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SUM OF TOEPLITZ PRODUCTS ON THE HARDY SPACE OVER THE POLYDISK

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ABSTRACT. In this paper, we obtain several sufficient and necessary conditions for a finite sum of Toeplitz products with form $\sum_{m=1}^{M} T_{f_m} T_{g_m}$ on the Hardy space over the polydisk to be zero. The methods used in this note are Berezin transform and the essential fiber dimension.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$. Its boundary is the unit circle $\mathbb{T}$. Throughout this paper, let N denote a fixed positive integer, and let $\mathbb{D}^N$ and $\mathbb{T}^N$ denote the Cartesian products of N copies of $\mathbb{D}$ and $\mathbb{T}$, respectively. For $1 \leq p \leq \infty$ let $L^p(\mathbb{T}^N)$ be the usual Lebesgue space on $\mathbb{T}^N$ with respect to $d\sigma$, the Haar measure on $\mathbb{T}^N$. Hardy space $H^p(\mathbb{D}^N)$ is the closure of the analytic polynomials in $L^p(\mathbb{T}^N)$. Let P be the orthogonal projection from $L^2(\mathbb{T}^N)$ onto $H^2(\mathbb{D}^N)$. The Toeplitz operator T_u with symbol u in $L^\infty(\mathbb{T}^N)$ is defined by

$$T_u(f) = P(uf)$$

for all $f \in H^2(\mathbb{D}^N)$. It is clear that T_u is a bounded linear operator on the $H^2(\mathbb{D}^N)$.

On the Hardy space of the unit disk, Brown and Halmos [4] firstly showed that two Toeplitz operators T_f and T_g commute with each other if and only if either both f and g are analytic, or both f and g are coanalytic, or a nontrivial linear combination of f and g is constant. They also proved that a Toeplitz product $T_f T_g$ equals some Toeplitz operator if and only if either g is analytic or f is co-analytic. In the case of the Bergman space, this problem is more

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subtle. Axler and Čučković [3] characterized commuting Toeplitz operators with bounded harmonic symbols on the Bergman space over the unit disk. After that, there were many subsequent works about algebraic properties of Toeplitz operators. Ahern and Čučkovič [1] discuss when a product of two Toeplitz operators with bounded harmonic symbols on the Bergman space equals some Toeplitz operator with a symbol such that some restrictions. On the Dirichlet space of the unit disk, Lee in [12] studied the commutativity of two Toeplitz operators with harmonic symbols. Then, Chen and Dieu [6] and Yu [15] extended, respectively, Lee's result to more general case.

For the function spaces of several variables, Zheng [17] characterized commuting Toeplitz operators with parahormonic symbols on Bergman space over the unit ball. Choe, Koo and Lee [5] characterized commuting Toeplitz operators with pluriharmonic symbols on the Bergman space over the polydisk. On the Hardy space over the polydisk, Lee [13] studied the commutativity of two Toeplitz operators of which one symbol is an arbitrarily bounded function and the other is a pluriharmonic function.

In the setting of several variables, even for Hardy space, it is interesting when two Toeplitz operators with general symbols are commuting. Gu and Zheng [9] obtained a condition for a product of two Toeplitz operators on the Hardy space over the bidisk equals a Toeplitz operator. They proved the following theorem.

THEOREM 1.1 (See [9]). *Let $f,g\in L^\infty(\mathbb{T}^2)$. The semi-commutator $T_{\underline{f}}T_g - T_{fg}$ equals zero on $H^2(\mathbb{D}^2)$ if and only if , for each i $(i=1,2)$, either $\overline{f}$ or g is analytic in the ith variable.*

Ding, Sun and Zheng [8] completely characterized the commuting Toeplitz operators on the Hardy space over the bidisk. In [8], the symbols f_{++} and f_{--} represent P_1P_2f and $(I-P_1)(I-P_2)f$, respectively, where P_i $(i=1,2)$ denotes the Szegö projection for the ith variable (their definitions can be seen later).

THEOREM 1.2 (See [8]). *Let $f,g\in L^\infty(\mathbb{T}^2)$. The Toeplitz operator T_f commutes with the Toeplitz operator T_g on $H^2(\mathbb{D}^2)$ if and only if the following conditions hold.*

(a) *for almost all $\xi_2\in\mathbb{T}$,*
   (a1) *$f(z_1,\xi_2)$ and $g(z_1,\xi_2)$ are both analytic in variable z_1 on $\mathbb{D}$, or*
   (a2) *$f(z_1,\xi_2)$ and $g(z_1,\xi_2)$ are both co-analytic in variable z_1 on $\mathbb{D}$, or*
   (a3) *there are $a_1(\xi_2)$ and $b_1(\xi_2)$, not both zero, such that*

$$a_1(\xi_2)f(z_1,\xi_2)+b_1(\xi_2)g(z_1,\xi_2)$$

   is a constant in variable z_1 on $\mathbb{D}$.

(b) *for almost all $\xi_1\in\mathbb{T}$,*
   (b1) *$f(\xi_1,z_2)$ and $g(\xi_1,z_2)$ are both analytic in variable z_2 on $\mathbb{D}$, or*

(b2) *$f(\xi_1, z_2)$ and $g(\xi_1, z_2)$ are both co-analytic in variable z_2 on $\mathbb{D}$, or*

(b3) *there are $a_2(\xi_1)$ and $b_2(\xi_1)$, not both zero, such that*

$$a_2(\xi_1)f(\xi_1, z_2) + b_2(\xi_1)g(\xi_1, z_2)$$

is a constant in variable z_2 on $\mathbb{D}$.

(c) *One of the following conditions holds:*

(c1)

$$f_{++}(z_1, z_2) = f_1(z_1) + f_2(z_2),$$
$$g_{++}(z_1, z_2) = g_1(z_1) + g_2(z_2),$$

where f_1, f_2, g_1 and g_2 are in $H^q(\mathbb{D})$ for every $q > 1$.

(c2)

$$f_{--}(z_1, z_2) = \bar{f}_1(z_1) + \bar{f}_2(z_2),$$
$$g_{--}(z_1, z_2) = \bar{g}_1(z_1) + \bar{g}_2(z_2),$$

where f_1, f_2, g_1 and g_2 are in $H^q(\mathbb{D})$ for every $q > 1$.

(c3) *There exist constants a, b, not both zero, such that*

$$af_{++}(z_1, z_2) + bg_{++}(z_1, z_2) = \bar{h}_1(z_1) + \bar{h}_2(z_2),$$
$$af_{--}(z_1, z_2) + bg_{--}(z_1, z_2) = \bar{r}_1(z_1) + \bar{r}_2(z_2),$$

where h_1, h_2, r_1 and r_2 are in $H^q(\mathbb{D})$ for every $q > 1$.

Motivated by the above results, we discuss the algebraic properties of Toeplitz operators on the Hardy space over the polydisk, and get the conditions for a finite sum of Toeplitz products with the form $\sum_{m=1}^{M} T_{f_m}T_{g_m}$ to be zero. It is clear that this will generalize the cases of commutator and semi-commutator of Toeplitz operators.

Let $K_z(\zeta) = \frac{1}{1-\bar{z}\zeta}$ denote the reproducing kernel of Hardy space $H^2(\mathbb{D})$ at the point $z \in \mathbb{D}$ and $k_z(\zeta) = K_z(\zeta)/\|K_z\| = \frac{\sqrt{1-|z|^2}}{1-\bar{z}\zeta}$ the normalized reproducing kernel of $H^2(\mathbb{D})$. Then, for $z = (z_1, z_2, \ldots, z_N) \in \mathbb{D}^N$ and $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_N) \in \mathbb{T}^N$, the normalized reproducing kernel of $H^2(\mathbb{D}^N)$ is given by

$$k_z(\zeta) = \prod_{i=1}^{N} k_{z_i}(\zeta_i).$$

For a bounded linear operator S on $H^2(\mathbb{D}^N)$ the Berezin transform $\widetilde{S}$ of S is defined by

$$\widetilde{S}(z) = \langle Sk_z, k_z\rangle = \int_{\mathbb{T}^N} (Sk_z)(\zeta)\overline{k_z(\zeta)}\, d\sigma(\zeta).$$

For a Toeplitz operator T_f on $H^2(\mathbb{D}^N)$ the Berezin transform of T_f is

$$\widetilde{T_f}(z) = \langle T_f k_z, k_z\rangle = \int_{\mathbb{T}^N} f(\zeta)\big|k_z(\zeta)\big|^2\, d\sigma(\zeta),$$

which is exactly the Poisson extension to the polydisk of f.

For $1 \le i \le N$ write ∂_i and $\bar{\partial}_i$ instead of $\frac{\partial}{\partial z_i}$ and $\frac{\partial}{\partial \bar{z}_i}$, respectively. Suppose f is a twice differentiable function in a open set of $\mathbb{C}^N$. Then f is called N-harmonic if $\partial_i \bar{\partial}_i f = 0$ for each $1 \le i \le N$.

With these notations we state the following result.

THEOREM 1.3. *Let $f_m, g_m \in L^\infty(\mathbb{T}^N)$ for $1 \le m \le M$. Then $\sum_{m=1}^M T_{f_m} \times T_{g_m} = 0$ on $H^2(\mathbb{D}^N)$ if and only if the Berezin transform of $\sum_{m=1}^M T_{f_m} T_{g_m}$ is N-harmonic in $\mathbb{D}^N$ and, for almost all $\xi \in \mathbb{T}^N$, $\sum_{m=1}^M f_m(\xi) g_m(\xi) = 0$.*

The equivalence between that a Toeplitz operator T_f commutes with T_g and that the Berezin transform of the commutator $T_f T_g - T_g T_f$ is N-harmonic was proved in [10]. In fact this equivalence was also contained in the proof of [8] and [16] for the cases of bidisk and polydisk, respectively. The proof in this note is different from those and will be used to the proof of Theorem 1.5.

Let $\hat{N}$ denote the set $\{1, 2, \dots, N\}$. For a subset $I = \{i_1, i_2, \dots, i_k\}$ of $\hat{N}$, let $I^c = \hat{N} \setminus I$, and

$$z_I = (z_{i_1}, z_{i_2}, \dots, z_{i_k})$$

and

$$(z_I, z_{I^c}) = (z_1, z_2, \dots, z_N).$$

The normalized *partial* reproducing kernel of $H^2(\mathbb{D}^N)$ at the point $z_I \in \mathbb{D}^k$ is defined by

$$k_{z_I}(\zeta_I) = \prod_{i \in I} k_{z_i}(\zeta_i).$$

DEFINITION 1.4. For $f \in L^1(\mathbb{T}^N)$ the *partial* harmonic extension of f with respect to I is defined by

$$\begin{aligned} \tilde{f}_I(z_I, \xi_{I^c}) &= \int_{\mathbb{T}^n} f(\xi_I, \xi_{I^c}) \prod_{i \in I} \frac{1 - |z_i|^2}{|1 - z_i \bar{\xi}_i|^2} \, d\sigma(\xi_I) \\ &= \int_{\mathbb{T}^n} f(\xi_I, \xi_{I^c}) \big| k_{z_I}(\xi_I) \big|^2 \, d\sigma(\xi_I) \\ &= \big\langle f(\xi_{I^c}, \cdot) k_{z_I}, k_{z_I} \big\rangle. \end{aligned}$$

It is clear that $\tilde{f}_I(z_I, \xi_{I^c})$ is k-harmonic in z_I for almost all $\xi_{I^c} \in \mathbb{T}^{N-k}$. In what follows, the partial harmonic extension $\hat{f}_I$ of f will also be denoted by f for short.

For a nonempty subset $I = \{i_1, i_2, \dots, i_k\}$ of $\hat{N}$ let $\partial_I = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_k}$. If $I = \emptyset$, let $\partial_\emptyset f = f$.

Let $|I|$ denote the cardinal number of a set I.

THEOREM 1.5. *Let $f_m, g_m \in L^\infty(\mathbb{T}^N)$, $1 \le m \le M$. Then $\sum_{m=1}^M T_{f_m} T_{g_m} = 0$ on $H^2(\mathbb{D}^N)$ if and only if for all $I \subset \hat{N}$,*

$$\sum_{m=1}^{M} \partial_I f_m(z_I, \xi_{I^c}) \bar{\partial}_I g_m(z_I, \xi_{I^c}) = 0 \tag{1.1}$$

for all $z_I \in \mathbb{D}^{|I|}$ and for almost all $\xi_{I^c} \in \mathbb{T}^{N-|I|}$.

REMARK 1.6. When $N = M = 2$, if putting $f_2 = f_1 g_1$ and $g_2 = -1$ in Theorem 1.5, then we obtain Theorem 1.3 in [8], and if putting $f_2 = g_1$ and $g_2 = -f_1$, then Theorem 1.4 of [8] follows.

Given $f \in L^1(\mathbb{T}^N)$ and $1 \le i \le N$, the Szegö projection P_i is defined by

$$(P_i f)(z_i, \xi_{\hat{N}\setminus\{i\}}) = \int_{\mathbb{T}} f(\xi) \frac{1}{1 - z_i \bar{\xi}_i} \, d\sigma(\xi_i), \tag{1.2}$$

where $z_i \in \mathbb{D}$, $\xi \in \mathbb{T}^N$. It is well known that P_i is bounded on $L^p(\mathbb{T}^N)$ $(1 < p < \infty)$, and P_i commutes with P_j for $i, j \in \hat{N}$, see, for example, [8].

DEFINITION 1.7. For $I = \{i_1, i_2, \ldots, i_k\} \subset \hat{N}$, let $P_I = P_{i_1} \cdots P_{i_k}$. For I, $J \subset \hat{N}$, $I \cap J = \emptyset$ and $f \in L^\infty(\mathbb{T}^N)$, let $f_{(I,J)}$ denote $(1 - P_I) P_J f$.

It is clear that $f_{(I,J)} \in \bigcap_{1<q<\infty} L^q(\mathbb{T}^N)$ and $f_{(I,J)}(z_I, z_J, \xi_{\hat{N}\setminus(I\cup J)})$ is co-analytic in $z_I \in \mathbb{D}^{|I|}$ and analytic in $z_J \in \mathbb{D}^{|J|}$ for almost all $\xi_{\hat{N}\setminus(I\cup J)} \in \mathbb{T}^{N-|I|-|J|}$.

THEOREM 1.8. *Let $f = (f_1, f_2, \ldots, f_M), g = (g_1, g_2, \ldots, g_M) \in L^\infty(\mathbb{T}^N) \otimes \mathbb{C}^M$. Then $\sum_{m=1}^M T_{f_m} T_{g_m} = 0$ on $H^2(\mathbb{D}^N)$ if and only if*

$$\sum_{m=1}^{M} f_m(\xi) g_m(\xi) = 0 \tag{1.3}$$

for almost all $\xi \in \mathbb{T}^N$. And for every nonempty set $I = \{n_1, n_2, \ldots, n_{|I|}\} \subset \hat{N}$ the following condition holds.

(*CIk*) *for almost all $\xi_{I^c} \in \mathbb{T}^{N-|I|}$, there exists a natural number $k(\xi_{I^c})$ $(0 \le k(\xi_{I^c}) \le M)$ and an orthonormal bases $\{e_1(\xi_{I^c}), e_2(\xi_{I^c}), \ldots, e_M(\xi_{I^c})\}$ of $\mathbb{C}^M$ such that*

$$\left\langle f_{(\emptyset,I)}(z_I, \xi_{I^c}), e_i(\xi_{I^c}) \right\rangle_{\mathbb{C}^M} = \sum_{j=1}^{|I|} f_{ij}^I(z_{I\setminus\{n_j\}}, \xi_{I^c}), \quad 1 \le i \le k(\xi_{I^c}) \tag{1.4}$$

and

$$\left\langle \bar{g}_{(I,\emptyset)}(z_I, \xi_{I^c}), e_i(\xi_{I^c}) \right\rangle_{\mathbb{C}^M} = \sum_{j=1}^{|I|} g_{ij}^I(z_{I\setminus\{n_j\}}, \xi_{I^c}), \quad k(\xi_{I^c}) + 1 \le i \le M \tag{1.5}$$

for $z_I \in \mathbb{D}^{|I|}$, *where all functions* $f^I_{ij}(z_{I\setminus\{n_j\}}, \xi_{I^c})$ *and* $g^I_{ij}(z_{I\setminus\{n_j\}}, \xi_{I^c})$ *are analytic in* $z_{I\setminus\{n_j\}}$.

Moreover, if $j \in I^c$, *then for almost all given* $\xi_{I^c\setminus\{j\}} \in \mathbb{T}^{N-|I|-1}$, $k(\xi_{I^c})$ *can be chosen independent to* ξ_j.

In Section 2, we introduce several notations as the partial harmonic extension and the essential fiber dimension. Theorems 1.3 and 1.5 will be proved in the Section 3. In Section 4, we give the proof of Theorems 1.8 and 4.4.

2. Preliminaries

2.1. Partial harmonic extension. In this section, we discuss the relationship of a function in $L^p(\mathbb{T}^N)$ and its partial harmonic extensions (see Definition 1.4).

LEMMA 2.1. *If* $1 \le p < \infty$, $f \in L^p(\mathbb{T}^N)$ *and* $I \subset \hat{N}$, *then*

(a) $\int_{\mathbb{T}^N} |f(r\xi_I, \xi_{I^c})|^p \, d\sigma(\xi) \le \|f\|_p^p$ *for* $0 < r < 1$;
(b) $\int_{\mathbb{T}^N} |f(r\xi_I, \xi_{I^c}) - f(\xi_I, \xi_{I^c})|^p \, d\sigma(\xi) \to 0$ *as* $r \to 1^-$;

If furthermore $f, g \in L^{2p}(\mathbb{T}^N)$, *then*

(c) $\langle f(r\xi_I, \cdot) g(r\xi_I, \cdot) k_{r\xi_{I^c}}, k_{r\xi_{I^c}} \rangle \to f(\xi) g(\xi)$ *as* $r \to 1^-$ *in the norm of* $L^p(\mathbb{T}^N)$.

Proof. The proof of (a) and (b) is similar to that of Theorem 2.1.3 in [14].
(a) By Jesen's inequality, we have

$$\begin{aligned}\int_{\mathbb{T}^N} \left|f(r\xi_I, \xi_{I^c})\right|^p d\sigma(\xi) &= \int_{\mathbb{T}^N} \left| \int_{\mathbb{T}^{|I|}} f(\zeta_I, \xi_{I^c}) \left|k_{r\xi_I}(\zeta_I)\right|^2 d\sigma(\zeta_I) \right|^p d\sigma(\xi) \\ &\le \int_{\mathbb{T}^N} \int_{\mathbb{T}^{|I|}} \left|f(\zeta_I, \xi_{I^c})\right|^p \left|k_{r\xi_I}(\zeta_I)\right|^2 d\sigma(\zeta_I)\, d\sigma(\xi).\end{aligned}$$

Since $\int_{\mathbb{T}^k} |k_{r\xi_I}(\zeta_I)|^2 \, d\sigma(\xi_I) = 1$, we get

$$\int_{\mathbb{T}^N} \left|f(r\xi_I, \xi_{I^c})\right|^p d\sigma(\xi) \le \|f\|_p^p.$$

(a) implies (b) since $C(\mathbb{T}^N)$ is dense in $L^p(\mathbb{T}^N)$ if $1 \le p < \infty$.

(c) Applying (a) and (b), we have that

$$\begin{aligned}
&\int_{\mathbb{T}^N} \big|\big\langle f(r\xi_I,\cdot)g(r\xi_I,\cdot)k_{r\xi_{I^c}},k_{r\xi_{I^c}}\big\rangle-(fg)(\xi_I,r\xi_{I^c})\big|^p\,d\sigma(\xi)\\
&\quad=\int_{\mathbb{T}^N}\big|\big\langle\big[f(r\xi_I,\cdot)g(r\xi_I,\cdot)-f(\xi_I,\cdot)g(\xi_I,\cdot)\big]k_{r\xi_{I^c}},k_{r\xi_{I^c}}\big\rangle\big|^p\,d\sigma(\xi)\\
&\quad\le\int_{\mathbb{T}^N}\int_{\mathbb{T}^{N-|I|}}\big|f(r\xi_I,\zeta_{I^c})g(r\xi_I,\zeta_{I^c})\\
&\qquad-f(\xi_I,\zeta_{I^c})g(\xi_I,\zeta_{I^c})\big|^p\big|k_{r\xi_{I^c}}(\zeta_{I^c})\big|^2\,d\sigma(\zeta_{I^c})\,d\sigma(\xi)\\
&\quad=\int_{\mathbb{T}^{N-|I|}}\int_{\mathbb{T}^{|I|}}\big|f(r\xi_I,\zeta_{I^c})g(r\xi_I,\zeta_{I^c})-f(\xi_I,\zeta_{I^c})g(\xi_I,\zeta_{I^c})\big|^p\,d\sigma(\xi_I)\,d\sigma(\zeta_{I^c})\\
&\quad\le 2^p\int_{\mathbb{T}^N}\big(\big|f(r\xi_I,\xi_{I^c})\big|^p\big|g(r\xi_I,\xi_{I^c})-g(\xi)\big|^p\\
&\qquad+\big|f(r\xi_I,\xi_{I^c})-f(\xi)\big|^p\big|g(\xi)\big|^p\big)\,d\sigma(\xi)\\
&\quad\le 2^p\bigg(\int_{\mathbb{T}^N}\big|f(r\xi_I,\xi_{I^c})\big|^{2p}\,d\sigma(\xi)\bigg)^{1/2}\bigg(\int_{\mathbb{T}^N}\big|g(r\xi_I,\xi_{I^c})-g(\xi)\big|^{2p}\,d\sigma(\xi)\bigg)^{1/2}\\
&\qquad+2^p\|g\|_{2p}^p\bigg(\int_{\mathbb{T}^N}\big|f(r\xi_I,\xi_{I^c})-f(\xi)\big|^{2p}\,d\sigma(\xi)\bigg)^{1/2}\\
&\quad\to 0\quad\text{as } r\to 1.
\end{aligned}$$

Applying (b) again, we have that $\langle f(r\xi_I,\cdot)g(r\xi_I,\cdot)k_{r\xi_{I^c}},k_{r\xi_{I^c}}\rangle$ tends to $f(\xi)g(\xi)$ in the norm of $L^p(\mathbb{T}^N)$ as $r\to 1^-$. □

2.2. Essential fiber dimension. For to obtain concrete forms of f and g when T_f commutes with T_g, we need to discuss the dimension of the space generated by the range of a vector-valued function. So we introduce a notation of essential fiber dimension which will be used in the proof of Theorem 1.8.

Let n be a positive integer, $H^2(\mathbb{D})\otimes\mathbb{C}^n$ the vector-valued Hardy space. For a subspace $\mathcal{M}$ of $H^2(\mathbb{D})\otimes\mathbb{C}^n$ and $z\in\mathbb{D}$, the fiber $\mathcal{M}(z)$ at z is defined by

$$\mathcal{M}(z)=\big\{f(z):f\in\mathcal{M}\big\}\subset\mathbb{C}^n,$$

and the fiber dimension $fd_{\mathbb{D}}(\mathcal{M})$ by

$$fd_{\mathbb{D}}(\mathcal{M})=\sup_{z\in\mathbb{D}}\dim\mathcal{M}(z).$$

The fiber dimension $fd_{\mathbb{D}}(\mathcal{M})$ is achieved for all $z\in\mathbb{D}$ except possibly a discrete subset. For the notation and applications of fiber dimension we refer to see [2], [7] for examples.

In following, we introduce the definition of *essential fiber dimension* at the unit circle. Suppose $\mathcal{A}$ is countable subset of $H^2(\mathbb{D})\otimes\mathbb{C}^n$. Taking a representative f for every element in $\mathcal{A}$, then for every $\xi\in\mathbb{T}$, $f(\xi)$ is a determined vector in $\mathbb{C}^n$. So for a given choice of representatives of $\mathcal{A}$, the fiber $\mathcal{A}(\xi)$ at ξ is defined by

$$\mathcal{A}(\xi) = \operatorname{span}\{f(\xi) : f \in \mathcal{A}\} \subset \mathbb{C}^n.$$

Note that for a given point $\xi \in \mathbb{T}$ the space $\mathcal{A}(\xi)$ perhaps changes with respect to the different choices of representatives of $\mathcal{A}$.

DEFINITION 2.2. The essential fiber dimension $fd_{\mathbb{T}}(\mathcal{A})$ of $\mathcal{A}$ is defined by

$$fd_{\mathbb{T}}(\mathcal{A}) = \inf_{E_0} \sup_{\xi} \{\dim \mathcal{A}(\xi) : \xi \in \mathbb{T} \setminus E_0, \sigma(E_0) = 0\}$$

for a choice of representatives of $\mathcal{A}$.

For two choices of representatives of $\mathcal{A}$, since $\mathcal{A}$ is countable set, this two choices is complete consistent except some subset of $\mathbb{T}$ with measure zero. So the essential fiber dimension $fd_{\mathbb{T}}(\mathcal{M})$ is well defined.

LEMMA 2.3. *If $\mathcal{A}$ is a countable subset of $H^2(\mathbb{D}) \otimes \mathbb{C}^n$, then, for every choice of representatives of $\mathcal{A}$, the essential fiber dimension $fd_{\mathbb{T}}(\mathcal{A})$ is achieved for almost all $\xi \in \mathbb{T}$, that is,*

$$fd_{\mathbb{T}}(\mathcal{A}) = \dim \mathcal{A}(\xi) \quad a.e.\ \xi \in \mathbb{T}. \tag{2.1}$$

Moreover, $fd_{\mathbb{T}}(\mathcal{A}) = fd_{\mathbb{D}}(\mathcal{A})$.

Proof. Fix a choice of representatives of $\mathcal{A}$. Let d denote the essential fiber dimension $fd_{\mathbb{T}}(\mathcal{A})$. By the definition of $fd_{\mathbb{T}}(\mathcal{A})$, there exists a subset E of $\mathbb{T}$ with positive measure such that

$$\dim \mathcal{A}(\xi) = d$$

for all $\xi \in E$. So there exist $f^{1,\xi}, f^{2,\xi}, \ldots, f^{d,\xi}$ in $\mathcal{A}$ and $i_{1,\xi}, i_{2,\xi}, \ldots, i_{d,\xi}$ in $\{1, 2, \ldots, n\}$ such that

$$\det \begin{pmatrix} f^{1,\xi}_{i_{1,\xi}}(\xi) & f^{2,\xi}_{i_{2,\xi}}(\xi) & \cdots & f^{d,\xi}_{i_{d,\xi}}(\xi) \\ f^{2,\xi}_{i_{1,\xi}}(\xi) & f^{2,\xi}_{i_{2,\xi}}(\xi) & \cdots & f^{2,\xi}_{i_{d,\xi}}(\xi) \\ \vdots & \vdots & \cdots & \vdots \\ f^{d,\xi}_{i_{1,\xi}}(\xi) & f^{d,\xi}_{i_{2,\xi}}(\xi) & \cdots & f^{d,\xi}_{i_{d,\xi}}(\xi) \end{pmatrix} \neq 0$$

for all $\xi \in E$. Because of $\mathcal{A}$ is countable, so is the set

$$\{f^{1,\xi}, f^{2,\xi}, \ldots, f^{d,\xi}, i_{1,\xi}, i_{2,\xi}, \ldots, i_{d,\xi} : \xi \in E\}.$$

Therefore, there exist $f^1, f^2, \ldots, f^d$ in $\mathcal{A}$, $i_1, i_2, \ldots, i_d$ in $\{1, 2, \ldots, n\}$ and $E' \subset E$ with $\sigma(E') > 0$ such that

$$F(\xi) := \det \begin{pmatrix} f^1_{i_1}(\xi) & f^2_{i_2}(\xi) & \cdots & f^d_{i_d}(\xi) \\ f^2_{i_1}(\xi) & f^2_{i_2}(\xi) & \cdots & f^2_{i_d}(\xi) \\ \vdots & \vdots & \cdots & \vdots \\ f^d_{i_1}(\xi) & f^d_{i_2}(\xi) & \cdots & f^d_{i_d}(\xi) \end{pmatrix} \neq 0$$

for all $\xi \in E'$. Let

$$F(z) = \det \begin{pmatrix} f^1_{i_1}(z) & f^2_{i_2}(z) & \cdots & f^d_{i_d}(z) \\ f^2_{i_1}(z) & f^2_{i_2}(z) & \cdots & f^2 i_d(z) \\ \vdots & \vdots & \cdots & \vdots \\ f^d_{i_1}(z) & f^d_{i_2}(z) & \cdots & f^d_{i_d}(z) \end{pmatrix}, \quad z \in \mathbb{D}.$$

Then F is a analytic function on $\mathbb{D}$ and, for almost all $\xi \in \mathbb{T}$,

$$F(z) \to F(\xi)$$

as $z \to \xi$ non-tangentially. By Privalov's uniqueness theorem [11, III:D], $F(\xi) \neq 0$ for almost all $\xi \in \mathbb{T}$. So the conclusion (2.1) holds and $fd_{\mathbb{T}}(\mathcal{A}) \leq fd_{\mathbb{D}}(\mathcal{A})$. The converse inequality is obvious and so $fd_{\mathbb{T}}(\mathcal{A}) = fd_{\mathbb{D}}(\mathcal{A})$. □

For a vector-valued function $f = (f_1, f_2, \ldots, f_m)$ on $\mathbb{D}^n \times \mathbb{T}$ and $\xi \in \mathbb{T}$, let $\bigvee_\xi(f)$ denote the subspace $\operatorname{span}\{(f_1(z,\xi), f_2(z,\xi), \ldots, f_m(z,\xi)) : z \in \mathbb{D}^n\}$ of $\mathbb{C}^m$. The dimension of $\bigvee_\xi(f)$ is denoted by $\mathrm{vd}_\xi(f)$.

Lemma 2.4. *Suppose $f = (f_1, f_2, \ldots, f_m)$ is a vector-valued function on $\mathbb{D}^n \times \mathbb{T}$. If for $i = 1, 2, \ldots, m$ $f_i(z,\xi)$ is analytic in $z \in \mathbb{D}^n$ for almost all $\xi \in \mathbb{T}$, and is in $\overline{H^2(\mathbb{T})}$ for all $z \in \mathbb{D}^n$ and $z \mapsto f(z,\cdot)$ define a continuous map from $\mathbb{D}^n$ to $\overline{H^2(\mathbb{T})}$. Then $\mathrm{vd}_\xi(f)$ is constant for almost all $\xi \in \mathbb{T}$.*

Proof. For $i = 1, 2, \ldots, m$ and almost all $\xi \in \mathbb{T}$ express f_i as a power-series

$$f_i(z,\xi) = \sum_{k \in \mathbb{Z}^n_+} a^{(i)}_k(\xi) z^k,$$

where $\mathbb{Z}_+$ denote the set of nonnegative integers, $k = (k_1, k_2, \ldots, k_n)$ a multi-index and $z^k := z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$. Then, for $0 < r < 1$ and $k \in \mathbb{Z}^n_+$, we have

$$\int_{\mathbb{T}^n} f_i(r\zeta, \xi) \bar{\zeta}^k \, d\sigma(\zeta) = r^{|k|} a^{(i)}_k(\xi),$$

where $|k| := k_1 + k_2 + \cdots + k_n$. The function $a^{(i)}_k$ is in $\overline{H^2(\mathbb{T})}$ since the map $z \mapsto f(z,\cdot)$ is continuous from $\mathbb{D}^n$ to $\overline{H^2(\mathbb{T})}$. Let a_k denote the vector-valued function $(a^{(1)}_k, a^{(2)}_k, \ldots, a^{(m)}_k) \in \overline{H^2(\mathbb{T})} \otimes \mathbb{C}^m$ for $k \in \mathbb{Z}^n_+$.

For given $\xi \in \mathbb{T}$ and a vector $e = (e_1, e_2, \ldots, e_m) \in \mathbb{C}^m$ it is clear that

$$\big\langle f(z,\xi), e \big\rangle = \sum_{i=1}^m \bar{e}_i \sum_{k \in \mathbb{Z}^n_+} a^{(i)}_k(\xi) z^k = 0$$

for all $z \in \mathbb{D}^n$ is equivalent to that $\langle a_k(\xi), e \rangle = 0$ for all $k \in \mathbb{Z}^n_+$. So

$$\bigvee_\xi(f) = \mathcal{A}(\xi),$$

where $\mathcal{A} = \{a_k : k \in \mathbb{Z}^n_+\}$. It follows from Lemma 2.3 that $\mathrm{vd}_\xi(f)$ is constant for almost all $\xi \in \mathbb{T}$. □

3. The proof of Theorems 1.3 and 1.5

Recall Definition 1.7, for $f \in L^\infty(\mathbb{T}^N)$, one can have a decomposition as following

$$f = \sum_{I \subset \hat{N}} f_{(I,I^c)}. \tag{3.1}$$

The following is a well-known fact.

$$T_{f_{(I,I^c)}} k_z = f_{(I,I^c)}(z_I, \cdot) k_z. \tag{3.2}$$

Proof of Theorem 1.3. The necessity is trivial. For the sufficiency, suppose the Berezin transform of $\sum_{m=1}^M T_{f_m} T_{g_m}$ is N-harmonic in $\mathbb{D}^N$ and, for almost all $\xi \in \mathbb{T}^N$, $\sum_{m=1}^M f_m(\xi) g_m(\xi) = 0$.

By equations (3.1) and (3.2), we have

$$\begin{aligned} \left\langle \left(\sum_{m=1}^M T_{f_m} T_{g_m} \right) k_z, k_z \right\rangle &= \sum_{m=1}^M \sum_{I \subset \hat{N}} \langle T_{f_m} T_{g_{m(I,I^c)}} k_z, k_z \rangle \\ &= \sum_{m=1}^M \sum_{I \subset \hat{N}} \left\langle T_{f_m} \left[g_{m(I,I^c)}(z_I, \cdot) k_z \right], k_z \right\rangle \\ &= \sum_{m=1}^M \sum_{I \subset \hat{N}} \left\langle f_m(z_I, \cdot) g_{m(I,I^c)}(z_I, \cdot) k_{z_{I^c}}, k_{z_{I^c}} \right\rangle. \end{aligned} \tag{3.3}$$

Applying Lemma 2.1, for $1 \le p < \infty$,

$$\left\langle \left(\sum_{m=1}^M T_{f_m} T_{g_m} \right) k_{r\xi}, k_{r\xi} \right\rangle$$

tends to

$$\sum_{m=1}^M \sum_{I \subset \hat{N}} f_m(\xi) g_{m(I,I^c)}(\xi) = \sum_{m=1}^M f_m(\xi) g_m(\xi)$$

as $r \to 1^-$ in the norm of $L^p(\mathbb{T}^N)$. Thus

$$\left\langle \left(\sum_{m=1}^M T_{f_m} T_{g_m} \right) k_z, k_z \right\rangle = 0, \quad z \in \mathbb{D}^N.$$

Since the Berezin transform is one-to-one, $\sum_{m=1}^M T_{f_m} T_{g_m} = 0$. This completes the proof. □

Proof of Theorem 1.5. For the necessity, by Theorem 1.3, suppose that the Berezin transform of $\sum_{m=1}^M T_{f_m} T_{g_m}$ is N-harmonic in $\mathbb{D}^N$ and, for almost all $\xi \in \mathbb{T}^N$, $\sum_{m=1}^M f_m(\xi) g_m(\xi) = 0$.

Note that if $i \in I^c$, then $\langle f_m(z_I,\cdot)g_{m(I,I^c)}(z_I,\cdot)k_{z_{I^c}},k_{z_{I^c}}\rangle$ is harmonic in z_i. Differentiating under the integral sign, it follows from formula (3.3) that for a set $I \subset \hat{N}$

$$\sum_{m=1}^{M}\sum_{I\subset J\subset \hat{N}}\big\langle \partial_I f_m(z_J,\cdot)\bar{\partial}_I g_{m(J,J^c)}(z_J,\cdot)k_{z_{J^c}},k_{z_{J^c}}\big\rangle = 0. \tag{3.4}$$

For a function $f \in \bigcap_{1<q<\infty} L^q(\mathbb{T}^N)$ and a set $I \subset \hat{N}$, we have

$$\begin{aligned}\partial_I f(z_I,\xi_{I^c}) &= \int_{\mathbb{T}^{|I|}} f(\xi_I,\xi_{I^c})\partial_I\big(\big|k_{z_I}(\xi_I)\big|^2\big)\,d\sigma(\xi_I)\\ &= \int_{\mathbb{T}^{|I|}} f(\xi)\prod_{i\in I}\frac{\bar{\xi}_i-\bar{z}_i}{(1-\bar{z}_i\xi_i)(1-z_i\bar{\xi}_i)^2}\,d\sigma(\xi_I)\end{aligned} \tag{3.5}$$

for almost all $\xi_{I^c} \in \mathbb{T}^{|I^c|}$. For given $z_I \in \mathbb{D}^{|I|}$, it is clear that

$$\partial_I f(z_I,\xi_{I^c}) \in \bigcap_{1<q<\infty} L^q\big(\mathbb{T}^{N-|I|}\big). \tag{3.6}$$

For $I \subset J \subset \hat{N}$ by differentiating under the integral sign it is easy to see

$$\begin{aligned}\partial_I f(z_J,\xi_{J^c}) &= \partial_I\big\langle f(z_I,\cdot,\xi_{J^c})k_{z_{J\setminus I}},k_{z_{J\setminus I}}\big\rangle\\ &= \big\langle \partial_I f(z_I,\cdot,\xi_{J^c})k_{z_{J\setminus I}},k_{z_{J\setminus I}}\big\rangle.\end{aligned} \tag{3.7}$$

Using the polar coordinates $z_{I^c} = r\xi_{I^c}$ in the left hand side of formula (3.4) and taking limit as $r \to 1^-$, by formulas (3.6) and (3.7) and Lemma 2.1, we get

$$\begin{aligned}0 &= \sum_{m=1}^{M}\sum_{I\subset J\subset \hat{N}} \partial_I f_m(z_I,\xi_{I^c})\bar{\partial}_I g_{m(J,J^c)}(z_I,\xi_{I^c})\\ &= \sum_{m=1}^{M} \partial_I f_m(z_I,\xi_{I^c})\bar{\partial}_I g_{m(I,\emptyset)}(z_I,\xi_{I^c})\\ &= \sum_{m=1}^{M} \partial_I f_m(z_I,\xi_{I^c})\bar{\partial}_I g_{m}(z_I,\xi_{I^c})\end{aligned}$$

for almost all $\xi_{I^c} \in \mathbb{T}^{|I^c|}$.

Sufficiency. Suppose equation (1.1) hold for all $I \subset \hat{N}$. Taking $I = \emptyset$, we have $\sum_{m=1}^{M} f_m(\xi)g_m(\xi) = 0$ for almost all $\xi \in \mathbb{T}^N$. By Theorem 1.3, we need to prove that the Berezin transform of $\sum_{m=1}^{M} T_{f_m}T_{g_m}$ is N-harmonic in $\mathbb{D}^N$, by formula (3.3), which is equivalent to

$$\sum_{m=1}^{M}\sum_{i\in I\subset \hat{N}}\big\langle \partial_i f_m(z_I,\cdot)\bar{\partial}_i g_{m(I,I^c)}(z_I,\cdot)k_{z_{I^c}},k_{z_{I^c}}\big\rangle = 0, \quad 1 \le i \le N. \tag{3.8}$$

We will prove it by induction. If $N=1$, taking $I=\{1\}$ in formula (1.1), then $\sum_{m=1}^{M}\partial_1 f_m(z_1)\bar{\partial}_1 g_m(z_1)=0$ which is equivalent to formula (3.8).

Fix $i\in\hat{N}$. By formula (1.1), for all $J\subset\hat{N}\setminus\{i\}$, we have

$$\sum_{m=1}^{M}\partial_J\partial_i f_m(z_i,z_J,\xi_{\hat{N}\setminus(J\cup\{i\})})\bar{\partial}_J\bar{\partial}_i g_m(z_i,z_J,\xi_{\hat{N}\setminus(J\cup\{i\})})=0$$

for almost all $\xi_{\hat{N}\setminus(J\cup\{i\})}\in\mathbb{T}^{N-|J|-1}$. By induction, for given $z_i\in\mathbb{D}$, we have therefore

$$\sum_{m=1}^{M}T_{\partial_i f_m(z_i,\cdot)}T_{\bar{\partial}_i g_m(z_i,\cdot)}=0.$$

It follows from formula (3.3) that

$$\begin{aligned}0&=\sum_{m=1}^{M}\sum_{J\subset\hat{N}\setminus\{i\}}\big\langle\partial_i f_m(z_i,z_J,\cdot)\bar{\partial}_i g_{m(J,\hat{N}\setminus(J\cup\{i\}))}(z_i,z_J,\cdot)k_{z_{\hat{N}\setminus(J\cup\{i\})}},\\&\qquad k_{z_{\hat{N}\setminus(J\cup\{i\})}}\big\rangle\\&=\sum_{m=1}^{M}\sum_{i\in I\subset\hat{N}}\big\langle\partial_i f_m(z_I,\cdot)\bar{\partial}_i g_{m(I,I^c)}(z_I,\cdot)k_{z_{I^c}},k_{z_{I^c}}\big\rangle.\end{aligned}$$

This completes the proof. $\square$

4. Proof of Theorem 1.8

LEMMA 4.1. *Suppose* $f=(f_1,f_2,\dots,f_n)$, $g=(g_1,g_2,\dots,g_n)\in H(\mathbb{D}^m)\otimes\mathbb{C}^n$ *such that*

$$\big\langle f(z),g(z)\big\rangle_{\mathbb{C}^n}=0 \tag{4.1}$$

for all $z\in\mathbb{D}^m$. *Then there exist a natural number* k $(0\le k\le n)$ *and an orthonormal base* $\{e_1,e_2,\dots,e_n\}$ *in* $\mathbb{C}^n$ *such that*

$$\big\langle f(z),e_i\big\rangle_{\mathbb{C}^n}=0,\quad 1\le i\le k$$

and

$$\big\langle g(z),e_i\big\rangle_{\mathbb{C}^n}=0,\quad k+1\le i\le n$$

for all $z\in\mathbb{D}^m$.

Proof. As in [1, Lemma 2] "complexifying" formula (4.1), we have

$$\big\langle f(z),g(w)\big\rangle_{\mathbb{C}^n}=0$$

for all $z,w\in\mathbb{D}^m$. Hence,

$$\operatorname{span}\big\{f(z):z\in\mathbb{D}^m\big\}\perp\operatorname{span}\big\{g(z):z\in\mathbb{D}^m\big\}.$$

So the conclusion follows. $\square$

Proof of Theorem 1.8. By Theorem 1.5, $\sum_{m=1}^{M} T_{f_m}T_{g_m}=0$ on $H^2(\mathbb{D}^N)$ if and only if for all $I\subset\hat{N}$,

$$\sum_{m=1}^{M}\partial_I f_m(z_I,\xi_{I^c})\bar{\partial}_I g_m(z_I,\xi_{I^c})=0 \tag{4.2}$$

for $z_I\in\mathbb{D}^{|I|}$ and almost all $\xi_{I^c}\in\mathbb{T}^{N-|I|}$. When $I=\emptyset$, formula (4.2) is equivalent to

$$\sum_{m=1}^{M} f_m(\xi)g_m(\xi)=0.$$

When $I\subset\hat{N}$ and $I\neq\emptyset$, Lemma 4.1 implies that formula (4.2) is equivalent to that, for almost all $\xi_{I^c}\in\mathbb{T}^{N-|I|}$, there exist a natural number $k(\xi_{I^c})$ ($0\le k(\xi_{I^c})\le M$) and an orthonormal base $\{e_1(\xi_{I^c}),e_2(\xi_{I^c}),\dots,e_M(\xi_{I^c})\}$ of $\mathbb{C}^M$ such that

$$\left\langle\partial_I f_{(\emptyset,I)}(z_I,\xi_{I^c}),e_i(\xi_{I^c})\right\rangle_{\mathbb{C}^M}=0,\quad 1\le i\le k(\xi_{I^c}) \tag{4.3}$$

and

$$\left\langle\overline{\partial_I g_{(I,\emptyset)}(z_I,\xi_{I^c})},e_i(\xi_{I^c})\right\rangle_{\mathbb{C}^M}=0,\quad k(\xi_{I^c})+1\le i\le M \tag{4.4}$$

for $z_I\in\mathbb{D}^{|I|}$. It is easy to see that conditions (4.3) and (4.4) are equivalent to the conditions (1.4) and (1.5) in Theorem 1.8, respectively.

Now suppose $j\in I^c$. Let $J=I\cup\{j\}$. The condition (CIk) implies, for almost all given $\xi_{J^c}\in\mathbb{T}^{N-|J|}$, there exist $k(\xi_{J^c})$ ($0\le k\le M$) and an orthonormal base $\{e_1(\xi_{J^c}),e_2(\xi_{J^c}),\dots,e_M(\xi_{J^c})\}$ of $\mathbb{C}^M$ such that

$$\left\langle\partial_J f(z_J,\xi_{J^c}),e_i(\xi_{J^c})\right\rangle_{\mathbb{C}^M}=0,\quad 1\le i\le k(\xi_{J^c})$$

and

$$\left\langle\partial_J\bar{g}(z_J,\xi_{J^c}),e_i(\xi_{J^c})\right\rangle_{\mathbb{C}^M}=0,\quad k(\xi_{J^c})+1\le i\le M$$

for $z_J\in\mathbb{D}^{|J|}$. Therefore, there exists a unitary matrix U of degree $M\times M$ such that

$$U\partial_J f(z_J,\xi_{J^c})\in\mathbb{C}^M\ominus E$$

and

$$U\partial_J\bar{g}(z_J,\xi_{J^c})\in E$$

for $z_J\in\mathbb{D}^{|J|}$, where E denotes the subspace of $\mathbb{C}^M$ consisting of all vectors vanish at the coordinates after the $k(\xi_{J^c})$th item. Let F and G denote $Uf(\cdot,\xi_{J^c})$ and $U\bar{g}(\cdot,\xi_{J^c})$, respectively. It is clear that

$$\left\langle\partial_I F(z_I,\xi_j),\partial_I G(z_I,\xi_j)\right\rangle_{\mathbb{C}^M}=0 \tag{4.5}$$

for $z_I\in\mathbb{D}^{|I|}$ and almost all $\xi_j\in\mathbb{T}$, and

$$P_E\partial_J F(z_J)=0\quad\text{and}\quad P_{\mathbb{C}^M\ominus E}\partial_J G(z_J)=0 \tag{4.6}$$

for $z_J\in\mathbb{D}^{|J|}$, where P_E denote the orthonormal projection from $\mathbb{C}^M$ onto E. Formula (4.6) implies that $P_E\partial_I F(z_I,\xi_j)$ is a vector-valued function on $\mathbb{D}^{|I|}\times$

$\mathbb{T}$ with which entries are analytic in $z_I \in \mathbb{D}^{|I|}$ for almost all $\xi_j \in \mathbb{T}$, and in $\overline{H^2(\mathbb{T})}$ for all $z_I \in \mathbb{D}^{|I|}$. By formula (3.5), it is easy to see that the map $z_I \mapsto P_E \partial_I F(z_I, \cdot)$ is continuous from $\mathbb{D}^{|I|}$ to $\overline{H^2(\mathbb{T})} \otimes E$. By Lemma 2.4, we have that the dimension of $\bigvee_{\xi_j}(P_E \partial_I F)$ is constant, say k_1, for almost all $\xi_j \in \mathbb{T}$. Similarly the dimension of $\bigvee_{\xi_j}(P_{\mathbb{C}^M \ominus E} \partial_I G)$ is constant, say k_2, for almost all $\xi_j \in \mathbb{T}$.

Fix a point $\xi_j \in \mathbb{T}$ such that $\mathrm{vd}_{\xi_j}(P_E \partial_I F) = k_1$, $\mathrm{vd}_{\xi_j}(P_{\mathbb{C}^M \ominus E} \partial_I G) = k_2$ and formula (4.5) holds. Let E_1 and E_2 denote $\bigvee_{\xi_j}(P_E \partial_I F)$ and $\bigvee_{\xi_j}(P_{\mathbb{C}^M \ominus E} \partial_I G)$, respectively. Then we have that

$$\partial_I F(z_I, \xi_j) = P_{E_1 \oplus E_2} \partial_I F(z_I, \xi_j) + P_{\mathbb{C}^M \ominus (E \oplus E_2)} \partial_I F(z_I, \xi_j) \tag{4.7}$$

and

$$\partial_I G(z_I, \xi_j) = P_{E_1 \oplus E_2} \partial_I G(z_I, \xi_j) + P_{E \ominus E_1} \partial_I G(z_I, \xi_j). \tag{4.8}$$

By "complexifying" formula (4.5), we have

$$\begin{aligned} 0 &= \left\langle \partial_I F(z_I, \xi_j), \partial_I G(w_I, \xi_j) \right\rangle_{\mathbb{C}^M} \\ &= \left\langle P_{E_1 \oplus E_2} \partial_I F(z_I, \xi_j), P_{E_1 \oplus E_2} \partial_I G(w_I, \xi_j) \right\rangle_{\mathbb{C}^M}. \end{aligned}$$

It follows that

$$\bigvee_{\xi_j}(P_{E_1 \oplus E_2} \partial_I F) \perp \bigvee_{\xi_j}(P_{E_1 \oplus E_2} \partial_I G). \tag{4.9}$$

Since

$$\dim \bigvee_{\xi_j}(P_{E_1 \oplus E_2} \partial_I F) \geq \dim \bigvee_{\xi_j}(P_{E_1} \partial_I F) = k_1$$

and $\dim \bigvee_{\xi_j}(P_{E_1 \oplus E_2} \partial_I G) \geq k_2$, formula (4.9) implies that

$$\mathrm{vd}_{\xi_j}(P_{E_1 \oplus E_2} \partial_I F) = k_1 \quad \text{and} \quad d_{\xi_j}(P_{E_1 \oplus E_2} \partial_I G) = k_2.$$

Applying formulas (4.7) and (4.8), we have

$$\begin{aligned} \mathrm{vd}_{\xi_j}(\partial_I F) &\leq \mathrm{vd}_{\xi_j}(P_{E_1 \oplus E_2} \partial_I F) + \mathrm{vd}_{\xi_j}(P_{\mathbb{C}^M \ominus (E \oplus E_2)} \partial_I F) \\ &= k_1 + M - k(\xi_{J^c}) - k_2 \end{aligned}$$

and

$$\mathrm{vd}_{\xi_j}(\partial_I G) \leq \mathrm{vd}_{\xi_j}(P_{E_1 \oplus E_2} \partial_I G) + \mathrm{vd}_{\xi_j}(P_{E \ominus E_1} \partial_I G) = k_2 + k(\xi_{J^c}) - k_1.$$

Let $k = k_2 + k(\xi_{J^c}) - k_1$. Then we can choose k which is independent to ξ_j excepting a set of measure zero as for $k(\xi_{I^c})$ in the condition (CIk). This completes the proof. $\square$

Corollary 4.2. *Let f and g are in $\in L^\infty(\mathbb{T}^N)$. Then $T_f T_g = T_g T_f$ on $H^2(\mathbb{D}^N)$ if and only if for all nonempty subset $I = \{n_1, n_2, \dots, n_{|I|}\}$ of $\hat{N}$ and almost all $\xi_{I^c} \in \mathbb{T}^{N-|I|}$, one of the following conditions holds.*

(aI)

$$f_{(\emptyset,I)}(z_I,\xi_{I^c})=\sum_{k=1}^{|I|} f_k^I(z_{I\setminus\{n_k\}},\xi_{I^c})$$

and

$$g_{(\emptyset,I)}(z_I,\xi_{I^c})=\sum_{k=1}^{|I|} g_k^I(z_{I\setminus\{n_k\}},\xi_{I^c}),$$

where all of $f_k^I(z_{I\setminus\{n_k\}},\xi_{I^c})$ *and* $g_k^I(z_{I\setminus\{n_k\}},\xi_{I^c})$ *are analytic in* $z_{I\setminus\{n_k\}}$.

(bI)

$$f_{(I,\emptyset)}(z_I,\xi_{I^c})=\sum_{k=1}^{|I|} \bar{f}_k^I(z_{I\setminus\{n_k\}},\xi_{I^c})$$

and

$$g_{(I,\emptyset)}(z_I,\xi_{I^c})=\sum_{k=1}^{|I|} \bar{g}_k^I(z_{I\setminus\{n_k\}},\xi_{I^c}),$$

where all of $f_k^I(z_{I\setminus\{n_k\}},\xi_{I^c})$ *and* $g_k^I(z_{I\setminus\{n_k\}},\xi_{I^c})$ *are analytic in* $z_{I\setminus\{n_k\}}$.

(cI) *There exist two function* $a(\xi_{I^c})$ *and* $b(\xi_{I^c})$ *with* $|a(\xi_{I^c})|^2+|b(\xi_{I^c})|^2=1$ *such that*

$$a(\xi_{I^c})f_{(\emptyset,I)}(z_I,\xi_{I^c})+b(\xi_{I^c})g_{(\emptyset,I)}(z_I,\xi_{I^c})=\sum_{k=1}^{|I|} h_k^I(z_{I\setminus\{n_k\}},\xi_{I^c})$$

and

$$a(\xi_{I^c})f_{(I,\emptyset)}(z_I,\xi_{I^c})+b(\xi_{I^c})g_{(I,\emptyset)}(z_I,\xi_{I^c})=\sum_{k=1}^{|I|} \bar{r}_k^I(z_{I\setminus\{n_k\}},\xi_{I^c}),$$

where all of $h_k^I(z_{I\setminus\{n_k\}},\xi_{I^c})$ *and* $r_k^I(z_{I\setminus\{n_k\}},\xi_{I^c})$ *are analytic in* $z_{I\setminus\{n_k\}}$. *Moreover, if* $j\in I^c$, *then for almost all given* $\xi_{I^c\setminus\{j\}}\in\mathbb{T}^{N-|I|-1}$, *one of three conditions above holds for almost all* $\xi_j\in\mathbb{T}$.

REMARK 4.3. If let $N=2$, then the above corollary implies Theorem 1.2.

THEOREM 4.4. *Let* f *and* g *are in* $\in L^\infty(\mathbb{T}^N)$. *Then* $T_fT_g=T_{fg}$ *on* $H^2(\mathbb{D}^N)$ *if and only if for each* $i\in\hat{N}$, *either* $f(z)$ *is co-analytic in* z_i *or* $g(z)$ *is analytic in* z_i *for all* $z_{\hat{N}\setminus\{i\}}\in\mathbb{D}^{N-1}$.

Proof. Sufficiency. Suppose $I\subset\hat{N}$ such that $f(z)$ is co-analytic in z_I and $g(z)$ is analytic in $z_{\hat{N}\setminus I}$. We have

$$\begin{aligned}\langle T_fT_gk_z,k_z\rangle &= \langle T_gk_z,\bar{f}k_z\rangle=\left\langle P_I(gk_z),\bar{f}k_z\right\rangle\\ &=\langle gk_z,\bar{f}k_z\rangle=\langle T_{fg}k_z,k_z\rangle.\end{aligned}$$

Since the Berezin transform is one-to-one, the equation $T_fT_g=T_{fg}$ holds.

Necessary. Suppose that the conclusion fails for some $i \in \hat{N}$. Then there exist two subset E_1 and F_1 of $\mathbb{T}^{N-1}$ with positive measure such that

$$\partial_i f(z_i, \xi_{\hat{N}\setminus\{i\}}) \neq 0$$

for $\xi_{\hat{N}\setminus\{i\}} \in E_1$ and almost all $z_i \in \mathbb{D}$, and

$$\bar{\partial}_i g(z_i, \xi_{\hat{N}\setminus\{i\}}) \neq 0$$

for $\xi_{\hat{N}\setminus\{i\}} \in F_1$ and almost all $z_i \in \mathbb{D}$. If there exists some $j \in \hat{N} \setminus \{i\}$ such that both $\partial_i f(z_i, \xi_{\hat{N}\setminus\{i\}})$ is not co-analytic and $\bar{\partial}_i g(z_i, \xi_{\hat{N}\setminus\{i\}})$ is not analytic in the variable ξ_j. Then there exist two subset E_2 and F_2 of $\mathbb{T}^{N-2}$ with positive measure such that

$$\partial_j \partial_i f(z_{\{i,j\}}, \xi_{\hat{N}\setminus\{i,j\}}) \neq 0$$

for $\xi_{\hat{N}\setminus\{i,j\}} \in E_2$ and almost all $z_{\{i,j\}} \in \mathbb{D}^2$, and

$$\bar{\partial}_j \bar{\partial}_i g(z_{\{i,j\}}, \xi_{\hat{N}\setminus\{i,j\}}) \neq 0$$

for $\xi_{\hat{N}\setminus\{i,j\}} \in F_2$ and almost all $z_{\{i,j\}} \in \mathbb{D}^2$. By induction, we can find a subset I of $\hat{N}$ and two subset E and F of $\mathbb{T}^{N-|I|}$ with positive measure such that

$$\partial_I f(z_I, \xi_{I^c}) \neq 0 \tag{4.10}$$

for $\xi_{I^c} \in E$ and almost all $z_I \in \mathbb{D}^{|I|}$, and

$$\bar{\partial}_I g(z_I, \xi_{I^c}) \neq 0$$

for $\xi_{I^c} \in F$ and almost all $z_I \in \mathbb{D}^{|I|}$, and either $\partial_I f(z_I, \xi_{I^c})$ is co-analytic or $\bar{\partial}_I g(z_I, \xi_{I^c})$ is analytic in the variable ξ_j for each $j \in I^c$.

Let J, K be two subset of $\hat{N} \setminus I$ such that $J \cup K = I^c$, $J \cap K = \emptyset$ and $\partial_I f(z_I, \xi_{I^c})$ is co-analytic in ξ_j for $j \in J$ and $\bar{\partial}_I g(z_I, \xi_{I^c})$ is analytic in the variable ξ_k for $k \in K$. By formula (4.10), there exist a subset E' of $\mathbb{T}^{|K|}$ with positive measure such that

$$\partial_I f(z_I, \xi_J, \xi_K) \neq 0$$

for ξ_J in a subset of $\mathbb{T}^{|J|}$ with positive measure and each $\xi_K \in E'$ and almost all $z_I \in \mathbb{D}^{|I|}$. Therefore we have that $\partial_I f(z_I, \cdot, \xi_K) \neq 0$ for almost all $(\xi_J, \xi_K) \in \mathbb{T}^{|J|} \times E'$ and almost all $z_I \in \mathbb{D}^{|I|}$. Similarly there exist a subset F' of $\mathbb{T}^{|J|}$ with positive measure such that $\bar{\partial}_I g(z_I, \xi_J, \xi_K) \neq 0$ for almost all $(\xi_J, \xi_K) \in F' \times \mathbb{T}^{|K|}$ and almost all $z_I \in \mathbb{D}^{|I|}$. So we have

$$\partial_I f(z_I, \xi_J, \xi_K) \bar{\partial}_I g(z_I, \xi_J, \xi_K) \neq 0$$

for almost all $(\xi_J, \xi_K) \in F' \times E'$ and almost all $z_I \in \mathbb{D}^{|I|}$. This contradicts to Theorem 1.5 and completes the proof. □

Remark 4.5. If let $N = 2$, then the above theorem implies Theorem 1.1.

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