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LEFSCHETZ THEORY ON FIBRE BUNDLES VIA GYSIN HOMOMORPHISM

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Abstract. For a pair of fibre preserving continuous functions $f,g : E_1 \to E_2$ between two compact smooth fibre bundles over B, we construct a transfer map $T(f,g) : H^*(B) \to H^*(B)$ that generalizes Lefschetz number $\lambda_{f,g}$ of the pair of maps. If the pair (f,g) is smooth satisfying a transversality condition and $T(f,g)$ is non-zero, then there is a surjective submersion from any connected component of $\{x \mid f(x) = g(x)\}$ to B. This yields a necessary and sufficient condition for a principal G-bundle over a simply connected compact manifold to be trivial and we also get a necessary condition for every smooth map from S^{2n+1} to S^1 for all $n \geq 1$.

1. Introduction

In [2], classical Lefschetz fixed point theorem [3] is generalized to a pair of continuous maps between compact oriented smooth manifolds via Gysin homomorphism. On the other hand, for a fibre preserving smooth map $f : E \to E$ of a fibre bundle $\pi : E \to B$, where E and B are compact oriented smooth manifolds, Lefschetz number is generalized in [4] as a transfer map by using Poincaré dual in the sense that when $B = \{pt\}$, the transfer map is the multiplication by Lefschetz number. In this manuscript, we combine these two generalizations to construct a grading preserving transfer map $T(f,g) : H^*(B) \to H^*(B)$ for a pair of fibre preserving maps $f,g : E_1 \to E_2$.

In Section 2, we establish relationships between Gysin homomorphism and Poincaré dual of a submanifold. For a pair of continuous maps $f,g : M \to N$ between compact oriented smooth manifolds, we verify a formula for the Lefschetz number $\lambda_{f,g}$ in terms of Poincaré dual of the diagonal Δ in $N \times N$.

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For a fibre preserving map $f: E_1 \to E_2$ between two fibre bundles over B, we give a description of Gysin homomorphism f_* in terms of integrations along the fibers.

In Section 3, we define the transfer map $T(f,g): H^*(B) \to H^*(B)$ for a pair of fibre preserving maps $f,g: E_1 \to E_2$ by using Gysin homomorphism. Our main result describes a formula of $T(f,g)$ in terms of Poincaré dual of the diagonal Δ in $E_2 \oplus E_2$ and as a corollary we show that the set $C = \{x \in E_1 \mid f(x) = g(x)\}$ is non-empty if $T(f,g)$ is a non-zero map. If in addition, the maps f,g are smooth and satisfy a transversality condition, then we show that the restriction of $\pi_1: E_1 \to B$ to any connected component of C is a surjective submersion to B. In Corollary 3.10, we give a necessary and sufficient condition for a principal G-bundle over a simply connected manifold to be trivial where G is a compact Lie group. This yields a necessary condition on every smooth map from $S^{2n+1} \to S^1$ for all $n \geq 1$.

2. Gysin homomorphism and Poicaré duality

Let $f: M^m \to N^n$ be a continuous map between two compact, connected, oriented and smooth manifolds. Then the Gysin homomorphism f_*^p induced by f is the unique homomorphism $f_*^p: H^p(M) \to H^{p+n-m}(N)$ such that $\langle f_*(x), y\rangle = \langle x, f^*(y)\rangle$ where $x \in H^p(M)$ and $y \in H^{m-p}(N)$. In other words, for $x = [\xi] \in H^p(M)$, $f_*([\xi]) = [\omega]$ such that $\int_N \omega \wedge \lambda = \int_M \xi \wedge f^*(\lambda)$ for each $\lambda \in Z^{m-p}(N)$ (see [2]).

In particular, let us consider the inclusion map $i: M \hookrightarrow N$ where M is a closed submanifold of N. Then $\int_N i_*(1_M) \wedge \lambda = \int_M i^*(\lambda)$ for each $\lambda \in Z^m(N)$. On the other hand, the Poincaré dual η_M of M in N is defined by $\int_M i^*(\lambda) = \int_N \lambda \wedge \eta_M$ for each $\lambda \in Z^m(N)$ (see [1]). Therefore we have the following lemma.

Lemma 2.1. *The Poincaré dual of a submanifold M in N is given by the formula*: $i_*(1_M) = (-1)^{m(n-m)}\eta_M$.

Hereafter, we use the same notation for a closed form and its cohomology class. One has the following explicit formulas for Gysin homomorphism and Poincaré dual in terms of dual bases. Let us choose dual bases $\langle b_i^p\rangle$ for $H^p(M)$, and $\langle \hat{b}_j^{m-p}\rangle$ for $H^{m-p}(M)$ via Poincaré duality, i.e., $\int_M b_i \wedge \hat{b}_j = \delta_{ij}$. Similarly let $\langle c_k^{p+n-m}\rangle$ and $\langle \hat{c}_l^{m-p}\rangle$ be dual bases for $H^{p+n-m}(N)$ and $H^{m-p}(N)$, respectively.

Lemma 2.2. *Let $f_{m-p}^*: H^{m-p}(N) \to H^{m-p}(M)$ induced by $f: M \to N$ be written with respect to above bases by $f_{m-p}^*(\hat{c}_l^{m-p}) = \sum_j \hat{f}_{lj}^{m-p}\hat{b}_j^{m-p}$. Then $f_*(b_i^p) = \sum_k \hat{f}_{ki}^{m-p} c_k^{p+n-m}$.*

Proof. Let $f_*(b_i^p) = \sum_k \alpha_{ik}^p c_k^{p+n-m}$. Then

$$\begin{aligned}\alpha_{il}^p &= \sum_k \alpha_{ik}^p \delta_{kl} = \sum_k \alpha_{ik}^p \int_N c_k^{p+n-m} \wedge \hat{c}_l^{m-p} \\ &= \int_N f_*\big(b_i^p\big) \wedge \hat{c}_l^{m-p} = \int_M b_i^p \wedge f^*\big(\hat{c}_l^{m-p}\big) \\ &= \sum_j \hat{f}_{lj}^{m-p} \int_M b_i^p \wedge \hat{b}_j^{m-p} = \hat{f}_{li}^{m-p}.\end{aligned}$$

□

COROLLARY 2.3. *The Poincaré dual η_M is given with respect to above bases by the formula $\eta_M = (-1)^{m(n-m)} i_*(1_M) = (-1)^{m(n-m)} \sum_k \hat{i}_k c_k^{n-m}$ where $\hat{i}_k$ are coefficients of the restriction map $i^* : H^m(N) \to H^m(M)$ with respect to dual bases, i.e., $i^*(\hat{c}_k^m) = \hat{i}_k \omega$ where $H^m(M) = \langle \omega \rangle$.*

DEFINITION 2.4. Let $f, g : M \to N$ be two continuous maps between compact smooth oriented manifolds of same dimension. Then the Lefschetz number of the pair (f, g) is defined as (see [2])

$$\lambda_{f,g} = \sum_{i=0}^{n} (-1)^i \operatorname{Tr}(f_i^* \circ g_*^i).$$

Consider the map $G : M \to N \times N$ given by $G(x) = (f(x), g(x))$.

PROPOSITION 2.5. *The Lefschetz number $\lambda_{f,g}$ of the pair (f, g) is given by the formula $\lambda_{f,g} = \int_M G^*(\eta_\Delta^{N\times N})$ where $\eta_\Delta^{N\times N}$ is the Poincaré dual of the diagonal Δ in $N \times N$.*

Proof. Now

$$\begin{aligned}\lambda_{f,g} &= \big\langle \Delta^* G_*(1_M), 1_N \big\rangle \quad \text{(see p. 295 in [2])} \\ &= \int_N \Delta^* G_*(1_M) = \int_{N\times N} G_*(1_M) \wedge \eta_\Delta^{N\times N} \\ &= \int_M G^*\big(\eta_\Delta^{N\times N}\big).\end{aligned}$$

□

Let $\pi : E^{n+r} \to B^n$ be a fibre bundle over B where both E and B are compact connected oriented smooth manifolds. Then the integration along the fibre is the map $\pi_! : H^{p+r}(E) \to H^p(B)$ (for $0 \le p \le n$) given by $\int_E \pi^*\eta \wedge \omega = \int_B \eta \wedge \pi_!\omega$ for each $\eta \in H^{n-p}(B)$.

PROPOSITION 2.6. *Let $\pi_1 : E_1^{n+r} \to B^n$ and $\pi_2 : E_2^{n+s} \to B^n$ be two fibre bundles and $f : E_1 \to E_2$ be a continuous map. Then the Gysin homomorphism $f_* : H^p(E_1) \to H^{p+s-r}(E_2)$ is uniquely given by*

$$(\pi_2)_!\big[f_*(\xi) \wedge \lambda\big] = (\pi_1)_!\big[\xi \wedge f^*(\lambda)\big]$$

for each $\lambda \in H^{n+r-p}(E_2)$.

Proof. We have

$$\begin{aligned} &(\pi_2)_!\big[f_*(\xi)\wedge\lambda\big] = (\pi_1)_!\big[\xi\wedge f^*(\lambda)\big] \\ \iff\quad & \int_B (\pi_2)_!\big[f_*(\xi)\wedge\lambda\big] = \int_B (\pi_1)_!\big[\xi\wedge f^*(\lambda)\big] \\ \iff\quad & \int_{E_2} f_*(\xi)\wedge\lambda = \int_{E_1} \xi\wedge f^*(\lambda). \end{aligned}$$

□

Remark 2.7. One can check that for a commutative diagram of fibre bundles

$$\begin{tikzcd} E_1^{n+r} \arrow[r, "f"] \arrow[d, "\pi_1"] & E_2^{m+s} \arrow[d, "\pi_2"] \\ B_1^n \arrow[r, "h"] & B_2^m \end{tikzcd}$$

we have, $h_*(\pi_1)_![\xi] = (-1)^{(n-p)(s+r)}(\pi_2)_! f_*[\xi]$.

3. Transfer map

Let $E_1^{n+r} \xrightarrow{\pi_1} B^n$ and $E_2^{n+s} \xrightarrow{\pi_2} B^n$ be two fibre bundles of fibre dimensions r and s, respectively. Let

$$E_2 \oplus E_2 = \{(u,v)\in E_2\times E_2 \mid \pi_2(u) = \pi_2(v)\}.$$

Then $E_2\oplus E_2$ is a $(n+2s)$-dimensional submanifold of $E_2\times E_2$ and $\tilde\pi_2 : E_2\oplus E_2 \to B$, defined by $\tilde\pi_2(u,v) = \pi_2(u) = \pi_2(v)$, is a fibre bundle. Let $f,g : E_1 \to E_2$ be two fibre preserving continuous maps. Let $G : E_1 \to E_2\oplus E_2$ be the map $G(u) = (f(u), g(u))$.

Definition 3.1. The transfer map $T(f,g) : H^*(B) \to H^{*+s-r}(B)$ is defined as the composition,

$$\begin{aligned} H^*(B) \xrightarrow{\pi_1^*} H^*(E_1) \xrightarrow{G_*} H^{*+2s-r}(E_2\oplus E_2) \\ \xrightarrow{\Delta^*} H^{*+2s-r}(E_2) \xrightarrow{(\pi_2)_!} H^{*+s-r}(B). \end{aligned}$$

Theorem 3.2. *For each $\alpha\in H^*(B)$, we have*

$$T(f,g)(\alpha) = \alpha\wedge(\pi_1)_!\big[G^*\big(\eta_\Delta^{E_2\oplus E_2}\big)\big],$$

where $\eta_\Delta^{E_2\oplus E_2}\in\Omega^s(E_2\oplus E_2)$ is the Poincaré dual of the diagonal Δ in $E_2\oplus E_2$.

Proof. For each $\alpha\in H^p(B)$ and $\mu\in H^{n-p-s+r}(B)$, we have

$$\begin{aligned} &\int_B \mu\wedge T(f,g)(\alpha) \\ &\quad = \int_B \mu\wedge(\pi_2)_!\big[\Delta^* G_*\pi_1^*(\alpha)\big] = \int_{E_2} \pi_2^*(\mu)\wedge\Delta^* G_*\pi_1^*(\alpha) \end{aligned}$$

$$
\begin{aligned}
&= \int_{E_2} \Delta^* \tilde{\pi}_2^*(\mu) \wedge \Delta^* G_* \pi_1^*(\alpha) = \int_{E_2} \Delta^* \left[\tilde{\pi}_2^*(\mu) \wedge G_* \pi_1^*(\alpha)\right] \\
&= \int_{E_2 \oplus E_2} \tilde{\pi}_2^*(\mu) \wedge G_* \pi_1^*(\alpha) \wedge \eta_\Delta^{E_2 \oplus E_2} \\
&= \int_B \mu \wedge \left(\tilde{\pi}_2^*\right)_! \left[G_* \pi_1^*(\alpha) \wedge \eta_\Delta^{E_2 \oplus E_2}\right].
\end{aligned}
$$

This implies that

$$
\begin{aligned}
T(f,g)(\alpha) &= \left(\tilde{\pi}_2^*\right)_! \left[G_* \pi_1^*(\alpha) \wedge \eta_\Delta^{E_2 \oplus E_2}\right] \\
&= (\pi_1)_! \left[\pi_1^*(\alpha) \wedge G^*\left(\eta_\Delta^{E_2 \oplus E_2}\right)\right] \quad \text{(by Proposition 2.6)} \\
&= \alpha \wedge (\pi_1)_! \left[G^*\left(\eta_\Delta^{E_2 \oplus E_2}\right)\right]. \qquad \square
\end{aligned}
$$

Corollary 3.3. *In particular,* $T(f,g)(1_B) = (\pi_1)_! [G^*(\eta_\Delta^{E_2 \oplus E_2})]$.

Notice that $(\pi_1)_! [G^*(\eta_\Delta^{E_2 \oplus E_2})] \in H^{s-r}(B)$. So the fiber dimensions $s \geq r$ is a necessary condition for the map $T(f,g)$ to be non-zero.

Corollary 3.4. *If $T(f,g)$ is a non-zero map, then the set*

$$C = G^{-1}(\Delta) = \left\{x \in E_1 \mid f(x) = g(x)\right\}$$

is non-empty.

We now consider the transversality condition for fibre preserving maps. For a fibre bundle $\pi : E \to B$, let $T_v E = \operatorname{Ker}(d\pi)$ be the vertical tangent bundle of E.

Definition 3.5. Let $f, g : E_1 \to E_2$ be any two fiber preserving smooth maps. The smooth map $G = (f,g) : E_1 \to E_2 \oplus E_2$ is said to be in fibrewise transversal with the diagonal $\Delta \subseteq E_2 \oplus E_2$ if

$$(dG)_x (T_v E_1)_x + (T_v \Delta)_{(y,y)} = T_v (E_2 \oplus E_2)_{(y,y)}$$

for each $x \in C = G^{-1}(\Delta)$ and $y = f(x) = g(x)$. In this case, we say that the maps f and g are fibrewise transversal to each other, or in consistent with classical notation, one may call the pair of fibrewise transversal maps (f,g) as Lefschetz fibration pair of maps.

Proposition 3.6. *A pair (f,g) is a Lefschetz fibration pair if and only if*

$$(df)_x - (dg)_x : (T_v E_1)_x \to (T_v E_2)_y$$

is an epimorphism for all $x \in C$.

Proof. Suppose that (f,g) is Lefschetz fibration pair. Let $\omega \in (T_v E_2)_y$. Now $(w,0) \in T_v(E_2 \oplus E_2)_{(y,y)}$ and hence $(w,0) = dG_x(u) + (\alpha,\alpha)$ for some $u \in (T_v E_1)_x$ and $\alpha \in (T_v E_2)_y$. Then one can check that $df_x(u) - dg_x(u) = w$.

Conversely suppose that $df_x - dg_x$ maps $(T_v E_1)_x$ onto $(T_v E_2)_y$ for each $x \in C$. Let $(w_1, w_2) \in T_v(E_2 \oplus E_2)_{(y,y)}$. Then $w_1 = df_x(u_1) - dg_x(u_1)$

and $w_2 = df_x(u_2) - dg_x(u_2)$ for some $u_1, u_2 \in (T_vE_1)_x$. One can check that $(w_1, w_2) = dG_x(u_1 - u_2) + (df_x(u_2) - dg_x(u_1), df_x(u_2) - dg_x(u_1)) \in dG_x(T_vE_1)_x + (T\Delta)_{(y,y)}$. □

Notice that if C is non-empty, we must have $r \geq s$ in order to satisfy the fibrewise transversality condition.

PROPOSITION 3.7. *If $f, g : E_1 \to E_2$ is a Lefschetz fibration pair, then $G = (f, g) : E_1 \to E_2 \oplus E_2$ is transversal to the diagonal Δ in the classical sense.*

Proof. Let $w \in T(E_2 \oplus E_2)_{(y,y)}$ and suppose that $(d\tilde{\pi}_2)_y(w) = \eta \neq 0$. Since $(d\pi_1)_x$ is a submersion, there exists $\alpha \in (TE_1)_x$ such that $(d\pi_1)_x(\alpha) = \eta$. Now $(d\tilde{\pi}_2)_y[w - dG_x(\alpha)] = \eta - \eta = 0$ implies that $w - dG_x(\alpha) \in T_v(E_2 \oplus E_2)_{(y,y)} = dG_x(T_vE_1)_x + (T_v\Delta)_{(y,y)}$. Hence, $w \in dG_x(TE_1)_x + (T_v\Delta)_{(y,y)}$. □

EXAMPLE 3.8. Consider the trivial bundle $S^1 \times S^1 \to S^1$ and the fibre preserving maps $f, g : S^1 \times S^1 \to S^1 \times S^1$ given by $f(z, w) = (z, \frac{z}{w})$ and $g(z, w) = (z, zw)$. One can see that f and g are fibrewise transversal to each other with $C = \{(z, \pm 1) \mid z \in S^1\}$. The map f is fibrewise transversal to the identity map with $C = \{(e^{i\theta}, \pm e^{i(\theta/2)}) \mid e^{i\theta} \in S^1\}$. On the other hand, the map g is not fibrewise transversal to the identity map, even though (g, id) is transversal to $\Delta \subseteq (S^1 \times S^1) \oplus (S^1 \times S^1)$ in the classical sense with $C = \{(1, w) \mid w \in S^1\}$.

Now one can generalize Theorem 4.18 and Corollary 4.20 of [4] as follows.

THEOREM 3.9. *If $(f, g) : E_1^{n+r} \to E_2^{n+s}$ is a Lefschetz fibration pair of maps and $C = G^{-1}(\Delta) = \{x \in E_1 \mid f(x) = g(x)\}$ is non-empty, then the restriction of π_1 to any connected component of C is a surjective submersion.*

Proof. If N is a non-empty connected component of C, then N is a connected compact $(n + r - s)$-dimensional submanifold of E_1 by transversality condition. Consider $(d\pi_1)_x : (TN)_x \to (TB)_b$ where $b = \pi_1(x)$. If $\alpha \in (TN)_x \cap (T_vE_1)_x$, then $f \equiv g$ on N implies that $df_x(\alpha) = dg_x(\alpha)$ and hence $\alpha \in \mathrm{Ker}(df_x - dg_x) \cap (T_vE_1)_x$. Since $df_x - dg_x$ maps $(T_vE_1)_x$ onto $(T_vE_2)_y$, dimension of $\mathrm{Ker}(df_x - dg_x) \cap (T_vE_1)_x$ is equal to $r - s$. So the dimension of $(T_vE_1)_x \cap (TN)_x \leq r - s$ and this implies that the restriction of $(d\pi_1)_x$ to $(TN)_x$ is a submersion, because the dimension of N is $n + r - s$. Since both N and B are compact and connected, the restriction of π_1 to N is surjective. □

COROLLARY 3.10. *Let G be a compact Lie group and let $G \to E_1 \xrightarrow{\pi} B$ be a principal G-bundle over a simply connected compact manifold B. Let $E_2 \to B$ be any compact fiber bundle of same dimension. If there exists a Lefschetz fibration pair $(f, g) : E_1 \to E_2$ such that $C = \{x \in E \mid f(x) = g(x)\}$ is non-empty, then $E_1 \xrightarrow{\pi} B$ is a trivial bundle.*

Proof. If N is a connected component of C, then $\pi : N \to B$ is a surjective local diffeomorphism and hence is a covering space. Now B is simply connected implies that $\pi : N \to B$ is a diffeomorphism and so there exists a section for $\pi : E \to B$. □

Recall that the transfer map $T(f,g)$ is non-zero is a sufficient condition for the set $C = \{x \in E \mid f(x) = g(x)\}$ to be nonempty.

COROLLARY 3.11. *Let G be a compact Lie group and $G \to E \xrightarrow{\pi} B$ be a principal G-bundle over a simply connected compact smooth manifold B. Then $G \to E \to B$ is a trivial bundle if and only if there exists a smooth map $\gamma : E \to G$ such that $d\gamma_x$ maps $(T_vE)_x$ isomorphically onto $(TG)_{\gamma(x)}$ for each $x \in E$.*

Proof. For a fixed $g \in \operatorname{Im}\gamma$, choose the constant map $\tau : E \to G$ such that $\tau(E) = \{g\}$. Consider the fibre preserving maps $f, g : E \to B \times G$ defined by $f(x) = (\pi(x), \gamma(x))$ and $g(x) = (\pi(x), \tau(x))$. Then the set $C = \{x \in E \mid f(x) = g(x)\}$ is non-empty. For each $x \in C$ and $u \in (T_vE)_x$, one can see that $df_x(u) - dg_x(u) = (0, (d\gamma)_x(u))$ and hence (f,g) is a Lefschetz fibration pair by Proposition 3.6. Now the result follows from Corollary 3.10. □

EXAMPLE 3.12. If B is not a simply connected manifold in the previous corollary, then we have the following example. Consider the bundle $S^1 \to U(2) \to SO(3) \cong \mathbb{R}P^3$. Every element of $U(2)$ is of the form

$$M_{\alpha\beta\sigma} = \begin{pmatrix} \cos\theta e^{i\alpha} & \sin\theta e^{i\sigma} \\ -\sin\theta e^{i(\beta-\sigma)} & \cos\theta e^{i(\beta-\alpha)} \end{pmatrix}.$$

One has the map $\gamma : U(2) \to S^1$ given by $\gamma(M_{\alpha\beta\sigma}) = e^{i\alpha} \in S^1$, but $S^1 \to U(2) \to SO(3) \cong \mathbb{R}P^3$ is a nontrivial bundle, indeed topologically, $U(2) \cong S^3 \times S^1$.

Consider $S^{2n+1} = \{x = (w_1, \ldots, w_{n+1}) \in \mathbb{C}^{n+1} \mid \sum_k |w_k|^2 = 1\}$ and the Hopf fibration $S^1 \to S^{2n+1} \to \mathbb{C}P^n$. For any given smooth map $\gamma : S^{2n+1} \to S^1$ and $z_0 \in \operatorname{Im}\gamma$, choose $x \in \gamma^{-1}(z_0)$. Consider the map $\gamma_x : S^1 \to S^1$ defined by $\gamma_x(e^{i\theta}) = \gamma(xe^{i\theta})$. One can easily calculate that $(d\gamma_x)_{e_0}(1) = (d\gamma)_x(xi)$ where e_0 is the unit of S^1. Then above corollary immediately yields the following corollary.

COROLLARY 3.13. *For each smooth map $\gamma : S^{2n+1} \to S^1$ and $z_0 \in \operatorname{Im}\gamma$, there exists an element $x \in \gamma^{-1}(z_0)$ such that $(d\gamma)_x(xi) = 0$.*

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