

THE WEAK LEFSCHETZ PROPERTY FOR \mathfrak{m} -FULL IDEALS AND COMPONENTWISE LINEAR IDEALS

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ABSTRACT. We give a necessary and sufficient condition for a standard graded Artinian ring of the form $K[x_1, \dots, x_n]/I$, where I is an \mathfrak{m} -full ideal, to have the weak Lefschetz property in terms of graded Betti numbers. This is a generalization of a theorem of Wiebe for componentwise linear ideals. We also prove that the class of componentwise linear ideals and that of completely \mathfrak{m} -full ideals coincide in characteristic zero and in positive characteristic, with the assumption that $\text{Gin}(I)$ w.r.t. the graded reverse lexicographic order is stable.

1. Introduction

In [12] Wiebe gave a necessary and sufficient condition, among other things, for a componentwise linear ideal to have the weak Lefschetz property in terms of the graded Betti-numbers. His result says that the following conditions are equivalent for a componentwise linear ideal I in the polynomial ring R .

- (i) R/I has the weak Lefschetz property.
- (ii) $\beta_{n-1, n-1+j}(I) = \beta_{0,j}(I)$ for all $j > d$.
- (iii) $\beta_{i, i+j}(I) = \binom{n-1}{i} \beta_{0,j}(I)$ for all $j > d$ and all $i \geq 0$.

Here d is the minimum of all j with $\beta_{n-1, n-1+j}(I) > 0$. It seems to be an interesting problem to consider if there exist classes of ideals other than componentwise linear ideals for which these three conditions are equivalent.

Recently Conca, De Negri and Rossi [2] obtained a result which says that componentwise linear ideals are \mathfrak{m} -full. Bearing this in mind, one might expect that for \mathfrak{m} -full ideals these three conditions might be equivalent. The outcome is not quite what the authors had expected; if I is \mathfrak{m} -full, then although (i)

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and (iii) are equivalent, the condition (ii) has to be strengthened slightly. This is stated in Theorem 10 below. What is interesting is that if we assume that both I and $I + (x)/(x)$ are \mathfrak{m} -full, where x is a general linear form, then it turns out that these three conditions are precisely equivalent. This is the first main result in this paper and is stated in Theorem 11. As a corollary we get that for completely \mathfrak{m} -full ideals the conditions (i)–(iii) are equivalent.

In the above cited paper [2] the authors in fact proved, without stating it, that if I is componentwise linear, it is completely \mathfrak{m} -full. We supply a new proof in Proposition 18, using some results of Conca–De Negri–Rossi ([2], Propositions 2.8 and 2.11). Hence one sees that Theorem 11 is a generalization of the theorem of Wiebe ([12], Theorem 3.1). Proposition 18 suggests investigating whether all completely \mathfrak{m} -full ideals should be componentwise linear. In Theorem 20, we prove that it is indeed true with the assumption that the generic initial ideal with respect to the graded reverse lexicographic order is stable. This is the second main theorem in this paper. Thus we have a somewhat striking fact that the class of completely \mathfrak{m} -full ideals and that of componentwise linear ideals coincide at least in characteristic zero, since the additional assumption is automatically satisfied in characteristic zero. We have been unable to prove it without the assumption that the generic initial ideal is stable, but we conjecture it is true. In Example 13 we provide a rather trivial example which shows that Theorem 11 is not contained in Wiebe’s result.

In Section 2, we give some remarks on a minimal generating set of an \mathfrak{m} -full ideal, and also review a result on graded Betti numbers obtained in [10]. These are needed for our proof of the main theorems. In Sections 3 and 4, we will prove the main theorems.

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Throughout this paper, we let $R = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over an infinite field K with the standard grading, and $\mathfrak{m} = (x_1, \dots, x_n)$ the homogeneous maximal ideal. All the ideals we consider are homogeneous.

2. Some properties of \mathfrak{m} -full ideals

We quickly review some basic properties of \mathfrak{m} -full ideals, which were mostly obtained in [10], and fix notation which we use throughout the paper. We start with a definition.

DEFINITION 1 ([9], Definition 4). An ideal I of R is said to be \mathfrak{m} -full if there exists an element x in R such that $\mathfrak{m}I : x = I$.

\mathfrak{m} -full ideals are studied in [2], [3], [4], [7], [9], [10] and [11].

NOTATION AND REMARK 2. (1) Suppose that I is an \mathfrak{m} -full ideal of R . Then the equality $\mathfrak{m}I : x = I$ holds for a general linear form x in R ([9], Remark 2(i)). Moreover, it is easy to see that, for any $x \in R$, if $\mathfrak{m}I : x = I$, then it implies that $I : \mathfrak{m} = I : x$. Let y_1, \dots, y_l be homogeneous elements in $I : \mathfrak{m}$ such that $\{\overline{y_1}, \dots, \overline{y_l}\}$ is a minimal generating set for $(I : \mathfrak{m})/I$, where $\overline{y_i}$ is the image of y_i in R/I . Then Proposition 2.2 in [4] implies that $\{xy_1, \dots, xy_l\}$ is a part of a minimal generating set of I . Write a minimal generating set of I as

$$xy_1, \dots, xy_l, z_1, \dots, z_m.$$

(2) Suppose that I is an \mathfrak{m} -primary \mathfrak{m} -full ideal of R . The socle of R/I is the ideal of R/I annihilated by the maximal ideal $\overline{\mathfrak{m}} = (\overline{x_1}, \dots, \overline{x_n})$. Hence

$$\text{Soc}(R/I) = \{a \in R/I \mid a\overline{\mathfrak{m}} = 0\} = \bigoplus_j \text{Soc}(R/I)_j \cong (I : \mathfrak{m})/I.$$

The socle degree of R/I is the maximum of integer j with $\text{Soc}(R/I)_j \neq 0$. We note that

$$\max\{\deg(xy_i) \mid i = 1, \dots, l\} = c + 1,$$

where c is the socle degree of R/I .

(3) Let x be a general linear form in R . Let δ be the minimum integer j with $(R/(I + xR))_j = 0$ and let d be the least of integers $\{\deg(xy_i) \mid i = 1, \dots, l\}$. Then $\delta - 1$ is equal to the socle degree of $R/(I + xR)$ and $d - 1$ is equal to the initial degree of $\text{Soc}(R/I)$, i.e., $d - 1 = \min\{j \mid \text{Soc}(R/I)_j \neq 0\}$. Note that δ is independent of x provided that it is sufficiently general. This is discussed in Remark 3 below.

(4) Let $\beta_{i,j}(I)$ be the (i, j) th graded Betti number of I as an R -module. Then we have $d = \min\{j \mid \beta_{n-1, n-1+j}(I) > 0\}$, as $\dim_K \text{Soc}(R/I)_j = \beta_{n-1, n+j}(I)$ ([12], Fact 3.3).

REMARK 3. Let δ be the integer defined in Remark 2(3). In this remark we want to prove that δ is independent of x as long as it is sufficiently general.

Let I be an \mathfrak{m} -primary ideal in $R = K[x_1, \dots, x_n]$ and set $A = R/I$. Let ξ_1, \dots, ξ_n be indeterminates over K . Let $K(\xi) = K(\xi_1, \dots, \xi_n)$ be the rational function field over K and put $A(\xi) = K(\xi) \otimes_K A$. It is easy to see that both $A(\xi)$ and A have the same Hilbert function. Put $Y = \xi_1 \overline{x_1} + \dots + \xi_n \overline{x_n} \in A(\xi)$. It is proved, in a more general setup in [9, Theorem A], that

$$(1) \quad \text{length}(A(\xi)/YA(\xi)) \leq \text{length}(A/yA)$$

for any linear form y of A and

$$(2) \quad \text{length}(A(\xi)/YA(\xi)) = \text{length}(A/yA)$$

for any sufficiently general linear form y of A

We show that $H(A(\xi)/YA(\xi), i) \leq H(A/yA, i)$ for every i , where $H(*, i)$ is the Hilbert function of a graded algebra. Let $A(\xi)_i$ be the homogeneous

part of $A(\xi)$ of degree i . Choose a homogeneous basis $\{v_\lambda\}$ for A as a vector space over K . Then $\{1 \otimes v_\lambda\}$ is a homogeneous basis for $A(\xi)$ over $K(\xi)$. We fix any pair of such bases and suppose that we write the homomorphisms $\times Y : A(\xi)_i \rightarrow A(\xi)_{i+1}$ and $\times y : A(\xi)_i \rightarrow A(\xi)_{i+1}$ as matrices M_i and N_i respectively over these bases. It is easy to see that the entries of M_i are homogeneous linear forms in ξ_1, \dots, ξ_n , and N_i is obtained from M_i by substituting (ξ_j) for (a_j) , where $y = a_1x_1 + \dots + a_nx_n$ with $a_i \in A$. Thus $\text{rank } M_i \geq \text{rank } N_i$ and consequently

$$(3) \quad H(A(\xi)/YA(\xi), i) \leq H(A(\xi)/yA(\xi), i) = H(A/yA, i)$$

for every i and every linear form $y \in A$. By (2) and (3), we see that both $A(\xi)/YA(\xi)$ and A/yA have the same Hilbert function if $y \in A$ is a sufficiently general linear form. This shows that δ is independent of a choice of sufficiently general linear form $y \in A$.

LEMMA 4. *Let I be an \mathfrak{m} -primary \mathfrak{m} -full ideal of R . Let x, z_1, \dots, z_m and δ be as in Notation 2. Let $\overline{z_i}$ be the image of z_i in R/xR and \overline{I} the image of I in R/xR . Then we have:*

- (1) $\{\overline{z_1}, \dots, \overline{z_m}\}$ is a minimal generating set of \overline{I} .
- (2) $\text{deg } z_i \leq \delta$ for $i = 1, 2, \dots, m$.
- (3) If \overline{I} is an \mathfrak{m} -full ideal of R/xR , then $\text{deg } z_i = \delta$ for some i .

Proof. (1) Suppose that $z_1 \in (z_2, \dots, z_m, x)$. Then $z_1 = f_2z_2 + \dots + f_mz_m + f_{m+1}x$ for some $f_i \in R$. Since $xf_{m+1} = z_1 - (f_2z_2 + \dots + f_mz_m) \in I$, we have

$$f_{m+1} \in I : x = I : \mathfrak{m} = (y_1, \dots, y_l, z_1, \dots, z_m).$$

Hence,

$$f_{m+1} = g_1y_1 + \dots + g_ly_l + h_1z_1 + \dots + h_mz_m$$

for some $g_i, h_j \in R$. Thus we obtain

$$z_1 - xh_1z_1 = (f_2 + xh_2)z_2 + \dots + (f_m + xh_m)z_m + g_1xy_1 + \dots + g_ly_l,$$

and $z_1 \in (xy_1, \dots, xy_l, z_2, \dots, z_m)$. This is a contradiction.

(2) Since $\delta - 1$ is equal to the socle degree of $R/(I + xR)$, (1) implies that

$$\text{deg } z_i = \text{deg } \overline{z_i} \leq \delta.$$

(3) This is proved by applying Remark 2(2) to the \mathfrak{m} -full ideal \overline{I} of R/xR . □

PROPOSITION 5 ([10], Corollary 8). *Let I be an \mathfrak{m} -full ideal of R (not necessarily \mathfrak{m} -primary) and let x be a general linear form of R satisfying $\mathfrak{m}I : x = I$. With Notation 2(1), let \overline{I} be the image of I in R/xR and let*

$\beta_{i,j}(\bar{I})$ be the (i, j) th graded Betti number of \bar{I} as an R/xR -module. Set $c_j = \#\{i \mid 1 \leq i \leq l, \deg(xy_i) = j\}$ for all j . Then

$$\beta_{i,i+j}(I) = \beta_{i,i+j}(\bar{I}) + \binom{n-1}{i} c_j$$

for all i and all j .

3. \mathfrak{m} -full ideals and the WLP

DEFINITION 6. Let I be an \mathfrak{m} -primary ideal of R and set $A = R/I = \bigoplus_{i=0}^c A_i$. We say that A has the *weak Lefschetz property* (WLP) if there exists a linear form $L \in A_1$ such that the multiplication map $\times L : A_i \rightarrow A_{i+1}$ has full rank for all $0 \leq i \leq c - 1$.

REMARK 7. Let I be an \mathfrak{m} -primary ideal of R and set $A = R/I = \bigoplus_{i=0}^c A_i$.

(1) Let h_0, h_1, \dots, h_c be the Hilbert function of A . Then it is easy to see that the following conditions are equivalent.

- (i) A has the WLP.
- (ii) There exists a linear form $L \in A_1$ such that

$$\dim_K \ker(\times L : A_i \rightarrow A_{i+1}) = \max\{0, h_i - h_{i+1}\}$$

for all $0 \leq i \leq c - 1$.

- (2) If A has the WLP, then for a general linear form L in A , the multiplication map $\times L : A_i \rightarrow A_{i+1}$ has full rank for all $0 \leq i \leq c - 1$.
- (3) When we deal with the WLP of Artinian algebras defined by \mathfrak{m} -full ideals, we need to select a linear form with both properties in the definition of \mathfrak{m} -full ideals and in the definition of WLP. This is possible since either property is satisfied by a sufficiently general linear form.

LEMMA 8. Let I be an \mathfrak{m} -primary \mathfrak{m} -full ideal of R . Then, with δ and d as defined in Notation 2, the following conditions are equivalent.

- (i) R/I has the WLP.
- (ii) $\delta \leq d$.

Proof. Let x be a general linear form in R . Note that δ is equal to the minimum of integers j such that the multiplication map $\times \bar{x} : (R/I)_{j-1} \rightarrow (R/I)_j$ is surjective. Furthermore note that $\times \bar{x} : (R/I)_{j-1} \rightarrow (R/I)_j$ is injective for all $j \leq d - 1$, since $\text{Soc}(R/I) = \ker(\times \bar{x} : (R/I) \rightarrow (R/I))$ and $d - 1$ is equal to the initial degree of $\text{Soc}(R/I)$. Hence (ii) \Rightarrow (i) as is easily seen. Assume that $d < \delta$. Then the map $\times \bar{x} : (R/I)_{d-1} \rightarrow (R/I)_d$ is neither surjective nor injective. Hence R/I does not have the WLP. This shows (i) \Rightarrow (ii). \square

LEMMA 9. Let I be an \mathfrak{m} -full ideal of R and \bar{I} the image of I in R/xR for a general linear form x in R . Let $\beta_{i,j}$ be as in Notation 2. Then the following conditions are equivalent.

- (i) R/I has the WLP.
- (ii) $\beta_{n-2,n-2+j}(\bar{I}) = 0$ for all $j > d$.

Proof. Recall that d is the minimum j such that $\beta_{n-1,n-1+j}(I) > 0$. (See Notation 2(3) and (4).) Since

$$\dim_K \text{Soc}(\bar{R}/\bar{I})_j = \beta_{n-2,n-1+j}(\bar{I})$$

for all j , it follows from Remark 2(3) that

$$\delta - 1 = \max\{j \mid \beta_{n-2,n-1+j}(\bar{I}) > 0\}.$$

Hence we have the equivalence:

$$\delta \leq d \iff \beta_{n-2,n-2+j}(\bar{I}) = 0 \text{ for all } j > d.$$

Thus the assertion follows from Lemma 8. □

Now we state the first theorem.

THEOREM 10. *Let $R = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over an infinite field K and $\mathfrak{m} = (x_1, \dots, x_n)$ the homogeneous maximal ideal. Let I be an \mathfrak{m} -primary \mathfrak{m} -full ideal of R and let d be the minimum of all j with $\beta_{n-1,n-1+j}(I) > 0$. Then the following conditions are equivalent.*

- (i) R/I has the WLP.
- (ii) $\beta_{n-1,n-1+j}(I) = \beta_{0,j}(I)$ and $\beta_{n-2,n-2+j}(I) = (n-1)\beta_{0,j}(I)$, for all $j > d$.
- (iii) $\beta_{i,i+j}(I) = \binom{n-1}{i}\beta_{0,j}(I)$ for all $j > d$ and all i .

Proof. We use Notation 2. Let \bar{I} be the image of I in R/xR . Noting that $\beta_{n-1,n-1+j}(\bar{I}) = 0$ for all j , we have from Proposition 5 that

$$(4) \quad c_j = \beta_{0,j}(I) \text{ for all } j > d \iff \beta_{n-1,n-1+j}(I) = \beta_{0,j}(I) \text{ for all } j > d.$$

(i) \Rightarrow (ii): By Lemmas 8 and 4(2), it follows that

$$(5) \quad c_j = \beta_{0,j}(I)$$

for all $j > d$. Hence we have the first equality by the equivalence (1). Moreover, by Lemma 9 and Proposition 5, it follows that $\beta_{n-2,n-2+j}(I) = (n-1)c_j$ for all $j > d$. Hence we have the second equality by the above equality (2).

(ii) \Rightarrow (i): By our assumption (ii), Proposition 5 and the equivalence (1), we have $\beta_{n-2,n-2+j}(\bar{I}) = 0$ for all $j > d$. Hence (ii) \Rightarrow (i) follows by Lemma 9.

(ii) \Rightarrow (iii): Since $\beta_{n-2,n-2+j}(\bar{I}) = 0$ for all $j > d$, we have $\beta_{i,i+j}(\bar{I}) = 0$ for all i and all $j > d$. Hence, noting that $\beta_{0,j}(I) = c_j$ for all $j > d$, we see that the desired equalities follow from Proposition 5.

(iii) \Rightarrow (ii) is trivial. □

The above theorem can be strengthened as follows.

THEOREM 11. *With the same notation as Theorem 10, suppose that $\mathfrak{m}I : x = I$ and $\bar{I} = (I + xR)/xR$ is \mathfrak{m} -full as an ideal of R/xR for some linear form x in R . Then the following conditions are equivalent.*

- (i) R/I has the WLP.
- (ii) $\beta_{n-1, n-1+j}(I) = \beta_{0,j}(I)$ for all $j > d$.
- (iii) $\beta_{i, i+j}(I) = \binom{n-1}{i} \beta_{0,j}(I)$ for all $j > d$ and all i .

Proof. In view of Theorem 10, it suffices to show the assertion (ii) \Rightarrow (i). We use Notation 2. Recall that $d = \min\{\deg(xy_s) \mid 1 \leq s \leq l\}$, as explained in Remark 2(3) and (4). By Proposition 5, we have $\beta_{n-1, n-1+j}(I) = c_j$ for all j , since $\beta_{n-1, n-1+j}(\bar{I}) = 0$ for all j . Now assume (ii). Then $\beta_{0,j}(I) = c_j$ for all $j > d$. Therefore, we have $\deg(z_t) \leq d$ for all $t = 1, 2, \dots, m$. On the other hand, since we assume that \bar{I} is \mathfrak{m} -full, we have $\max\{\deg(z_t) \mid 1 \leq t \leq m\} = \delta$ by Lemma 4(3). Thus $\delta \leq d$, and R/I has the WLP by Lemma 8. \square

COROLLARY 12. *With the same notation as Theorem 10, let I be an \mathfrak{m} -primary completely \mathfrak{m} -full ideal of R (see Definition 15 in the next section). Then the following conditions are equivalent.*

- (i) R/I has the WLP.
- (ii) $\beta_{n-1, n-1+j}(I) = \beta_{0,j}(I)$ for all $j > d$.
- (iii) $\beta_{i, i+j}(I) = \binom{n-1}{i} \beta_{0,j}(I)$ for all $j > d$ and for all i .

Proof. This follows from Theorem 11. \square

Proposition 18 in the next section says that Theorem 11 is a generalization of Theorem 3.1 in [12] due to Wiebe. Furthermore, the following example shows that Theorem 11 is not contained in Wiebe’s result.

EXAMPLE 13. Let $R = K[w, x, y, z]$ be the polynomial ring in four variables. Let $I = (w^3, x^3, x^2y) + (w, x, y, z)^4$. Then, it is easy to see that I is not componentwise linear, but I and $I + (z)/(z)$ are \mathfrak{m} -full ideals.

Finally in this section, we give an example of \mathfrak{m} -full ideal where conditions (ii) and (iii) of Theorem 11 are not equivalent. In other words, the condition “ $I + gR/gR$ is \mathfrak{m} -full” cannot be dropped in this theorem.

EXAMPLE 14. Let $R = K[x, y, z]$ be the polynomial ring in three variables. Let $I = (x^3, x^2y, x^2z, y^3) + (x, y, z)^4$. Then, it is easy to see that I is \mathfrak{m} -full and $I + (g)/(g)$ is not \mathfrak{m} -full, where g is a general linear form of R . The Hilbert function of R/I is $1 + 3t + 6t^2 + 6t^3$ and that of $R/I + gR$ is $1 + 2t + 3t^2 + t^3$. Thus R/I does not have the WLP. The minimal free resolution of I is:

$$0 \rightarrow R(-5) \oplus R(-6)^6 \rightarrow R(-4)^3 \oplus R(-5)^{13} \rightarrow R(-3)^4 \oplus R(-4)^6 \rightarrow I \rightarrow 0.$$

Note that $d = 3$, $\beta_{2,6} = \beta_{0,4}$ and $\beta_{1,5} > 2\beta_{0,4}$.

4. Complete \mathfrak{m} -fullness and componentwise linearity

DEFINITION 15 ([10], Definition 2). Let $R = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over an infinite field K , and I a homogeneous ideal of R . We define the *completely \mathfrak{m} -full ideals* recursively as follows.

- (1) If $n = 0$ (i.e., if R is a field), then the zero ideal is completely \mathfrak{m} -full.
- (2) If $n > 0$, then I is completely \mathfrak{m} -full if $\mathfrak{m}I : x = I$ and $(I + xR)/xR$ is completely \mathfrak{m} -full as an ideal of R/xR , where x is a general linear form. (The definition makes sense by induction on n .)

DEFINITION 16. A monomial ideal I of $R = K[x_1, \dots, x_n]$ is said to be *stable* if I satisfies the following condition: for each monomial $u \in I$, the monomial $x_i u / x_{m(u)}$ belongs to I for every $i < m(u)$, where $m(u)$ is the largest index j such that x_j divides u .

EXAMPLE 17. Typical examples of completely \mathfrak{m} -full ideals are stable monomial ideals. Let I be a stable monomial ideal of $R = K[x_1, \dots, x_n]$. First we show that $\mathfrak{m}I : x_n = I$. Let $w \in \mathfrak{m}I : x_n$ be a monomial. Since $x_n w \in \mathfrak{m}I$, we have $x_n w = x_i u$ for some x_i and a monomial $u \in I$. Hence $w = x_i u / x_n \in I$, as I is stable. Therefore $\mathfrak{m}I : x_n \subset I$, and other inclusion is clear. Furthermore, since $\bar{I} = (I + x_n R) / x_n R$ is stable in $\bar{R} = R / x_n R$ again, our assertion follows by inductive argument.

PROPOSITION 18. *Every componentwise linear ideal of $R = K[x_1, \dots, x_n]$ is a completely \mathfrak{m} -full ideal.*

Proof. Let $I = \bigoplus_{j \geq 0} I_j$ be a componentwise linear ideal of $R = \bigoplus_{j \geq 0} R_j$ and let $I_{\langle d \rangle} = \bigoplus_{j \geq d} (I_{\langle d \rangle})_j$ be the ideal generated by all homogeneous polynomials of degree d belonging to I . Then it follows by Lemma 8.2.10 in [6] that $I_{\langle d \rangle}$ is a componentwise linear ideal for all d . Recall the result of Conca–De Negri–Rossi [2] that every componentwise linear ideal is \mathfrak{m} -full. Hence I and $I_{\langle d \rangle}$ are \mathfrak{m} -full ideals for all d .

First we show the following: there exists a common linear form x in R such that $\mathfrak{m}I : x = I$ and $\mathfrak{m}I_{\langle d \rangle} : x = I_{\langle d \rangle}$ for all d . Note that there is a positive integer k such that $R_i I_k = I_{k+i}$ for all $i \geq 0$, where $R_i I_k = \{ \sum_{\lambda} r_{\lambda} v_{\lambda} \mid r_{\lambda} \in R_i, v_{\lambda} \in I_k \}$. Write $I_{\langle k \rangle} = \bigoplus_{j \geq k} (I_{\langle k \rangle})_j$. Then we have $I_{\langle k+i \rangle} = \bigoplus_{j \geq k+i} (I_{\langle k \rangle})_j$ for all $i \geq 0$. Hence, it is easy to see that if $\mathfrak{m}I_{\langle k \rangle} : z = I_{\langle k \rangle}$ for some linear form z then $\mathfrak{m}I_{\langle k+i \rangle} : z = I_{\langle k+i \rangle}$ for all $i \geq 0$, as $(I_{\langle k+i \rangle})_j = (I_{\langle k \rangle})_j$ for all $j \geq k+i$. Therefore if x is a sufficiently general linear form, we have both $\mathfrak{m}I : x = I$ and $\mathfrak{m}I_{\langle d \rangle} : x = I_{\langle d \rangle}$ for all d .

To prove this proposition, it suffices to show that $\bar{I} = (I + xR) / xR$ is also a componentwise linear ideal of R/xR , because if so, then \bar{I} is \mathfrak{m} -full by the above result stated in [2]. Therefore our assertion follows by inductive argument. Let $\bar{I}_{\langle d \rangle} = \bigoplus_{j \geq d} (\bar{I}_{\langle d \rangle})_j$ be the ideal generated by all homogeneous polynomials of degree d belonging to \bar{I} . We have to show that $\bar{I}_{\langle d \rangle}$ has a linear resolution for all d . We use the same notation as Proposition 5 for $I_{\langle d \rangle}$. Since $\bar{I}_{\langle d \rangle} = (I_{\langle d \rangle} + xR) / xR$, it follows by Proposition 5 that

$$\beta_{i,i+j}(I_{\langle d \rangle}) = \beta_{i,i+j}(\bar{I}_{\langle d \rangle}) + \binom{n-1}{i} c_j$$

for all i and all j . Hence $\overline{I}_{\langle d \rangle}$ has a linear resolution, as $I_{\langle d \rangle}$ does. This completes the proof. \square

From the preceding proof, we obtain an immediate consequence.

PROPOSITION 19. *Let I be a componentwise linear ideal of $R = K[x_1, \dots, x_n]$. Then $\overline{I} = (I + xR)/xR$ is a componentwise linear ideal of $\overline{R} = R/xR$ for a general linear form x in R .*

We conjecture that a completely \mathfrak{m} -full ideal is componentwise linear. We have already proved that a componentwise linear ideal is completely \mathfrak{m} -full. We prove the converse with the assumption that the generic initial ideal is stable.

THEOREM 20. *Let I be a homogeneous ideal of $R = K[x_1, \dots, x_n]$ and $\text{Gin}(I)$ the generic initial ideal of I with respect to the graded reverse lexicographic order induced by $x_1 > \dots > x_n$. Assume that $\text{Gin}(I)$ is stable. Then I is completely \mathfrak{m} -full if and only if I is componentwise linear.*

Proof. The “if” part is proved in Proposition 18. So we show the “only if” part. Set $J = \text{Gin}(I)$. Since J is stable, it suffices to show that $\beta_0(I) = \beta_0(J)$ by Theorem 2.5 in [8], that is, the minimal number of generator of I coincides with that of J . We use induction on the number n of variables. The case where $n = 1$ is obvious. Let $n \geq 2$. Since the minimal number of generators of $(I : \mathfrak{m})/I$ is equal to the dimension of $(I : \mathfrak{m})/I$ as a K -vector space, it follows by Proposition 5 and Remark 2(1) that

$$\beta_0(I) = \beta_0(\overline{I}) + \dim_K((I : \mathfrak{m})/I),$$

where set $\overline{I} = (I + xR)/xR$ for a general linear form x in R . Similarly, since $\mathfrak{m}J : x_n = J$, J is stable and consequently completely \mathfrak{m} -full by Example 17, we have

$$\beta_0(J) = \beta_0(\overline{J}) + \dim_K((J : \mathfrak{m})/J),$$

where $\overline{J} = (J + x_nR)/x_nR$. First we show that

$$\dim_K((I : \mathfrak{m})/I) = \dim_K((J : \mathfrak{m})/J).$$

From the exact sequence

$$0 \rightarrow (I : x)/I \rightarrow R/I \xrightarrow{\times x} (I + xR)/I \rightarrow 0,$$

it follows that

$$\dim_K((I : x)/I)_j = \dim_K(R/I)_j - \dim_K(R/I)_{j+1} + \dim_K(R/(I + xR))_{j+1}$$

for all j . Similarly, we get

$$\begin{aligned} \dim_K((J : x_n)/J)_j &= \dim_K(R/J)_j - \dim_K(R/J)_{j+1} + \dim_K(R/(J + x_nR))_{j+1} \end{aligned}$$

for all j . Here we recall the well-known facts:

- $\dim_K(R/I)_j = \dim_K(R/J)_j$ for all j .
- $\dim_K(R/(I + xR))_j = \dim_K(R/(J + x_nR))_j$ for all j (Lemma 1.2 in [1]).

Therefore we have

$$\dim_K((I : x)/I)_j = \dim_K((J : x_n)/J)_j$$

for all j . Thus, since I and J are \mathfrak{m} -full, it follows that $I : \mathfrak{m} = I : x$ and $J : \mathfrak{m} = J : x_n$, and hence we see that $\dim_K((I : \mathfrak{m})/I) = \dim_K((J : \mathfrak{m})/J)$. Furthermore, since both \bar{I} and \bar{J} are completely \mathfrak{m} -full again and $\text{Gin}(I + xR) = \text{Gin}(I) + x_nR$ (Corollary 2.15 in [5]), it follows by the inductive assumption that

$$\beta_0(\bar{I}) = \beta_0(\bar{J}).$$

This completes the proof. □

REFERENCES

- [1] A. Conca, *Reduction number and initial ideals*, Proc. Amer. Math. Soc. **131** (2003), 1015–1020. [MR 1948090](#)
- [2] A. Conca, E. De Negri and M. E. Rossi, *Integrally closed and componentwise linear ideals*, Math. Z. **265** (2010), 715–734. [MR 2644317](#)
- [3] S. Goto, *Integral closedness of complete intersection ideals*, J. Algebra **108** (1987), 151–160. [MR 0887198](#)
- [4] S. Goto and F. Hayasaka, *Finite homological dimension and primes associated to integrally closed ideals*, Proc. Amer. Math. Soc. **130** (2002), 3159–3164. [MR 1912992](#)
- [5] M. L. Green, *Generic initial ideals*, Six Lectures on Commutative Algebra, Progress in Math., vol. 166, Birkhäuser, Basel, 1998, pp. 119–186. [MR 1648665](#)
- [6] J. Herzog and T. Hibi, *Monomial ideals*, Springer, London, 2011. [MR 2724673](#)
- [7] J. Hong, H. Lee, S. Noh and D. E. Rush, *Full ideals*, Comm. Algebra **37** (2009), 2627–2639. [MR 2543509](#)
- [8] U. Nagel and T. Römer, *Criteria for componentwise linearity*, preprint, 2011, available at [arXiv:1108.3921v2](#).
- [9] J. Watanabe, *\mathfrak{m} -full ideals*, Nagoya Math. J. **106** (1987), 101–111. [MR 0894414](#)
- [10] J. Watanabe, *The syzygies of \mathfrak{m} -full ideals*, Math. Proc. Cambridge Philos. Soc. **109** (1991), 7–13. [MR 1075117](#)
- [11] J. Watanabe, *\mathfrak{m} -full ideals II*, Math. Proc. Cambridge Philos. Soc. **111** (1992), 231–240. [MR 1142742](#)
- [12] A. Wiebe, *The Lefschetz property for componentwise linear ideals and Gotzmann ideals*, Comm. Algebra **32** (2004), 4601–4611. [MR 2111103](#)

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