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# A SPECTRAL IDENTITY FOR SECOND MOMENTS OF EISENSTEIN SERIES OF $\mathrm{O}(n,1)$

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ABSTRACT. Let $H=\mathrm{O}(n)\times\mathrm{O}(1)$ be an anisotropic subgroup of $G=\mathrm{O}(n,1)$ and let $\mathbb{A}$ be the adele ring of $k=\mathbb{Q}$. Consider the periods

$$(E_\varphi,F)_H=\int_{H_k\backslash H_{\mathbb{A}}} E_\varphi\cdot\overline{F},$$

of an Eisenstein series $E_\varphi$ on $G$ against a form $F$ on $H$. Relying on a variant of Levi–Sobolev spaces, we describe certain Poincaré series as fundamental solutions for the Laplacian, and use them to establish a spectral identity concerning the second moments (in $F$-aspect) of $E_\varphi$.

## Introduction

Let $k=\mathbb{Q}$. Consider the form represented by

$$\begin{pmatrix}1 & & \\ & \mathrm{id} & \\ & & -1\end{pmatrix}$$

(here and elsewhere, omitted entries are zero) with respect to the decomposition $k^{n+1}=k^n\oplus(k\cdot e_-)=(k\cdot e_+)\oplus k^{n-1}\oplus(k\cdot e_-)$. Let $G=\mathrm{O}(n+1)$, $H=\mathrm{O}(n)\times\mathrm{O}(1)$, and $\Theta=\mathrm{O}(n-1)$, and note that $H$ and $\Theta$ are $k$-anisotropic.

As the form is isotropic, we consider the hyperbolic pair $e'=\frac{1}{2}e_+-\frac{1}{2}e_-$ and $e=e_++e_-$. Changing coordinates, we see the form is represented by

$$\begin{pmatrix} & & 1\\ & \mathrm{id} & \\ 1 & & \end{pmatrix}$$

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with respect to $(k \cdot e') \oplus k^{n-1} \oplus (k \cdot e)$. We use these new coordinates for the remainder of the Introduction, and observe that while $H$ has no simple description in these coordinates, $\Theta$ can still be identified with $\mathrm{O}(n-1)$.

Write

$$m_\lambda = \begin{pmatrix} \lambda & & \\ & \mathrm{id} & \\ & & \lambda^{-1} \end{pmatrix} \quad \text{and} \quad n_a = \begin{pmatrix} 1 & a & -\frac{1}{2}aa^{\mathrm{t}} \\ & \mathrm{id} & -a^{\mathrm{t}} \\ & & 1 \end{pmatrix}.$$

The parabolic $P$ stabilizing the isotropic line $k \cdot e$ can be written as $P = NM$, with unipotent radical $N = \{n_a\}$ and Levi component $M = \{m_\lambda\} \cdot \Theta$. The modular function on $P$ is given by $\delta_P(m_\lambda) = |\lambda|^n$.

Let $\mathbb{A}$ be the adele ring of $k = \mathbb{Q}$. At non-archimedean $v$, choose a maximal (open) compact $K_v$. At the archimedean place $v = \infty$, put $K_\infty = H_\infty$; it is a maximal compact in $G_\infty$. Write $K = \prod K_v$; it is a maximal compact in $G_\mathbb{A}$. Let us recapitulate briefly the most salient points about the spectral decomposition of (right $K$-invariant) functions in $L^2(G_k \backslash G_\mathbb{A}/K)$.

The *constant term* of $f \in L^2(G_k \backslash G_\mathbb{A}/K)$ is

$$\mathrm{c}f(g) = \int_{N_k \backslash N_\mathbb{A}} f(ng)\, \mathrm{d}n.$$

We say $f$ is a *cuspform* if $\mathrm{c}f = 0$; the space $L^2_0(G_k \backslash G_\mathbb{A}/K)$ of (right $K$-invariant) cuspforms decomposes discretely [18] into joint eigenfunctions of the center $\mathscr{Z}(\mathfrak{g}_\infty)$ of the universal enveloping algebra.

The constant term $\mathrm{c}f$ is left $N_\mathbb{A} M_k$-invariant. If $\varphi \in \mathscr{D}(N_\mathbb{A} M_k \backslash G_\mathbb{A}/K)$ is a test function, we have

$$\begin{aligned} \int_{N_\mathbb{A} M_k \backslash G_\mathbb{A}/K} \mathrm{c}f(g)\varphi(g)\, \mathrm{d}g &= \int_{N_\mathbb{A} M_k \backslash G_\mathbb{A}/K} \int_{N_k \backslash N_\mathbb{A}} f(ng)\, \mathrm{d}n\, \varphi(g)\, \mathrm{d}g \\ &= \int_{P_k \backslash G_\mathbb{A}/K} f(g)\varphi(g)\, \mathrm{d}g \\ &= \int_{G_k \backslash G_\mathbb{A}/K} f(g) E_\varphi(g)\, \mathrm{d}g, \end{aligned}$$

where

$$E_\varphi(g) = \sum_{\gamma \in P_k \backslash G_k} \varphi(\gamma g)$$

(the sum has finitely many non-zero terms) is a *pseudo-Eisenstein series*. Observing that $N_\mathbb{A} M_k = M_k N_\mathbb{A}$ and taking the Iwasawa decomposition $G_\mathbb{A} = N_\mathbb{A} M_\mathbb{A} K$ into account, we see the right $K$-invariant functions on $N_\mathbb{A} M_k \backslash G_\mathbb{A}$ are the right $K \cap M_\mathbb{A}$-invariant functions on $M_k \backslash M_\mathbb{A}$.

Recall that $M \cong \Theta \times \mathrm{GL}(1)$ and that, because $\Theta$ is $k$-anisotropic, $\Theta_k \backslash \Theta_\mathbb{A}$ is compact. Let $\Psi$ run over an orthonormal basis of $L^2(\Theta_k \backslash \Theta_\mathbb{A}/(K \cap \Theta_\mathbb{A}))$.

Let also $\lambda \mapsto \delta_P(\lambda)^s$ be a character of GL(1) (with $k = \mathbb{Q}$, there are no other characters to account for). Extend

$$\varphi_{s,\Psi}(m_\lambda \theta) = \delta_P(\lambda)^s \cdot \Psi(\theta) = |\lambda|^{ns} \cdot \Psi(\theta)$$

by left $N_{\mathbb{A}}$- and right $K$-invariance, and define the *Eisenstein series* as the meromorphic continuation of

$$E_{s,\Psi}(g) = \sum_{\gamma \in P_k \backslash G_k} \varphi_{s,\Psi}(\gamma g)$$

to all $\mathbb{C}$. (We will not go into the details, but the sum converges if $\operatorname{Re} s > 1$ and does have a meromorphic extension [31].) If $\Psi = 1$, we write simply $\varphi_s = \varphi_{s,\Psi}$ and $E_s = E_{s,\Psi}$.

Given a function $f$ in $L^2(G_k \backslash G_{\mathbb{A}}/K)$, we have [1], [31], [37]

$$f = \sum_{\Phi} \langle f, \Phi \rangle \cdot \Phi + \frac{1}{4\pi i} \sum_{\Psi} \int_{\operatorname{Re} s = \frac{1}{2}} \langle f, E_{s,\Psi} \rangle \cdot E_{s,\Psi} \, \mathrm{d}s + \sum_{R} \langle f, R \rangle \cdot R$$

with $\Phi$ running over an orthonormal basis of $L_0^2(G_k \backslash G_{\mathbb{A}}/K)$, $\Psi$ over an orthonormal basis of $L^2(\Theta_k \backslash \Theta_{\mathbb{A}}/(K \cap \Theta_{\mathbb{A}}))$, and $R$ over an orthonormal basis of residues of Eisenstein series to the right of $\operatorname{Re} s = \frac{1}{2}$. (The inner products are integrals over $G_k \backslash G_{\mathbb{A}}$.)

We choose each component to be a joint eigenvector of $\mathscr{Z}(\mathfrak{g}_\infty)$. The corresponding Plancherel identity is

$$\|f\|_{L^2}^2 = \sum_{\Phi} \bigl|\langle f, \Phi \rangle\bigr|^2 + \frac{1}{4\pi i} \sum_{\Psi} \int_{\operatorname{Re} s = \frac{1}{2}} \bigl|\langle f, E_{s,\Psi} \rangle\bigr|^2 \, \mathrm{d}s + \sum_{R} \bigl|\langle f, R \rangle\bigr|^2.$$

(We note that the Eisenstein series themselves are not in $L^2$, therefore the inner product and integral are obtained by isometric extension.)

In what follows, we shorten these formulas to read

$$f = \int^{\oplus} \langle f, \Phi \rangle \cdot \Phi \, \mathrm{d}\Phi \quad \text{and} \quad \|f\|_{L^2}^2 = \int^{\oplus} \bigl|\langle f, \Phi \rangle\bigr|^2 \, \mathrm{d}\Phi \tag{1}$$

(when writing thusly, $\Phi$ runs over all relevant spectral components).

We may consider the periods

$$(\Phi, F)_H = \int_{H_k \backslash H_{\mathbb{A}}} \Phi \cdot \overline{F}$$

of spectral components $\Phi$ on $G$ against cuspforms $F$ on $H$, or even

$$(\Phi)_H = (\Phi, 1)_H = \int_{H_k \backslash H_{\mathbb{A}}} \Phi.$$

Such periods contain information about the underlying representations. These same periods (called there global Shintani functions) were used by Katu, Murase, and Sugano [29], [38] to obtain and study integral expressions for standard $L$-functions of the orthogonal group. And the Gross–Prasad conjecture [19], [20], [21] predicts that a representation of $\mathrm{O}(n)$ occurs in a representation of $\mathrm{O}(n+1)$ if and only if the corresponding tensor product $L$-function is non-zero on $\operatorname{Re} s = \frac{1}{2}$. Ichino and Ikeda [24] discuss further details and broader context is provided in papers by Gross, Reeder [22], Jacquet, Lapid, Offen, and/or Rogawski [27], [33], [32], Jiang [28] and Sakellaridis and Venkatesh [39], [40].

The periods also help study the asymptotics of moments of automorphic $L$-functions. Often, the Phragmén–Lindelöf principle yields (so-called) *convex* bounds for such asymptotics [4], [26]. Diaconu and Garrett [9], [10] used a specific spectral identity to first break convexity for the asymptotics of second moments of automorphic forms in $\mathrm{GL}(2)$, over *any* number field $k$. In fact, their strategy produces families of spectral identities, explored in other papers by them and/or Goldfeld [10], [11], [12] and used by Letang [34]. In the present paper, we carry out that strategy to obtain a spectral identity for second moments of Eisenstein series of $\mathrm{O}(n, 1)$.

Given a function $f \in L^2(G_k \backslash G_{\mathbb{A}}/K)$, the spectral decomposition (1) above invites us to consider the effect of an operator $X \in \mathscr{Z}(\mathfrak{g}_\infty)$:

$$(2) \quad Xf = \int^{\oplus} \langle f, \Phi \rangle \cdot \lambda_{X,\Phi} \cdot \Phi \, \mathrm{d}\Phi \quad \text{and} \quad \|Xf\|_{L^2}^2 = \int^{\oplus} \bigl|\langle f, \Phi \rangle\bigr|^2 |\lambda_{X,\Phi}|^2 \, \mathrm{d}\Phi,$$

where $\lambda_{X,\Phi}$ is the $X$-eigenvalue of $\Phi$ (if $X = \Omega$, we write simply $\lambda_\Phi = \lambda_{\Omega,\Phi}$). The conditions for these decompositions to converge (even in the sense of isometric extensions) are most naturally discussed in the context of automorphic Sobolev spaces. The literature on *automorphic* Sobolev spaces is scarce; it includes papers by Bernstein and Reznikov [2], [3], Krötz and Stanton [30] and Michel and Venkatesh [36], as well as Garrett's [17] notes and DeCelles's [8] very detailed discussion. We discuss them (and their zonal counterparts) in Sections 1 and 2, following the approach in the author's dissertation [5].

The *automorphic Sobolev spaces* we discuss in Section 1 are closures (with respect to the relevant norms) of the space $\mathscr{D}(G_k \backslash G_{\mathbb{A}})$ of *global* test functions. Even though we only take into account the eigenvalues of $\Omega$ in their definition, we rely on a global spectral decomposition, and the norms are defined from integrals over $G_k \backslash G_{\mathbb{A}}$. So we should see these spaces as spaces of global functions.

A crucial point is that, using a pre-trace kernel, we can obtain an estimate

$$\int^{\oplus}_{|\lambda_\Phi| < T^2} \bigl|\Phi(g)\bigr|^2 \ll T^n$$

similar to Weyl's Law, from which we can characterize an automorphic delta $\delta_{\mathbb{A}}$. Then, it is just a matter of using the techniques one habitually uses with classical Sobolev spaces to obtain fundamental solutions of PDEs.

By contrast, the *zonal Sobolev spaces* we discuss in Section 2 are closures of test functions on $K_\infty \backslash G_\infty / K_\infty$; these are local (archimedean) functions. From them, we shall obtain a different construction of the (global) fundamental solutions just mentioned, which will help us extract some archimedean information.

In Section 3, we use those techniques to obtain fundamental solutions (following Diaconu and Garrett [9], we call them *Poincaré series*) for certain polynomials in $\Omega$. The spectral decomposition of these Poincaré series Pé involves the periods $(\Phi)_H$ discussed above. Given an automorphic function $f \otimes f'$ on $G \times G$, we expand $\langle f \cdot f', \mathrm{Pé} \rangle_G$ in two distinct ways, yielding an identity between a spectral expansion (along $G$) and a moment expansion (in $F$-aspect, with $F$ running over an orthonormal basis of cuspforms on $H$).

In Section 4, we apply those ideas to Eisenstein series. In particular, we see how the moment expansion involves the second moments of the Eisenstein series in $F$-aspect, as well as the periods of Eisenstein series. (Elsewhere [5], [6], [7], this author has computed these periods at non-archimedean primes. As discussed there, for the cases used in the present paper, the local factor at the archimedean place is 1.)

In the Appendix, we explain the regularization used in Section 4.

## 1. Automorphic Sobolev spaces

In the continuation, we will rely heavily on some $L^2$ Sobolev spaces, adapted to the automorphic case. Classically, the Sobolev space of order $\ell$ is defined as the space of functions whose weak derivatives up to order $\ell$ are square-integrable. The topology induced by that family of seminorms (one for each derivative up to order $\ell$) can also be described by a norm obtained from Plancherel formula. For example, in $\mathbb{R}^n$, we set

$$\|f\|^2_{H^\ell} = \int_{\mathbb{R}^n} \left|\widehat{f}(\xi)\right|^2 \left(1 + |\xi|^2\right)^\ell \mathrm{d}\xi.$$

Under Fourier transform, the Laplacian $\Delta$ acts (up to a constant) by multiplication by $|\xi|^2$. In the Plancherel identity, the effect of $\Delta$ is as described in (2).

In our case, the effect of the Casimir element $\Omega$ of $G_\infty$ on the Plancherel identity is also as in (2). Thus, with inner products obtained from integrals over $G_k \backslash G_{\mathbb{A}}$ (or by isometric extension), we define the automorphic Sobolev norm by

$$\|f\|^2_\ell = \int^{\oplus} \left|\langle f, \Phi\rangle\right|^2 \left(1 + |\lambda_\Phi|\right)^\ell \mathrm{d}\Phi$$

and the *automorphic Sobolev space* as

$$H^\ell_{\text{auto}} = \text{closure of } \mathscr{D}(G_k \backslash G_{\mathbb{A}}) \text{ with respect to } \| \, \|_\ell. \tag{3}$$

We are specifically interested in the effect of the center $\mathscr{Z}(\mathfrak{g}_\infty)$ of the universal enveloping algebra (and the corresponding differential operators), so the only modification to the usual $L^2$ norm involves only archimedean information. However, the norm itself depends on the global automorphic spectral decomposition.

For $\ell > 0$, as usual, $H^{-\ell}_{\text{auto}}$ is the dual of $H^{\ell}_{\text{auto}}$. Let $f \in \mathscr{D}(G_k\backslash G_{\mathbb{A}}) \cap H^{-\ell}_{\text{auto}}$ and $\varphi \in \mathscr{D}(G_k\backslash G_{\mathbb{A}}) \cap H^{\ell}_{\text{auto}}$. In the expanded notation, we define $\langle f, \varphi\rangle$ by

$$\sum_{\Phi} \langle f, \Phi\rangle \overline{\langle \varphi, \Phi\rangle} + \frac{1}{4\pi i} \sum_{\Psi} \int_{\operatorname{Re} s = \frac{1}{2}} \langle f, E_{s,\Psi}\rangle \overline{\langle \varphi, E_{s,\Psi}\rangle}\, \mathrm{d}s + \sum_{R} \langle f, R\rangle \overline{\langle \varphi, R\rangle}.$$

From Cauchy–Schwarz–Bunyakowsky, we obtain (now in the compressed notation)

$$\begin{aligned}
\langle f, \varphi\rangle &= \int^{\oplus} \langle f, \Phi\rangle \overline{\langle \varphi, \Phi\rangle}\, \mathrm{d}\Phi \\
&\ll \int^{\oplus} \big|\langle f, \Phi\rangle\big| \big(1 + |\lambda_\Phi|\big)^{-\ell/2} \cdot \big|\langle \varphi, \Phi\rangle\big| \big(1 + |\lambda_\Phi|\big)^{\ell/2}\, \mathrm{d}\Phi \\
&\ll \sqrt{\int^{\oplus} \big|\langle f, \Phi\rangle\big|^2 \big(1 + |\lambda_\Phi|\big)^{-\ell}\, \mathrm{d}\Phi} \cdot \sqrt{\int^{\oplus} \big|\langle \varphi, \Phi\rangle\big|^2 \big(1 + |\lambda_\Phi|\big)^{\ell}\, \mathrm{d}\Phi} \\
&= \|f\|_{-\ell} \cdot \|\varphi\|_{\ell}.
\end{aligned}$$

Proposition 1. *With $X = G_\infty/K_\infty$ and $n = \dim_{\mathbb{R}} X$, we have*

$$\int^{\oplus}_{|\lambda_\Phi| < T^2} \big|\Phi(g)\big|^2 \ll T^n.$$

(This is unsurprising, in light of Weyl's Law [13], [25], [35].)

*Proof of Proposition 1.* We follow Garrett [15], with adjustments to a different group and attempting to avoid tedious computations.

We use the "ball"

$$B = \left\{ n_a m_\lambda :\ \max |a_i| < \frac{1}{T} \text{ and } |\log \lambda| < \frac{1}{T} \right\}$$

of radius $1/T$ in $P_\infty$. Then $BK_\infty$ is a tubular neighborhood of $K_\infty$ in $G_\infty$. Considering the action by $\eta = \mathrm{ch}_{BK_\infty} \otimes \bigotimes_{v<\infty} \mathrm{ch}_{K_v}$, we have

$$\begin{aligned}
(\eta \cdot f)(g) &= \int_{G_{\mathbb{A}}} \eta(h) f(gh)\, \mathrm{d}h = \int_{G_{\mathbb{A}}} \eta\big(g^{-1}h\big) f(h)\, \mathrm{d}h \\
&= \int_{G_k\backslash G_{\mathbb{A}}} \sum_{\gamma \in G_k} \eta\big(g^{-1}\gamma h\big) f(h)\, \mathrm{d}h = \langle \eta_g, \overline{f}\rangle,
\end{aligned}$$

where

$$\eta_g(h) = \sum_{\gamma \in G_k} \eta\big(g^{-1}\gamma h\big).$$

If the radius $1/T$ is sufficiently small, this sum has one single term. (Note that $\eta$ is not smooth; such a choice avoids cut-off functions.) Still following Garrett,

$$\begin{aligned}\|\eta_g\|^2 &= \int_{G_k\backslash G_{\mathbb{A}}} \eta_g(h) \sum_{\gamma\in G_k} \overline{\eta}\big(g^{-1}\gamma h\big)\,\mathrm{d}h = \int_{G_{\mathbb{A}}} \eta_g(h)\overline{\eta}\big(g^{-1}h\big)\,\mathrm{d}h \\ &= \int_{G_{\mathbb{A}}} \sum_{\gamma\in G_k} \eta\big(g^{-1}\gamma gh\big)\overline{\eta}(h)\,\mathrm{d}h.\end{aligned}$$

We are led to

$$\|\eta_g\|^2 \ll \mathrm{radius}^n = \frac{1}{T^n}.$$

On the other hand, because $\varPhi$ is right $K$-invariant and generates an irreducible representation, it must be that

$$\langle \eta_g, \varPhi\rangle = (\eta\cdot\overline{\varPhi})(g) = C\overline{\varPhi}(g),$$

where the constant $C$ depends only on $\eta$ and the archimedean parameters of $\varPhi$. Let $s$ be the archimedean parameter, seen as the parameter of a principal series representation $\varphi_s$ at $\infty$. Then, if $n_a m_\lambda \in B$, $k\in K_\infty$, and $s \ll T$ (which is the case if $|\lambda_\varPhi| < T^2$, as $\lambda_\varPhi \asymp s^2$), we have

$$\varphi_s(n_a m_\lambda k) = \delta(m_\lambda) = |\lambda|^{ns} < e^{ns/T} \ll 1.$$

In particular, assume that $g$ lies in a fixed compact and the radius $1/T$ is sufficiently small. Then

$$(\eta\cdot\overline{\varPhi})(g) = \int_{G_{\mathbb{A}}} \eta(h)\overline{\varPhi}(gh)\,\mathrm{d}h \gg \int_{G_{\mathbb{A}}} \eta(h)\,\mathrm{d}h\cdot\overline{\varPhi}(g) \asymp \mathrm{radius}^n\cdot\overline{\varPhi}(g)$$

and we see that

$$C \gg \mathrm{radius}^n = \frac{1}{T^n}.$$

Combining all this information, we conclude

$$\frac{1}{T^n} \gg \|\eta_g\|^2 = \int^{\oplus} \big|\langle\eta_g,\varPhi\rangle\big|^2\,\mathrm{d}\varPhi \geq \int^{\oplus}_{|\lambda_\varPhi|<T^2} \big|\langle\eta_g,\varPhi\rangle\big|^2 \gg \int^{\oplus}_{|\lambda_\varPhi|<T^2} \frac{|\varPhi(g)|^2}{T^{2n}}. \qquad \square$$

LEMMA 2. *Let $\delta_{\mathbb{A}}$ be the distribution defined, for right $K$-invariant $f$, by*

$$\int_{G_k\backslash G_{\mathbb{A}}} f\cdot\delta_{\mathbb{A}} = f(1).$$

*We have $\delta_{\mathbb{A}} \in H_{\mathrm{auto}}^{-n/2-\varepsilon}$, for any $\varepsilon > 0$.*

(As the definition of $H^\ell_{\mathrm{auto}}$ depends on the *global* spectral decomposition, it is not possible to reduce this to classical lemmas, of which it is a direct analogue.)

*Proof of Lemma 2.* Indeed, let

$$a_N = \int_{|\lambda_\Phi|<N^2}^{\oplus} \big|F(1)\big|^2.$$

Then

$$\begin{aligned}
\int^{\oplus} \frac{|\langle \delta_{\mathbb{A}}, \Phi\rangle|^2}{(1+|\lambda_\Phi|)^{n/2+\varepsilon}} \,\mathrm{d}\Phi &\ll \sum_{N\ge 0} \frac{a_{N+1}-a_N}{(1+N)^{n+2\varepsilon}} \\
&= \sum_{N\ge 0} a_N \left(\frac{1}{N^{n+2\varepsilon}} - \frac{1}{(N+1)^{n+2\varepsilon}}\right) \\
&\ll \sum_{N\ge 0} \frac{a_N}{N^{n+1+\varepsilon}} \ll \sum_{N\ge 0} \frac{N^n}{N^{n+1+\varepsilon}} < \infty.
\end{aligned}$$

Therefore, in $H_{\mathrm{auto}}^{-n/2-\varepsilon}$, we have

$$\delta_{\mathbb{A}} = \int^{\oplus} \Phi(1)\cdot \Phi \,\mathrm{d}\Phi. \qquad \square$$

PROPOSITION 3. *If* $|\lambda_\Phi - \lambda| \ge r > 0$ *for all* $\Phi$, *then* $(\Omega-\lambda) : H_{\mathrm{auto}}^{\ell} \to H_{\mathrm{auto}}^{\ell-2}$ *is an isomorphism.*

*Proof.* For $\lambda \in \mathbb{C}$ and $f \in H_{\mathrm{auto}}^{\ell}$, we have

$$\begin{aligned}
\big\|(\Omega-\lambda)f\big\|_{\ell-2}^2 &= \int^{\oplus} \big|\langle f,\Phi\rangle(\lambda_\Phi-\lambda)\big|^2 \big(1+|\lambda_\Phi|\big)^{\ell-2}\,\mathrm{d}\Phi \\
&\ll \int^{\oplus} \big|\langle f,\Phi\rangle\big|^2\big(1+|\lambda_\Phi|\big)^{\ell}\,\mathrm{d}\Phi = \|f\|_\ell^2,
\end{aligned}$$

showing that $(\Omega-\lambda) : H_{\mathrm{auto}}^{\ell} \to H_{\mathrm{auto}}^{\ell-2}$ is continuous and injective. On the other hand, if $|\lambda_\Phi-\lambda| \ge r > 0$ for all $\Phi$, then

$$1+|\lambda_\Phi| \le 1+|\lambda_\Phi-\lambda|+|\lambda| \ll |\lambda_\Phi-\lambda|. \qquad \square$$

For example, there is a unique solution $u_{\mathbb{A}}$ of

$$(\Omega-\lambda)^N u_{\mathbb{A}} = \delta_{\mathbb{A}},$$

for $\delta_{\mathbb{A}}$ as defined above. In $H_{\mathrm{auto}}^{2N-n/2-\varepsilon}$, it can be expressed as

$$u_{\mathbb{A}} = \int^{\oplus} \frac{\Phi(1)}{(\lambda_\Phi-\lambda)^N}\cdot\Phi\,\mathrm{d}\Phi. \tag{4}$$

## 2. Zonal spherical functions

In this section, we work at the single archimedean place (suppressed). The facts we need on spherical functions were taken from the monographs by Helgason [23] and Gangolli and Varadarajan [14]. In this *summary*, we follow mostly Helgason, as well as Garrett [15], with adaptations for the rank one case.

Let $X = G_\infty/K_\infty$, $n = \dim_{\mathbb{R}} X$, and $\Delta$ be the image of the Casimir element $\Omega$ on $X$. A smooth function $f$ on $K\backslash G/K$ is a *zonal spherical function* if it is an eigenfunction of $\Delta$ normalized by $f(1) = 1$. For any given eigenvalue $\lambda$, there is only one such $f$.

Recall that we defined

$$\varphi_s(n_a m_\lambda k) = \delta_P(m_\lambda)^s = |\lambda|^s.$$

By a theorem of Harish-Chandra, all zonal spherical functions are of the form

$$\psi_s(g) = \int_K \varphi_s(kg)\,\mathrm{d}k,$$

for some $s \in \mathbb{C}$.

The *spherical transform* is defined for $f \in L^2(K\backslash G/K)$ by

$$\widetilde{f}(s) = \int_G f \cdot \psi_{1-s}.$$

The inversion formula (up to a constant) is

$$f(g) = \int_{\operatorname{Re} s=\frac{1}{2}} \frac{\widetilde{f}(s)\cdot\psi_s(g)}{|\mathbf{c}(s)|^2}\,\mathrm{d}s,$$

with corresponding Plancherel identity

$$\|f\|_{L^2}^2 = \int_{\operatorname{Re} s=\frac{1}{2}} \frac{|\widetilde{f}(s)|^2}{|\mathbf{c}(s)|^2}\,\mathrm{d}s.$$

(We need to use the Plancherel identity to establish an isometric extension.) The Harish-Chandra function $\mathbf{c}(s)$ is given [23] by the Gindikin–Karpelevič formula, which, in our case, is

$$\mathbf{c}(s) = \frac{\Gamma((s-\frac{1}{2})\frac{n-1}{2})\cdot\Gamma(\frac{3(n-1)}{4})}{\Gamma((s+\frac{1}{2})\frac{n-1}{2})\cdot\Gamma(\frac{n-1}{4})}.$$

(Helgason's $i\lambda$ relates to our $s$ by $\rho + i\lambda = 2\rho s$. The positive simple root $\alpha$ has $\langle\alpha,\alpha\rangle = n-1$ and multiplicity $(n-1)$. Therefore, $2\rho = (n-1)\alpha$.) The main fact we need is that

$$\mathbf{c}(s) \asymp |s|^{-\frac{n-1}{2}}.$$

We define the zonal Sobolev norm by

$$\|f\|_\ell^2 = \int_{\operatorname{Re} s=\frac{1}{2}} \frac{|\widetilde{f}(s)|^2}{|\mathbf{c}(s)|^2}\big(1+|\lambda_s|\big)^\ell\,\mathrm{d}s,$$

where $\lambda_s = \lambda_{\psi_s} \asymp |s|^2$. We define the *zonal spherical Sobolev space* as

$$H^{\ell}_{\text{zonal}} = \text{closure of } \mathscr{D}(K\backslash G/K) \text{ with respect to } \| \, \|_{\ell}.$$

LEMMA 4. *If $\delta_\infty$ is the delta distribution centered at $1 \cdot K$, we have $\delta_\infty \in H^{-n/2-\ell}_{\text{zonal}}$, for any $\varepsilon > 0$.*

(This is compatible with the outcome for $\delta_{\mathbb{A}}$, in Lemma 2.)

*Proof of Lemma 4.* We have

$$\widetilde{\delta_\infty}(s) = \psi_{1-s}(1) = \int_K \varphi_{1-s}(k)\,\mathrm{d}k = 1.$$

On the other hand,

$$\frac{(1+|\lambda_s|)^{-\ell}}{|\mathbf{c}(s)|^2} \asymp \frac{|s|^{-2\ell}}{|s|^{-(n-1)}}$$

and the requirement for

$$\int_{\operatorname{Re} s=\frac{1}{2}} \frac{(1+|\lambda_s|)^{-\ell}}{|\mathbf{c}(s)|^2}\,\mathrm{d}s < \infty$$

is $2\ell - (n-1) > 1$, or $\ell > n/2$. □

The spherical expansion of $\delta_\infty$, valid in $H^{-n/2-\varepsilon}_{\text{zonal}}$, is

$$\delta_\infty = \int_{\operatorname{Re} s=\frac{1}{2}} \frac{\psi_s}{|\mathbf{c}(s)|^2}\,\mathrm{d}s.$$

Exactly as in Proposition 3, $(\Delta - \lambda) : H^{\ell}_{\text{zonal}} \to H^{\ell-2}_{\text{zonal}}$ is an isomorphism provided $\lambda$ is away from all eigenvalues of $\Delta$ (with $s$ lying on $\operatorname{Re} s = \frac{1}{2}$ this is not at all an issue). In that case, also as before, there is a solution $u_\infty$ of $(\Delta - \lambda)^N u_\infty = \delta_\infty$. In $H^{2N-n/2-\varepsilon}_{\text{zonal}}$,

$$u_\infty = \int_{\operatorname{Re} s=\frac{1}{2}} \frac{\psi_s}{(\lambda_s - \lambda)^N \cdot |\mathbf{c}(s)|^2}\,\mathrm{d}s.$$

## 3. Poincaré series and spectral identities

We return to the global picture and follow Diaconu and Garrett [9], [10], [15], [16].

At non-archimedean $v$, let $u_v$ be the characteristic function of $H_v \cdot K_v$ and $u = u_\infty \otimes \bigotimes_{v<\infty} u_v$. Noting that $u_\infty$ inherits the left $H_\infty$-invariance of $\delta_\infty$, we define the *Poincaré series*

$$\text{Pé}(g) = \sum_{\gamma \in H_k\backslash G_k} u(\gamma g).$$

(This is a function on $G_k\backslash G_{\mathbb{A}}$, while $u_\infty$ is a function on $G_\infty$.)

For brevity, write $P(\Omega)=(\Omega-\lambda)^N$. Require $P(\lambda_\Phi)\gg 0$. Note that Pé is left $G_k$-invariant and that $\Omega$ acts only on the archimedean information. It is clear that the Poincaré series is a solution of

$$P(\Omega)\,\mathrm{Pé}=\delta_{\mathbb{A}}.$$

This same solution was shown in Section 1 to be unique in the automorphic Sobolev space. Therefore, it must be that

$$\sum_{\gamma\in H_k\backslash G_k} u(\gamma g)=\mathrm{Pé}=\int^{\oplus}\frac{\Phi(1)}{P(\lambda_\Phi)}\cdot\Phi\,\mathrm{d}\Phi.$$

(Unremarkably, Pé has a larger support than $\delta_{\mathbb{A}}$. The same phenomenon occurs already with fundamental solutions of $\Delta$ in $\mathbb{R}^n$, whose support is all of $\mathbb{R}^n$, while the support of $\delta$ is only $\{0\}$.)

On the other hand, let $f: G_k\backslash G_{\mathbb{A}}\to\mathbb{C}$ be an eigenfunction of $\Omega$ with eigenvalue $\lambda_f\neq\lambda$. Then

$$\begin{aligned}\langle f,\mathrm{Pé}\rangle_G &= \int_{G_k\backslash G_{\mathbb{A}}} f\cdot\overline{\mathrm{Pé}}=\int_{H_{\mathbb{A}}\backslash G_{\mathbb{A}}}\int_{H_k\backslash H_{\mathbb{A}}} f\cdot\overline{u}\\ &=\int_{H_{\mathbb{A}}\backslash G_{\mathbb{A}}}\left(\int_{H_k\backslash H_{\mathbb{A}}}\frac{P(\Omega)}{P(\lambda_f)}f\right)\cdot\overline{u}=\int_{H_{\mathbb{A}}\backslash G_{\mathbb{A}}}\int_{H_k\backslash H_{\mathbb{A}}} f\cdot\frac{P(\Omega)}{P(\lambda_f)}\overline{u}\\ &=\frac{1}{P(\lambda_f)}\int_{H_{\mathbb{A}}\backslash G_{\mathbb{A}}}\int_{H_k\backslash H_{\mathbb{A}}} f\cdot\left(\delta_\infty\otimes\bigotimes_{v<\infty}u_v\right)=\frac{1}{P(\lambda_f)}\int_{H_k\backslash H_{\mathbb{A}}} f\\ &=\frac{(f)_H}{P(\lambda_f)}.\end{aligned}$$

That is, the Poincaré series can be used to extract information about periods.

In $H^{2N-n/2-\varepsilon}_{\mathrm{auto}}$, we decompose

$$\mathrm{Pé}=\int^{\oplus}\langle\mathrm{Pé},\Phi\rangle\cdot\Phi\,\mathrm{d}\Phi=\int^{\oplus}\frac{(\overline{\Phi})_H}{P(\lambda_{\overline{\Phi}})}\cdot\Phi\,\mathrm{d}\Phi=\int^{\oplus}\frac{(\Phi)_H}{P(\lambda_\Phi)}\cdot\overline{\Phi}\,\mathrm{d}\Phi.$$

Diaconu and Garrett [9] discuss a similar decomposition for $G=\mathrm{GL}_2$.

**Application to spectral identities.** Still following Diaconu and Garrett [9], consider two chains of inclusions: $H^\Delta\subseteq G^\Delta\subseteq G\times G$ and $H^\Delta\subseteq H\times H\subseteq G\times G$, where $G^\Delta$ denotes the image of $G\to G\times G: g\mapsto(g,g)$, and similarly for $H^\Delta$.

Let $f\otimes f'$ be an automorphic function on $G\times G$. The two inclusions suggest two different evaluations of

$$\langle f\cdot f',\mathrm{Pé}\rangle_G=\int_{H_k\backslash G_{\mathbb{A}}} f\cdot f'\cdot u;$$

a spectral decomposition along $G^\Delta$ (we will call it the *spectral expansion*) or along $H\times H$ (we will call it the *moment expansion*).

Decomposing along $G$ (the spectral expansion), we have

$$\begin{aligned}\langle f\cdot f',\mathrm{P\acute{e}}\rangle_G &= \int^{\oplus}_{\Phi\text{ on }G}\langle f\cdot f',\Phi\rangle\langle\Phi,\mathrm{P\acute{e}}\rangle \\ &= \int^{\oplus}\frac{(\Phi)_H}{P(\lambda_\Phi)}\int_{G_k\backslash G_{\mathbb{A}}} f\cdot f'\cdot\overline{\Phi}\,\mathrm{d}\Phi,\end{aligned}\tag{5}$$

involving triple products as well as the periods $(\Phi)_H$ of each component $\Phi$.

Note that $f$ has a discrete decomposition along $H$. Writing $(g\cdot f)(h) = f(hg)$:

$$g\cdot f = \sum_F (g\cdot f, F)_H\cdot F,$$

with $F$ running over an orthonormal basis of eigenfunctions of $\mathscr{Z}(\mathfrak{h}_\infty)$. We obtain the moment expansion:

$$\begin{aligned}\langle f\cdot f',\mathrm{P\acute{e}}\rangle_G &= \int_{H_k\backslash G_{\mathbb{A}}} f\cdot f'\cdot u = \int_{H_{\mathbb{A}}\backslash G_{\mathbb{A}}}\int_{H_k\backslash H_{\mathbb{A}}} f(hg)f'(hg)u(g)\,\mathrm{d}h\,\mathrm{d}g \\ &= \int_{H_{\mathbb{A}}\backslash G_{\mathbb{A}}}(g\cdot f,\overline{g\cdot f'})_H u(g)\,\mathrm{d}g \\ &= \int_{H_{\mathbb{A}}\backslash G_{\mathbb{A}}}\sum_F\big(g\cdot f,F\big)_H\big(g\cdot f',\overline{F}\big)_H u(g)\,\mathrm{d}g.\end{aligned}\tag{6}$$

Often, it can be rewritten in the form

$$\sum_F (f,F)_H\big(f',\overline{F}\big)_H\cdot\mathrm{weight}\big(f_\infty,f'_\infty,F_\infty\big),$$

with the weight depending only on the archimedean parameters. In the next section, we show the details of such a rewriting when $f$ and $f'$ are spherical Eisenstein series, unramified at non-archimedean places. For applications, one would need to study its asymptotics.

In sum, we establish the following theorem.

THEOREM 5. *Let $f\otimes f'$ be an automorphic function on $G\times G$, where $f$ and $f'$ are spherical Eisenstein series, unramified at non-archimedean places. Then $\langle f\cdot f',\mathrm{P\acute{e}}\rangle_G$ has two expansions: the spectral expansion*

$$\int^{\oplus}_{\Phi}\frac{(\Phi)_H}{P(\lambda_\Phi)}\int_{G_k\backslash G_{\mathbb{A}}} f\cdot f'\cdot\overline{\Phi}\,\mathrm{d}\Phi$$

*is a decomposition along $G$, while the moment expansion*

$$\begin{aligned}&\sum_F\int_{H_{\mathbb{A}}\backslash G_{\mathbb{A}}}(g\cdot f,F)_H\big(g\cdot f',\overline{F}\big)_H u(g)\,\mathrm{d}g \\ &\quad= \sum_F (f,F)_H\big(f',\overline{F}\big)_H\cdot\mathrm{weight}\big(f_\infty,f'_\infty,F_\infty\big)\end{aligned}$$

*is a decomposition along $H$.*

## 4. Eisenstein series and their second moments

We want to specialize to spherical, unramified, Eisenstein series $f = E_a$ and $f' = E_b$. Here, $a, b \in \mathbb{C}$, and $E_a$ and $E_b$ are parametrized as discussed in the Introduction.

One first obstacle is that $f \cdot f'$ is not in $L^2(G_k \backslash G_{\mathbb{A}})$ and it is unclear whether we can integrate

$$\langle f \cdot f', \mathrm{Pé} \rangle_G = \int_{G_k \backslash G_{\mathbb{A}}} f \cdot f' \cdot \mathrm{Pé}$$

directly. It is possible to subtract finitely many singular terms from $f \cdot f'$ so that the difference is square-integrable; we discuss that in the Appendix.

The exact choice of singular terms will depend on where $a$ or $b$ lie. For definiteness, say

$$\mathcal{F} = E_a E_b + \sum_s c_s E_s$$

(with finitely many $s$ occurring) is the regularized expression.

For the spectral expansion, we have, as in (5),

$$\langle \mathcal{F}, \mathrm{Pé} \rangle_G = \int_{\Phi \text{ on } G}^{\oplus} \langle \mathcal{F}, \Phi \rangle \langle \Phi, \mathrm{Pé} \rangle = \int^{\oplus} \langle \mathcal{F}, \Phi \rangle \frac{(\Phi)_H}{P(\lambda_\Phi)} \, \mathrm{d}\Phi.$$

The moment expansion starts as (6),

$$\langle \mathcal{F}, \mathrm{Pé} \rangle_G = \int_{G_k \backslash G_{\mathbb{A}}} \mathcal{F} \cdot \mathrm{Pé} = \int_{H_k \backslash G_{\mathbb{A}}} \mathcal{F} \cdot u = \int_{H_{\mathbb{A}} \backslash G_{\mathbb{A}}} \int_{H_k \backslash H_{\mathbb{A}}} \mathcal{F}(hg) \, \mathrm{d}h \cdot u(g) \, \mathrm{d}g,$$

where the convergence of the inner integral is justified by the compactness of $H_k \backslash H_{\mathbb{A}}$. Recall that at non-archimedean $v$ we chose $u_v = \mathrm{ch}_{H_v \cdot K_v}$, so we assume $g_v \in H_v \cdot K_v$. Therefore, we can simplify further:

$$\langle \mathcal{F}, \mathrm{Pé} \rangle_G = \int_{H_\infty \backslash G_\infty} \int_{H_k \backslash H_{\mathbb{A}}} \mathcal{F}(hg) \, \mathrm{d}h \, u_\infty(g) \, \mathrm{d}g. \tag{7}$$

The inner integral is

$$\int_{H_k \backslash H_{\mathbb{A}}} \mathcal{F}(hg) \, \mathrm{d}h = \int_{H_k \backslash H_{\mathbb{A}}} E_a(hg) E_b(hg) \, \mathrm{d}h + \sum_s c_s \int_{H_k \backslash H_{\mathbb{A}}} E_s(hg) \, \mathrm{d}h. \tag{8}$$

**The “main” part.** For the $E_a E_b$ summand, we have, as before,

$$\int_{H_k \backslash H_{\mathbb{A}}} E_a(hg) E_b(hg) \, \mathrm{d}h = \sum_F (g \cdot E_a, F)_H (g \cdot E_b, \overline{F})_H. \tag{9}$$

We remark that

$$\begin{aligned}(g \cdot E_s, F)_H &= \int_{H_k \backslash H_{\mathbb{A}}} E_s(hg) \overline{F}(h) \, \mathrm{d}h = \int_{\Theta_k \backslash H_{\mathbb{A}}} \varphi_s(hg) \overline{F}(h) \, \mathrm{d}h \\ &= \int_{\Theta_{\mathbb{A}} \backslash H_{\mathbb{A}}} \varphi_s(hg) \int_{\Theta_k \backslash \Theta_{\mathbb{A}}} \overline{F}(\theta h) \, \mathrm{d}\theta \, \mathrm{d}h.\end{aligned}$$

The function

$$F_\Theta(h) = \int_{\Theta_k\backslash\Theta_{\mathbb{A}}} F(\theta h)\,\mathrm{d}\theta$$

is a spherical vector in $\operatorname{Ind}_\Theta^H 1$, normalized by $F_\Theta(1) = (F)_\Theta$. Therefore, with $\eta$ a spherical vector normalized by $\eta(1) = 1$, we obtain

$$(g\cdot E_s, F)_H = \int_{\Theta_{\mathbb{A}}\backslash H_{\mathbb{A}}} \varphi_s(hg)\overline{F}_\Theta(h)\,\mathrm{d}h = (\overline{F})_\Theta \cdot \int_{\Theta_{\mathbb{A}}\backslash H_{\mathbb{A}}} \varphi_s(hg)\overline{\eta}(h)\,\mathrm{d}h.$$

Recalling that $g_v \in H_v\cdot K_v$ for non-archimedean $v$, we see that all but the archimedean factor are independent of $g$ and

$$\begin{aligned}&(g\cdot E_s, F)_H\\ &\quad= \left(\int_{\Theta_\infty\backslash H_\infty} \varphi_{s,\infty}(hg_\infty)\overline{F}_\Theta(h)\,\mathrm{d}h\right)\cdot\prod_{v<\infty}\left(\int_{\Theta_v\backslash H_v}\varphi_{s,v}(h)\overline{F}_\Theta(h)\,\mathrm{d}h\right).\end{aligned}$$

We abbreviate this as follows:

$$\begin{aligned}\psi_{s,F}(g_\infty) &= \int_{\Theta_\infty\backslash H_\infty}\varphi_{s,\infty}(hg_\infty)\overline{F}_\Theta(h)\,\mathrm{d}h;\\ (E_s,F)'_H &= \prod_{v<\infty}\int_{\Theta_v\backslash H_v}\varphi_{s,v}(h)\overline{F}_\Theta(h)\,\mathrm{d}h;\\ (g\cdot E_s,F)_H &= \psi_{s,F}(g_\infty)(E_s,F)'_H.\end{aligned}$$

Combining this with (9), we see that the “main” part of the moment expansion (7) is

$$\sum_F (E_a,F)'_H(E_b,\overline{F})'_H\int_{H_\infty\backslash G_\infty}\psi_{a,F}(g)\psi_{b,\overline{F}}(g)u_\infty(g)\,\mathrm{d}g. \tag{10}$$

Suppress for a moment the $\infty$ indices, and use $G = HP$, with measure $\mathrm{d}(hp) = \mathrm{d}h\,\mathrm{d}p$ and $\mathrm{d}p$ being a *right* Haar measure. We have

$$\int_{H\backslash G}\psi_{a,F}\cdot\psi_{b,\overline{F}}\cdot u = \int_H\int_H\int_P \varphi_a(hp)\overline{F}_\Theta(h)\cdot\varphi_b\big(h'p\big)F_\Theta\big(h'\big)\cdot u(p)\,\mathrm{d}p\,\mathrm{d}h\,\mathrm{d}h'.$$

With a nod to Diaconu and Garrett [9], set

$$X_{a,b}\big(h,h'\big) = \int_P \varphi_a(hp)\varphi_b\big(h'p\big)u_\infty(p)\,\mathrm{d}p$$

and conclude

$$\int_{H\backslash G}\psi_{a,F}\cdot\psi_{b,\overline{F}}\cdot u = \int_{\Theta\backslash H}\int_{\Theta\backslash H}\overline{F}_\Theta(h)F_\Theta\big(h'\big)X_{a,b}\big(h,h'\big)\,\mathrm{d}h\,\mathrm{d}h'.$$

Resuming (10), we see that the “main” part of the moment expansion is

$$\sum_F (E_a,F)'_H(E_b,\overline{F})'_H\int_{\Theta_\infty\backslash H_\infty}\int_{\Theta_\infty\backslash H_\infty}\overline{F}_\Theta(h)F_\Theta\big(h'\big)X_{a,b}\big(h,h'\big)\,\mathrm{d}h\,\mathrm{d}h'. \tag{11}$$

Recalling that $F_\Theta(h) = (F)_\Theta \eta_F(h)$, where $\eta_F$ is a spherical vector in $\mathrm{Ind}_\Theta^H 1$ normalized by $\eta_F(1) = 1$, we can make the periods even more apparent.

**The "singular" part.** For the other summands in (8), we observe that, by Witt's lemma, $P_k \backslash G_k$ is the space of isotropic lines in $k^{n+1}$, on which $H_k$ acts transitively. As $\Theta = H \cap P$, we have $P_k \backslash G_k = \Theta_k \backslash H_k$ and

$$\int_{H_k \backslash H_{\mathbb{A}}} E_s(hg)\,\mathrm{d}h = \int_{H_k \backslash H_{\mathbb{A}}} \sum_{\gamma \in \Theta_k \backslash H_k} \varphi_s(\gamma hg)\,\mathrm{d}h = \int_{\Theta_k \backslash H_{\mathbb{A}}} \varphi_s(hg)\,\mathrm{d}h$$
$$= \mathrm{vol}(\Theta_k \backslash \Theta_{\mathbb{A}}) \int_{\Theta_{\mathbb{A}} \backslash H_{\mathbb{A}}} \varphi_s(hg)\,\mathrm{d}h.$$

Normalizing $\mathrm{vol}(\Theta_k \backslash \Theta_{\mathbb{A}}) = 1$ and recalling that $g_v \in H_v \cdot K_v$ for non-archimedean $v$, we obtain

$$\int_{H_k \backslash H_{\mathbb{A}}} E_s(hg)\,\mathrm{d}h = \left( \int_{\Theta_\infty \backslash H_\infty} \varphi_{s,\infty}(hg_\infty)\,\mathrm{d}h \right) \cdot \prod_{v<\infty} \left( \int_{\Theta_v \backslash H_v} \varphi_{s,v}(h)\,\mathrm{d}h \right)$$
$$= \psi_s(g)(E_s)_H.$$

Additionally, because $u_\infty$ is a solution of $P(\Delta)u_\infty = (\Delta - \lambda)^N u_\infty = \delta_\infty$, we have

$$\int_{H_\infty \backslash G_\infty} \psi_s \cdot u_\infty = \int_{H_\infty \backslash G_\infty} \frac{P(\Delta)\psi_s}{P(\lambda_s)} \cdot u_\infty$$
$$= \int_{H_\infty \backslash G_\infty} \psi_s \cdot \frac{P(\Delta)u_\infty}{P(\lambda_s)} = \int_{H_\infty \backslash G_\infty} \psi_s \cdot \frac{\delta_\infty}{P(\lambda_s)} = \frac{1}{P(\lambda_s)}.$$

Therefore, the "singular" part of the moment expansion (7) becomes

$$\int_{H_\infty \backslash G_\infty} \sum_s c_s (g \cdot E_s, 1)_H = \sum_s c_s (E_s)_H \int_{H_\infty \backslash G_\infty} \psi_s \cdot u_\infty \tag{12}$$
$$= \sum_s c_s \frac{(E_s)_H}{P(\lambda_s)}.$$

Combining this with the "main" part (11), we obtain the complete moment expansion:

Proposition 6. *Let*

$$\mathcal{F} = E_a E_b + \sum_s c_s E_s$$

*be the regularized expression* (*with finitely many $s$ occurring*) *for $E_a E_b$, and*

$$X_{a,b}(h,h') = \int_{P_\infty} \varphi_{a,\infty}(hp)\varphi_{b,\infty}(h'p)u_\infty(p)\,\mathrm{d}p,$$
$$\mathrm{weight}_{a,b,F} = \left|(F)_\Theta\right|^2 \int_{H_\infty} \int_{H_\infty} \overline{\eta}_F(h)\eta_F(h') X_{a,b}(h,h')\,\mathrm{d}h\,\mathrm{d}h',$$

*where $\eta_F$ is a spherical vector in* $\mathrm{Ind}_{\Theta}^{H} 1$ *normalized by* $\eta_F(1) = 1$. *Then the moment expansion of* $\langle \mathcal{F}, \mathrm{Pé} \rangle_G$ *is*

$$\langle \mathcal{F}, \mathrm{Pé} \rangle_G = \sum_F (E_a, F)'_H (E_b, \overline{F})'_H \cdot \mathrm{weight}_{a,b,F} + \sum_s c_s \frac{(E_s)_H}{P(\lambda_s)}.$$

The actual computation of $X_{a,b}$ and $\mathrm{weight}_{a,b,F}$ can get quite involved, as illustrated, for example, in the $\mathrm{GL}(r) \times \mathrm{GL}(r-1)$ case discussed by Diaconu, Garrett and Goldfeld [12].

## Appendix: Regularizing functions not of rapid decay

In the previous section, we needed the spectral expansion of $E_a E_b$, and observed that one difficulty was that the product is not in $L^2(G_k \backslash G_{\mathbb{A}}/K)$. However, it is possible to subtract a linear combination of Eisenstein series (the singular part), so that the difference is an $L^2$ function.

The idea, which I learned from Garrett [15], [16] and he traces to Zagier [41], uses the constant terms of the Eisenstein series to guide the choice of singular terms, so as to assure cancellation of non-$L^2$ terms. We articulate the details in our specific case $G = \mathrm{O}(n,1)$.

We saw in the Introduction that $M = \Theta A$, where $A = \{m_\lambda\} \cong \mathrm{GL}(1)$. We will always write the elements of $P = N\Theta A$ in the form $p = n\theta m_\lambda$. Because $\mathrm{d}n\,\mathrm{d}\theta\,\mathrm{d}(m_\lambda)$ is a right invariant measure, $\mathrm{d}p = \delta_P(m_\lambda)^{-1} \cdot \mathrm{d}n\,\mathrm{d}\theta\,\mathrm{d}(m_\lambda)$ is a *left* invariant measure on $P$. In the same manner, we always write the elements of $G = PK = N\Theta AK$ in the form $g = pk = n\theta m_\lambda k$, in which case $\mathrm{d}g = \mathrm{d}p\,\mathrm{d}k = \delta_P(m_\lambda)^{-1} \cdot \mathrm{d}n\,\mathrm{d}\theta\,\mathrm{d}(m_\lambda)\,\mathrm{d}k$ is a Haar measure on $G$.

Recall now that we can choose a compact $C \subset N_{\mathbb{A}}\Theta_{\mathbb{A}}$ and a real $t_0 > 0$ such that the Siegel set

$$\mathfrak{S} = \big\{ g = n\theta m_\lambda k : n\theta \in C \text{ and } \delta_P(m_\lambda) \geq t_0 \big\}$$

satisfies $G_k \mathfrak{S} = G_{\mathbb{A}}$. We assume such a choice was made.

Supposing

$$f(g) \ll \delta_P(m_\lambda)^{\sigma}$$

for some real $\sigma$, we have

$$\begin{aligned} \int_{G_k \backslash G_{\mathbb{A}}} f \leq \int_{\mathfrak{S}} f &\ll \int_K \int_{t_0}^{\infty} \int_C \big| f(n\theta m_\lambda k) \big| \cdot \delta_P(m_\lambda)^{-1} \cdot \mathrm{d}(n\theta) \frac{\mathrm{d}\lambda}{\lambda}\,\mathrm{d}k \\ &\ll \int_{t_0}^{\infty} \delta_P(m_\lambda)^{\sigma-1} \frac{\mathrm{d}\lambda}{\lambda} = \int_{t_0}^{\infty} |\lambda|^{n(\sigma-1)-1}\,\mathrm{d}\lambda \end{aligned}$$

(in the last step, we used $\delta_P(m_\lambda) = |\lambda|^n$). This last integral converges when $\sigma < 1$. We have thus shown that $f$ is integrable over $G_k \backslash G_{\mathbb{A}}$ provided $\sigma < 1$. For $L^2$ integrability, we need $\sigma < \frac{1}{2}$.

Recall next that a function $f$ on $P_k \backslash G_{\mathbb{A}}$ is of moderate growth if

$$f(g) \ll \delta_P(m_\lambda)^{\sigma} \quad \text{for some } \sigma > 0$$

and of rapid decay if

$$f(g) \ll \delta_P(m_\lambda)^\sigma \quad \text{for all } \sigma < 0.$$

From the discussion above, it is apparent that if $f$ is right $G_k$-invariant and of rapid decay, then it is integrable over $G_k \backslash G_{\mathbb{A}}$.

We also know [31], [37] that, choosing the normalization $\mathrm{vol}(N_k \backslash N_{\mathbb{A}}) = 1$, the constant term of the Eisenstein series is

$$\mathrm{c}E_s(g) = \delta_P(m_\lambda)^s + c_s \cdot \delta_P(m_\lambda)^{1-s},$$

where $c_s$ is the same constant as in the functional equation

$$E_{1-s} = c_{1-s} \cdot E_s.$$

Moreover, it is a standard fact that $f - \mathrm{c}f$ is of rapid decay, so we can write

$$E_s(g) = \delta_P(m_\lambda)^s + c_s \cdot \delta_P(m_\lambda)^{1-s} + \text{fn rapid decay}.$$

We return to the case $E_a \cdot E_b$ with $a, b \in \mathbb{C}$. Clearly,

$$\begin{aligned} E_a(g) \cdot E_b(g) = \delta_P(m_\lambda)^{a+b} &+ c_a \cdot \delta_P(m_\lambda)^{1-a+b} \\ &+ c_b \cdot \delta_P(m_\lambda)^{a+1-b} + c_a \cdot c_b \cdot \delta_P(m_\lambda)^{1-a+1-b} \\ &+ \text{fn rapid decay}. \end{aligned}$$

As we know that exponents less than $\frac{1}{2}$ assure $L^2$ integrability, we usually can say more.

For example, if $\mathrm{Re}\, a > 1$ and $\mathrm{Re}\, b = \frac{1}{2}$,

$$E_a(g) \cdot E_b(g) = \delta_P(m_\lambda)^{a+b} + c_b \cdot \delta_P(m_\lambda)^{a+1-b} + L^2 \text{ function}.$$

Moreover,

$$\begin{aligned} E_{a+b}(g) &= \delta_P(m_\lambda)^{a+b} + c_{a+b} \cdot \delta_P(m_\lambda)^{1-a-b} + \text{fn rapid decay} \\ &= \delta_P(m_\lambda)^{a+b} + L^2 \text{ function}. \end{aligned}$$

In the same manner,

$$\begin{aligned} E_{a+1-b}(g) &= \delta_P(m_\lambda)^{a+1-b} + c_{a+1-b} \cdot \delta_P(m_\lambda)^{-a+b} + \text{fn rapid decay} \\ &= \delta_P(m_\lambda)^{a+1-b} + L^2 \text{ function}. \end{aligned}$$

Therefore,

$$E_a \cdot E_b - E_{a+b} - c_b \cdot E_{a+1-b} = L^2 \text{ function}.$$

We may well have more than two singular terms. For example, if $\mathrm{Re}\, a = \mathrm{Re}\, b = \frac{1}{2}$, we obtain:

$$\begin{aligned} E_a(g) \cdot E_b(g) = \delta_P(m_\lambda)^{a+b} &+ c_a \cdot \delta_P(m_\lambda)^{1-a+b} \\ &+ c_b \cdot \delta_P(m_\lambda)^{a+1-b} + c_a \cdot c_b \cdot \delta_P(m_\lambda)^{2-a-b} + L^2 \text{ function}. \end{aligned}$$

Here all exponents have real part equal to 1. But the important point is that if one exponent in

$$E_s(g) = \delta_P(m_\lambda)^s + c_s \cdot \delta_P(m_\lambda)^{1-s} + L^2 \text{ function}$$

has real part greater than $\frac{1}{2}$, the other one will have it less than $\frac{1}{2}$. In our case, we have

$$\begin{aligned} E_{a+b}(g) &= \delta_P(m_\lambda)^{a+b} + L^2 \text{ function;} \\ E_{1-a+b}(g) &= \delta_P(m_\lambda)^{1-a+b} + L^2 \text{ function;} \\ E_{a+1-b}(g) &= \delta_P(m_\lambda)^{a+1-b} + L^2 \text{ function;} \\ E_{2-a-b}(g) &= \delta_P(m_\lambda)^{2-a-b} + L^2 \text{ function.} \end{aligned}$$

Therefore,

$$E_a \cdot E_b - E_{a+b} - c_a \cdot E_{1-a+b} - c_b \cdot E_{a+1-b} - c_a \cdot c_b \cdot E_{2-a-b} = L^2 \text{ function.}$$

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