.41}$$

Substituting (4.41) into (4.39), we obtain

$$\begin{aligned}\langle U(t,\cdot),h\rangle &= -\int_{\mathbb{R}} dsg_1(s)\left(\int_{\mathbb{R}}\operatorname{sgn}(t-s)e^{-i(t-s)\mu}d\langle E_{ac}(\mu)g_2,h\rangle\right)\\ &= \int_{-\infty}^t dsg_1(s)\langle e^{-i(t-s)L}g_2,h\rangle-\int_t^\infty dsg_1(s)\langle e^{-i(t-s)L}g_2,h\rangle.\end{aligned}$$

Thus we complete the proof of Lemma 4.5. $\square$

Proof of Lemma 2.3. Since $\mathcal{S}_\otimes(\mathbb{R}^2)$ is dense in $L_x^1L_t^2$, it suffices to prove (2.11) for $g\in\mathcal{S}_\otimes(\mathbb{R}^2)$. As in the proof of Lemma 2.2, we have

$$\|\langle x\rangle^{-1}\partial_x^jU(\cdot,x)\|_{L_x^\infty L_t^2}\le\|\langle x\rangle^{-1}\partial_x^j\{R(\lambda-i0)+R(\lambda+i0)\}Q\mathcal{F}_t^{-1}g(\lambda,\cdot)\|_{L_x^\infty L_\lambda^2}.$$

Applying Plancherel's theorem and Minkowski's inequality, we have

$$\begin{aligned}&\|\langle x\rangle^{-1}\partial_x^jU(\cdot,x)\|_{L_x^\infty L_t^2}\\ \le&\left\|\|\langle\cdot\rangle^{-1}\partial_x^j\{R(\lambda-i0)+R(\lambda+i0)\}Q\langle\cdot\rangle^{-1}\|_{B(L_x^1,L_x^\infty)}\|\langle\cdot\rangle\mathcal{F}_t^{-1}g(\lambda,\cdot)\|_{L_x^1}\right\|_{L_\lambda^2}\\ \le&\sup_\lambda\|\langle\cdot\rangle^{-1}\partial_x^j\{R(\lambda-i0)+R(\lambda+i0)\}Q\langle\cdot\rangle^{-1}\|_{B(L_x^1,L_x^\infty)}\|\langle x\rangle g\|_{L_x^1L_t^2}\end{aligned}$$

for $j = 0, 1$. Hence it follows from Lemmas 4.2 and 4.4 that

$$\|\langle x\rangle^{-1}U\|_{L_x^\infty L_t^2} + \|\langle x\rangle^{-1}\partial_x U\|_{L_x^\infty L_t^2} \lesssim \|\langle x\rangle g\|_{L_x^1 L_t^2}. \tag{4.42}$$

For $I = [0,\infty)$ and $I = (-\infty, 0]$, we have

$$\begin{aligned}
&\int_I e^{-i(t-s)L}Qg(s)ds \\
&\quad = \int_{\mathbb{R}} ds\chi_I(s)\left(\int_{\mathbb{R}} e^{-i(t-s)\lambda}dE_{ac}(\lambda)g(s,\cdot)\right) \\
&\quad = \frac{1}{2\pi i}\int_{\mathbb{R}^2} d\lambda ds e^{-i(t-s)\lambda}\{R(\lambda - i0) - R(\lambda + i0)\}Q\chi_I(s)g(s,\cdot) \\
&\quad = -i\mathcal{F}_\lambda\{(R(\lambda - i0) - R(\lambda + i0))Q\mathcal{F}_s^{-1}(\chi_I(s)g)(\lambda,\cdot)\}(t).
\end{aligned}$$

By Plancherel's identity and Minkowski's inequality, we have

$$\begin{aligned}
&\left\|\langle x\rangle^{-1}\partial_x^j\int_I e^{-i(t-s)L}Qg(s)ds\right\|_{L_x^\infty L_t^2} \\
&\le \left\|\langle x\rangle^{-1}\partial_x^j(R(\lambda - i0) - R(\lambda + i0))Q\mathcal{F}_s^{-1}(\chi_I(s)g)(\lambda,\cdot)\right\|_{L_x^\infty L_\lambda^2} \\
&\le \sup_\lambda \|\langle\cdot\rangle^{-1}\partial_x^j\{R(\lambda - i0) - R(\lambda + i0)\}Q\langle\cdot\rangle^{-1}\|_{B(L_x^1, L_x^\infty)}\|\langle x\rangle g\|_{L_x^1 L_t^2}
\end{aligned} \tag{4.43}$$

for $j = 0, 1$. Combining (4.42)–(4.43) with Lemma 4.5, we obtain (2.11). Since (2.12) can be obtained in exactly the same way, we omit the proof. □

Finally, we will prove Lemma 2.4. To prove Lemma 2.4, we will use a lemma of Christ and Kiselev [10].

Proof of Lemma 2.4. Let $(q, p) = (4, \infty)$ or $(q, p) = (\infty, 2)$ and let

$$Tg(t) = \int_{\mathbb{R}} ds e^{-i(t-s)L}Qg(s).$$

Lemmas 2.1 and 2.2 imply $f := \int_{\mathbb{R}} ds e^{isL}Qg(s) \in L^2(\mathbb{R})$ and

$$\|Tg(t)\|_{L_t^q L_x^p} \lesssim \|f\|_{L_x^2} \lesssim \|\langle x\rangle^{3/2}g\|_{L_x^1 L_t^2}.$$

Thus by Schwarz's inequality, we see that there exists a $C > 0$ such that for every $g \in \mathcal{S}(\mathbb{R}^2)$,

$$\|Tg(t)\|_{L_t^q L_x^p} \le C\|g\|_{L_t^2 L_x^2(\mathbb{R},\langle x\rangle^5 dx)}. \tag{4.44}$$

Since $q > 2$, it follows from Lemma 3.1 in [40] and (4.44) that

$$\left\|\int_{s<t} ds e^{-i(t-s)L}Qg(s)\right\|_{L_t^q L_x^p} \lesssim \|g\|_{L_t^2 L_x^2(\mathbb{R},\langle x\rangle^5 dx)}. \tag{4.45}$$

Thus we prove Lemma 2.4. □

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