Illinois Journal of Mathematics
Volume 56, Number 3, Fall 2012, Pages 795–804
S 0019-2082

LOCALIZATION OF COMPACTNESS OF HANKEL OPERATORS ON PSEUDOCONVEX DOMAINS

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Abstract. We prove the following localization for compactness of Hankel operators on Bergman spaces. Assume that Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$, p is a boundary point of Ω, and $B(p,r)$ is a ball centered at p with radius r so that $U = \Omega \cap B(p,r)$ is connected. We show that if the Hankel operator H^{Ω}_{ϕ} with symbol $\phi \in C^1(\overline{\Omega})$ is compact on $A^2(\Omega)$ then $H^U_{R_U(\phi)}$ is compact on $A^2(U)$ where R_U denotes the restriction operator on U, and $A^2(\Omega)$ and $A^2(U)$ denote the Bergman spaces on Ω and U, respectively.

Let V be a domain in $\mathbb{C}^n$ and $A^2(V)$ denote the Bergman space on V, the space of square integrable holomorphic functions on V with respect to the Lebesgue measure $d\lambda$ in $\mathbb{C}^n$. Let P^V denote the Bergman projection, the orthogonal projection from $L^2(V)$ onto $A^2(V)$. The Hankel operator, H^V_ϕ, with symbol $\phi \in L^\infty(V)$ is defined as $H^V_\phi(f) = \phi f - P^V(\phi f)$ for $f \in A^2(V)$.

A Hankel operator is the commutator $[M_\phi, P^V]$ of a multiplication operator with the Bergman projection. Such commutators play important roles in some problems in several complex variables (see, for example, [CD97]).

Compactness is an important concept in analysis. In this paper, we are interested in the localization of compactness of Hankel operators. More precisely, we are interested in the following question:

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $\phi \in L^\infty(\Omega)$, and $p \in b\Omega$ where $b\Omega$ denotes the boundary of Ω. Assume that $U = \Omega \cap B(p,r)$ is con-

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Received July 1, 2011; received in final form January 19, 2013.

The author is supported in part by the University of Toledo's Summer Research Awards and Fellowships Program.

2010 *Mathematics Subject Classification.* 32W05, 47B35.

nected, R_U denotes the restriction onto U, and H^{Ω}_{ϕ} is compact on $A^2(\Omega)$. Is $H^{U}_{R_U(\phi)}$ compact on $A^2(U)$?

We are not able to answer the question in general. Using the $\overline{\partial}$-Neumann operator, we show that the answer is yes when the symbol is C^1 on the closure of the domain. For more information about the $\overline{\partial}$-Neumann problem, see [CS01], [Str10] and consult [Zhu07] about the theory of Hankel operators on domains in $\mathbb{C}$.

It would be interesting to know if Theorem 1 below is still true without the C^1 differentiability requirement. We note that in dimension one, regularity of the symbol can be relaxed. For example, one can choose the symbol to be continuous up to the boundary (see Proposition 1 below). However, in that case localization is trivial as compactness is not due to localization. The following proposition is probably known, although we cannot provide a reference. We therefore include a proof that was suggested in [Str11].

Proposition 1. *Let Ω be a bounded domain in $\mathbb{C}$ and $\phi \in C(\overline{\Omega})$. Then the Hankel operator H^{Ω}_{ϕ} is compact on $A^2(\Omega)$.*

The main result of this paper is the following theorem.

Theorem 1. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $p \in b\Omega$, and $B(p,r)$ be a ball centered at p with radius $r > 0$ so that $U = \Omega \cap B(p,r)$ is connected. Assume that $\phi \in C^1(\overline{\Omega})$ and H^{Ω}_{ϕ} is compact on $A^2(\Omega)$. Then $H^{U}_{R_U(\phi)}$ is compact on $A^2(U)$.*

We note that in the theorem above no regularity of $b\Omega$ is assumed. That is, the boundary of Ω may be very irregular. Also $\phi \in C^1(\overline{\Omega})$ means that the function ϕ and all of its first partial derivatives have continuous extensions up to the boundary.

Localization is an important technique in analysis. So we believe that such results can be useful in studying compactness of Hankel operators in connection to boundary geometry (see, for example, [ÇŞ], [ČŞ09]). This particular localization can be useful in the following way: when one studies compactness of Hankel operators in relation to the boundary geometry of a smooth bounded pseudoconvex domain, usually a local holomorphic change of coordinates is needed to simplify the boundary geometry while preserving the compactness of the operator. Theorem 1 guarantees that this is possible when the local domain is an intersection with a ball and the symbol is sufficiently regular.

The converse of Theorem 1 is known to be true (see, for example, ii in Proposition 1 in [ČŞ09]). Hence, we have the following corollary.

Corollary 1. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $\phi \in C^1(\overline{\Omega})$, and $B(q,r)$ denote a ball centered at $q \in b\Omega$ with radius $r > 0$.*

i. *If* $U=\Omega\cap B(p,r)$ *is connected for some* $p\in b\Omega$, $r>0$, *and* H^{Ω}_{ϕ} *is compact on* $A^2(\Omega)$ *then* $H^{U}_{R_U(\phi)}$ *is compact on* $A^2(U)$.
ii. *Assume that for any* $p\in b\Omega$ *there exists* $r>0$ *such that* $U=\Omega\cap B(p,r)$ *is connected and* $H^{U}_{R_U(\phi)}$ *is compact on* $A^2(U)$. *Then* H^{Ω}_{ϕ} *is compact on* $A^2(\Omega)$.

REMARK 1. The proof of Theorem 1 shows that the localization of compactness of Hankel operators is still true on the intersection of the domain Ω with strongly pseudoconvex domains. Whether Theorem 1 holds on the intersection of Ω with domains with compact $\overline{\partial}$-Neumann operator is still open. However, it may not hold on the intersection of Ω with a general pseudoconvex domain. For example, let $U=\Omega\cap V$ where V is a smooth bounded convex domain and $bV\cap\Omega$ contains a nontrivial analytic disc D, and $\phi\in C^{\infty}(\overline{\Omega})$ such that $\phi\equiv 0$ on $b\Omega$ and $\phi\circ\beta$ is not holomorphic for some holomorphic mapping $\beta:\{z\in\mathbb{C}:|z|<1\}\to D$. Then one can use the facts that the product operator $M_{\phi}:A^2(\Omega)\to L^2(\Omega)$ is compact and the Hankel operator H^{Ω}_{ϕ} is a composition of the projection on the orthogonal complement of the Bergman space with M_{ϕ} to show that H^{Ω}_{ϕ} is compact. Moreover, since $\phi\circ\beta$ is not holomorphic for some holomorphic mapping $\beta:\{z\in\mathbb{C}:|z|<1\}\to D$ Theorem 2 in [ČŞ09] implies that $H^{U}_{R_U(\phi)}$ is not compact (even though, [ČŞ09, Theorem 2] is stated for smooth domains its proof is still valid on U). Therefore, H^{Ω}_{ϕ} is compact on $A^2(\Omega)$ while $H^{U}_{R_U(\phi)}$ is not compact on $A^2(U)$.

In the following examples, we show that boundedness and pseudoconvexity of the domain are necessary in Theorem 1.

EXAMPLE 1. This example shows that boundedness of the domain Ω is necessary. Let us denote $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$, $\Omega=\mathbb{D}\times\mathbb{C}$, $p=(1,0)$, and $\phi(z,w)=\xi(|w|)$ where $\xi\in C^{\infty}_0(-1,1)$ and $\xi(0)=1$. Let $f\in A^2(\Omega)$ then

$$\int_{\Omega}\left|f(z,w)\right|^2 d\lambda(z,w)=\int_{\mathbb{D}}\int_{\mathbb{C}}\left|f(z,w)\right|^2 d\lambda(w)\,d\lambda(z)<\infty.$$

Fubini's theorem implies that the set $\Gamma=\{z\in\mathbb{D}:\int_{\mathbb{C}}|f(z,w)|^2\,d\lambda(w)=\infty\}$ has measure zero. Hence, $f(z,w)=0$ for $z\notin\Gamma$ and $w\in\mathbb{C}$. This implies that $A^2(\Omega)=\{0\}$ and $H_{\phi}=0$. In particular, H_{ϕ} is compact. However, since there is an analytic disc through p in the boundary of $U=\Omega\cap B(p,1)$ [ČŞ09, Theorem 1] (see the last sentence in Remark 1) implies that the operator $H^{U}_{R_U(\phi)}$ is not compact on $A^2(U)$.

EXAMPLE 2. This example shows that pseudoconvexity of the domain is necessary (for more information on pseudoconvexity, see [Kra01], [Ran86]). In [ÇŞ], Çelik and the author constructed an annulus type domain $\Omega\subset\mathbb{C}^3$ (that is, $\Omega=\Omega_1\setminus\overline{\Omega}_2$ where $\overline{\Omega}_2\subset\Omega_1$, and Ω_1 and Ω_2 are smooth bounded pseudoconvex domains) such that H^{Ω}_{ϕ} is compact on $A^2(\Omega)$ for all $\phi\in C(\overline{\Omega})$.

However, they show that there exist $p \in b\Omega$, on the inner boundary of Ω, and $r > 0$ such that $U = \Omega \cap B(p,r)$ is a convex domain and there exists a disc through p in the boundary of U. Hence, N^U is not compact (see [FS98, Theorem 1.1]). Furthermore, there exists $\phi \in C^\infty(\overline{U})$ such that H^U_ϕ is not compact on $A^2(U)$ because, on a convex domain V, the Hankel operator H^V_ϕ is compact for all $\phi \in C^\infty(\overline{V})$ if and only if N^V is compact (see [FS98]).

Proof of Theorem 1 and Proposition 1

We use the $\overline{\partial}$-Neumann problem in the proof of Theorem 1. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and $\square^\Omega = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ be defined on square integrable $(0,1)$-forms, $L^2_{(0,1)}(\Omega)$, where $\overline{\partial}^*$ is the Hilbert space adjoint of $\overline{\partial}$. Kohn [Koh63] and Hörmander [Hör65] showed that (since Ω is a pseudoconvex domain) $\square$ has a solution operator, denoted by N^Ω, on $L^2_{(0,1)}(\Omega)$. Kohn [Koh63] also showed that $P^\Omega = I - \overline{\partial}^* N^\Omega \overline{\partial}$. Therefore, $H^\Omega_\phi(f) = \overline{\partial}^* N^\Omega(f\overline{\partial}\phi)$ for $f \in A^2(\Omega)$ and $\phi \in C^1(\overline{\Omega})$. We note that $H^\Omega_\phi(f)$ is the canonical solution for $\overline{\partial}u = f\overline{\partial}\phi$. That is, $H^\Omega_\phi(f)$ is the solution that is orthogonal to $A^2(\Omega)$ (or equivalently, it is the solution with the smallest norm in $L^2(\Omega)$). We refer the reader to [CS01], [Str10] and [ČŞ09] (and references therein) for more information about the $\overline{\partial}$-Neumann problem and compactness of Hankel operators on Bergman spaces.

We use a series of lemmas for the proof of Theorem 1. We note that the following lemma is an immediate corollary of [D'A02, Proposition V.2.3] (see also [Str10, Lemma 4.3]).

Lemma 1. *Let $T : X \to Y$ be a linear operator between two Hilbert spaces X and Y. Then T is compact if and only if for every $\varepsilon > 0$ there exist a compact operator $K_\varepsilon : X \to Y$ so that*

$$\|T(h)\|_Y \le \varepsilon \|h\|_X + \|K_\varepsilon(h)\|_Y \quad \text{for } h \in X.$$

In the proof of Theorem 1, we will need to apply Lemma 1 in the following set-up.

Lemma 2. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $\phi \in C^1(\overline{\Omega})$, and $X_\phi(\Omega)$ be the closure of $\{f\overline{\partial}\phi \in L^2_{(0,1)}(\Omega) : f \in A^2(\Omega)\}$ in $L^2_{(0,1)}(\Omega)$. Then H^Ω_ϕ is compact on $A^2(\Omega)$ if and only if for every $\varepsilon > 0$ there exists a compact operator $K_\varepsilon : X_\phi(\Omega) \to L^2(\Omega)$ such that*

$$\|\overline{\partial}^* N^\Omega(f\overline{\partial}\phi)\| \le \varepsilon \|f\overline{\partial}\phi\| + \|K_\varepsilon(f\overline{\partial}\phi)\| \quad \text{for all } f \in A^2(\Omega). \tag{1}$$

Proof. Assume that H^Ω_ϕ is compact on $A^2(\Omega)$. Then $\overline{\partial}^* N^\Omega$ is compact on a dense subset of $X_\phi(\Omega)$ which implies that it is compact on $X_\phi(\Omega)$. Then applying Lemma 1 with $T = \overline{\partial}^* N^\Omega$ and $X = X_\phi(\Omega)$ we get the following estimate: for every $\varepsilon > 0$ there exists a compact operator $K_\varepsilon : X_\phi(\Omega) \to L^2(\Omega)$ so that

$$\left\|\overline{\partial}^* N^\Omega (f\overline{\partial}\phi)\right\| \le \varepsilon \|f\overline{\partial}\phi\| + \left\|K_\varepsilon(f\overline{\partial}\phi)\right\| \quad \text{for } f \in A^2(\Omega).$$

On the other hand, if we assume that we have (1) then Lemma 1 implies that $\overline{\partial}^* N^\Omega$ is a compact operator on $X_\phi(\Omega)$. Hence, H^Ω_ϕ is compact on $A^2(\Omega)$. This completes the proof of Lemma 2. $\square$

The following famous theorem of Hörmander [Hör90, Theorem 4.4.2] will be used.

THEOREM (Hörmander). *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ and ψ be a continuous plurisubharmonic function on Ω. Assume that $u = \sum_{j=1}^n u_j \, d\bar{z}_j \in L^2_{(0,1)}(\Omega, e^{-\psi})$ such that $\overline{\partial} u = 0$. Then there exists $f \in L^2(\Omega, e^{-\psi})$ such that $\overline{\partial} f = u$ and*

$$\int_\Omega \frac{|f(z)|^2}{(1 + \sum_{j=1}^n |z_j|^2)^2} e^{-\psi(z)} \, d\lambda(z) \le \int_\Omega \sum_{j=1}^n \left|u_j(z)\right|^2 e^{-\psi(z)} \, d\lambda(z),$$

where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$.

We include the following standard lemma and its proof for convenience of the reader.

LEMMA 3. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $B(p,r)$ be the ball centered at $p \in b\Omega$ with radius r, and $\Omega(p,r) = B(p,r) \cap \Omega$. For $\varepsilon > 0$ and $0 < \delta < r$ there exists a bounded operator $E_{\varepsilon,\delta} : A^2(\Omega(p,r)) \to A^2(\Omega)$ such that*

$$\left\|f - E_{\varepsilon,\delta}(f)\right\|_{L^2(\Omega(p,r-\delta))} \le \varepsilon \|f\|_{L^2(\Omega(p,r-\delta))} \quad \textit{for } f \in A^2\big(\Omega(p,r)\big).$$

The following proof will use Hörmander's theorem in a similar fashion as in the proof of [Jup03, TheoremVI.3] where Jupiter shows that a pseudoconvex domain in $\mathbb{C}^n$ is a Runge domain if and only if it is polynomially convex.

Proof of Lemma 3. The crucial step in the proof is constructing a sequence of weight functions that will allow us to get the desired norm estimates. To that end, let us choose positive numbers δ, r_1, and r_2 so that $0 < r - \delta = r_1 < r_2 < r$ and define a function ψ as

$$\psi(z) = -r_2^2 + \sum_{j=1}^n |z_j - p_j|^2,$$

where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. Furthermore, we choose a smooth cut-off function $\chi \in C^\infty_0(B(p,r))$ such that $\chi \equiv 1$ in a neighborhood of $\overline{B(p,r_2)}$. We note that ψ is a continuous plurisubharmonic function on $\mathbb{C}^n$ that satisfies the following crucial property: $\psi(z) < 0$ for $z \in B(p,r_2)$ and $\psi(z) > 0$ for $z \in \mathbb{C}^n \setminus \overline{B(p,r_2)}$. Since ψ is bounded on Ω, the Hilbert spaces $L^2(\Omega)$ and $L^2(\Omega, e^{-k\psi})$ are equal for all k as sets. Then Hörmander's theorem implies that for every k there exists $u_k \in L^2(\Omega)$ such that $\overline{\partial} u_k = f\overline{\partial}\chi$ with

$$\int_\Omega \left|u_k(z)\right|^2 e^{-k\psi(z)}\, d\lambda(z) \le C \int_\Omega \left|f(z)\right|^2 \sum_{j=1}^n \left|\frac{\partial \chi(z)}{\partial \bar{z}_j}\right|^2 e^{-k\psi(z)}\, d\lambda(z), \tag{2}$$

where C is a positive real number that depends only on Ω. We note that $\psi < -r_2^2 + r_1^2 < 0$ on $B(p, r_1)$ and ψ is strictly positive on a neighborhood of the support of the $\overline{\partial}\chi$. Hence, the right-hand side of (2) goes to zero as k goes to infinity and we have

$$\begin{aligned} \int_{\Omega\cap B(p,r_1)} \left|u_k(z)\right|^2 d\lambda(z) &\le \int_\Omega \left|u_k(z)\right|^2 e^{-k\psi(z)}\, d\lambda(z) \\ &\le C \int_\Omega \left|f(z)\right|^2 \sum_{j=1}^n \left|\frac{\partial \chi(z)}{\partial \bar{z}_j}\right|^2 e^{-k\psi(z)}\, d\lambda(z). \end{aligned} \tag{3}$$

Then depending on ε and δ (and using (3)), we can choose $C_{\varepsilon,\delta} > 0$ and k so that $\|u_k\|_{L^2(\Omega(p,r_1))} \le \varepsilon \|f\|_{L^2(\Omega(p,r_1))}$ and $\|u_k\|_{L^2(\Omega)} \le C_{\varepsilon,\delta} \|f\|_{L^2(\Omega(p,r))}$. Therefore, we can define $E_{\varepsilon,\delta}$ as $E_{\varepsilon,\delta}(f) = \chi f - u_k$. □

Now we are ready to prove Theorem 1.

Proof of Theorem 1. To simplify the notation in this proof, we will denote the norm $\|\cdot\|_{L^2(U)}$ by $\|\cdot\|$ and the operator $H^U_{R_U(\phi)}$ by H^U_ϕ. We note that $\langle\cdot,\cdot\rangle$ denotes the inner product on U and $A \lesssim B$ means that $A \le cB$ for some constant c that is independent of the parameters of interest and its value can change at every appearance. For $f \in A^2(U)$, we have

$$\begin{aligned} \left\|H^U_\phi(f)\right\|^2 &= \left\langle \overline{\partial}^* N^U(f\overline{\partial}\phi), \overline{\partial}^* N^U(f\overline{\partial}\phi)\right\rangle \\ &= \left\langle f\overline{\partial}\phi, N^U \overline{\partial}\,\overline{\partial}^* N^U(f\overline{\partial}\phi)\right\rangle \\ &= \left\langle f\overline{\partial}\phi, N^U(f\overline{\partial}\phi)\right\rangle. \end{aligned}$$

In the last equality above, we used the facts that $N^U(\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}) = I$ and $\overline{\partial} N^U \overline{\partial} = 0$. Now we will construct a smooth bounded function λ that has a large Hessian on the boundary of the ball $B(p,r)$. Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a smooth, non-decreasing, convex function such that $-1 \le \gamma(t) \le 0$ for $t \le 0, \gamma(0) = 0$, and $\gamma'(0) \ge 2$. Furthermore, let us define

$$\rho_\varepsilon(z) = \frac{1}{\varepsilon}\left(-r^2 + \sum_{j=1}^n |z_j - p_j|^2\right)$$

for $r, \varepsilon > 0$ and $\psi_\varepsilon(z) = \gamma(\rho_\varepsilon(z))$. Then one can check that ψ_ε is a smooth plurisubharmonic function on $\mathbb{C}^n$, such that $-1 \le \psi_\varepsilon(z) \le 0$ for $z \in B(p,r)$.

Also, by continuity, there exists $\delta > 0$ such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \psi_\varepsilon(z)}{\partial z_j \, \partial \overline{z}_k} w_j \overline{w}_k \geq \frac{1}{\varepsilon} \sum_{j=1}^{n} |w_j|^2$$

for $z \in K = \overline{B(p,r) \setminus B(p,r-\delta)}$ and $(w_1, \ldots, w_n) \in \mathbb{C}^n$. Then (ii) in [Str10, Corollary 2.13] implies that

$$\begin{aligned} \frac{1}{e\varepsilon} \int_{K \cap U} \left|h(z)\right|^2 d\lambda(z) &\leq \sum_{j,k=1}^{n} \int_U e^{\psi_\varepsilon(z)} \frac{\partial^2 \psi_\varepsilon(z)}{\partial z_j \, \partial \overline{z}_k} h_j(z) \overline{h_k(z)} \, d\lambda(z) \\ &\leq \|\overline{\partial} h\|^2 + \left\|\overline{\partial}^* h\right\|^2 \end{aligned}$$

for $h = \sum_{j=1}^{n} h_j d\overline{z}_j \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*) \subset L^2_{(0,1)}(U)$. Let $\chi \in C^\infty(\overline{B(p,r)})$ such that $\chi \equiv 1$ on a neighborhood of $bB(p,r)$, and $\chi \equiv 0$ on $B(p,r-\delta)$. Then

$$\begin{aligned} \left\|H^U_\phi(f)\right\|^2 &\leq \left|\left\langle f\overline{\partial}\phi, \chi N^U(f\overline{\partial}\phi)\right\rangle\right| + \left|\left\langle f\overline{\partial}\phi, (1-\chi) N^U(f\overline{\partial}\phi)\right\rangle\right| \\ &\leq \|f\overline{\partial}\phi\| \left\|\chi N^U(f\overline{\partial}\phi)\right\| + \left|\left\langle (1-\chi) f\overline{\partial}\phi, N^U(f\overline{\partial}\phi)\right\rangle\right|. \end{aligned}$$

Then (4) implies that

$$\begin{aligned} \left\|\chi N^U(f\overline{\partial}\phi)\right\|^2 &\lesssim \varepsilon\left(\left\|\overline{\partial} N^U(f\overline{\partial}\phi)\right\|^2 + \left\|\overline{\partial}^* N^U(f\overline{\partial}\phi)\right\|^2\right) \\ &\lesssim \varepsilon \|f\|^2 \end{aligned}$$

for $f \in A^2(U)$. Let us denote $\chi_1 = 1 - \chi$ and choose $\widetilde{\chi} \in C^\infty_0(B(p,r))$ such that $0 \leq \widetilde{\chi} \leq 1$ and $\widetilde{\chi} \equiv 1$ on the support of χ_1. Then Lemma 3 implies that there exists a bounded operator $E_{\varepsilon,\delta} : A^2(U) \to A^2(\Omega)$ such that $\|\widetilde{\chi}(R_U E_{\varepsilon,\delta}(f) - f)\| \leq \varepsilon \|f\|$. Since δ depends on ε in the following calculation we will use the following notation: $E_\varepsilon = E_{\varepsilon,\delta}$, $F_\varepsilon = E_\varepsilon(f)$. Let M_ε denote the norm of the operator E_ε.

We note that in the following inequalities $\overline{\partial}^*_\Omega$ and $\overline{\partial}^*$ denote the Hilbert space adjoints of $\overline{\partial}$ on Ω and on U, respectively. A $(0,1)$-form f is in the domain of $\overline{\partial}^*$ if there exists a square integrable function g such that $\langle f, \overline{\partial} h\rangle = \langle g, h\rangle$ for all h in the domain of $\overline{\partial}$. Furthermore, if a $(0,1)$-form $f = \sum_{j=1}^{n} f_j \, d\overline{z}_j$ is in the domain of $\overline{\partial}^*$ then $\overline{\partial}^* f = -\sum_{j=1}^{n} \frac{\partial f_j}{\partial z_j}$ in the sense of distributions (see Chapter 4.2 in [CS01] for more information). The fact that $\overline{\partial}^* N$ is a solution operator for $\overline{\partial}$ (that is, $\overline{\partial}\overline{\partial}^* N f = f$ if f is a $\overline{\partial}$-closed form) implies that $F_\varepsilon \overline{\partial}\phi = \overline{\partial}(F_\varepsilon \phi) = \overline{\partial}\overline{\partial}^*_\Omega N^\Omega F_\varepsilon \overline{\partial}\phi$. We will use this equality as well as the Cauchy–Schwarz inequality to pass from the first line to the second line below.

$$\begin{aligned} &\left|\left\langle \chi_1(f\overline{\partial}\phi), N^U(f\overline{\partial}\phi)\right\rangle\right| \\ &\quad \leq \left|\left\langle \chi_1(f - F_\varepsilon)\overline{\partial}\phi, N^U(f\overline{\partial}\phi)\right\rangle\right| + \left|\left\langle \chi_1 F_\varepsilon \overline{\partial}\phi, N^U(f\overline{\partial}\phi)\right\rangle\right| \\ &\quad \lesssim \left\|\chi_1(f - F_\varepsilon)\right\| \|f\| + \left|\left\langle \chi_1 \overline{\partial}\overline{\partial}^*_\Omega N^\Omega(F_\varepsilon \overline{\partial}\phi), N^U(f\overline{\partial}\phi)\right\rangle\right| \end{aligned}$$

$$\lesssim \left\|\widetilde{\chi}(f-F_\varepsilon)\right\|\|f\| + \left|\left\langle \overline{\partial}^*_\Omega N^\Omega(F_\varepsilon\overline{\partial}\phi), \overline{\partial}^*\chi_1 N^U(f\overline{\partial}\phi)\right\rangle\right|$$
$$\lesssim \varepsilon\|f\|^2 + \widetilde{C}_\varepsilon\left\|\overline{\partial}^*_\Omega N^\Omega(F_\varepsilon\overline{\partial}\phi)\right\|_{L^2(\Omega)}\|f\|,$$

where $\widetilde{C}_\varepsilon$ is a constant that is independent of f. Now we will use the fact that H^Ω_ϕ is compact on $A^2(\Omega)$ and $\|F_\varepsilon\|_{L^2(\Omega)} \le M_\varepsilon\|f\|_{L^2(U)}$. Lemma 2 implies that for any $\varepsilon' > 0$ there exists a compact operator $K_{\varepsilon'}$ on $X_\phi(\Omega)$ such that

$$\left\|\overline{\partial}^*_\Omega N^\Omega(F_\varepsilon\overline{\partial}\phi)\right\|_{L^2(\Omega)} \lesssim \varepsilon'\|F_\varepsilon\|_{L^2(\Omega)} + \left\|K_{\varepsilon'}\Pi_{\overline{\partial}\phi}(F_\varepsilon)\right\|_{L^2(\Omega)},$$

where $\Pi_{\overline{\partial}\phi} : A^2(\Omega) \to X_\phi(\Omega)$ denotes the (bounded) multiplication operator by $\overline{\partial}\phi$. That is, $\Pi_{\overline{\partial}\phi}h = h\overline{\partial}\phi$ for $h \in A^2(\Omega)$. Therefore, for $f \in A^2(U)$ we have the following inequality

$$\begin{aligned}\left\|H^U_\phi(f)\right\|^2 &\lesssim \left(\varepsilon + \sqrt{\varepsilon} + \varepsilon' M_\varepsilon\widetilde{C}_\varepsilon\right)\|f\|^2 + \widetilde{C}_\varepsilon\|f\|\left\|K_{\varepsilon'}\Pi_{\overline{\partial}\phi}E_\varepsilon(f)\right\|_{L^2(\Omega)}\\ &\le \left(\varepsilon + \sqrt{\varepsilon} + \varepsilon' M_\varepsilon\widetilde{C}_\varepsilon + \varepsilon'\widetilde{C}_\varepsilon\right)\|f\|^2\\ &\quad + \left(\widetilde{C}_\varepsilon + \frac{\widetilde{C}_\varepsilon}{\varepsilon'}\right)\left\|K_{\varepsilon'}\Pi_{\overline{\partial}\phi}E_\varepsilon(f)\right\|^2_{L^2(\Omega)}.\end{aligned}$$

For any $0 < \varepsilon < 1$, there exists $\varepsilon' > 0$ so that $\varepsilon + \sqrt{\varepsilon} + \varepsilon' M_\varepsilon\widetilde{C}_\varepsilon \le 2\sqrt{\varepsilon}$. Then the above inequality combined with fact that $x^2 + y^2 \le (x+y)^2$ for $x, y \ge 0$ imply the following: for any $0 < \varepsilon < 1$ there exists a compact operator $K_\varepsilon = (\widetilde{C}_\varepsilon + \widetilde{C}_\varepsilon/\varepsilon')^{1/2}K_{\varepsilon'}\Pi_{\overline{\partial}\phi}E_\varepsilon$ such that

$$\left\|H^U_\phi(f)\right\| \lesssim \varepsilon^{1/4}\|f\| + \left\|K_\varepsilon(f)\right\| \quad \text{for } f \in A^2(U).$$

Now Lemma 1 implies that H^U_ϕ is compact on $A^2(U)$. □

Proof of Proposition 1. Since functions that are smooth up to the boundary of Ω are dense in $C(\overline{\Omega})$ and the sequence $\{H^\Omega_{\psi_n}\}$ converges to H^Ω_ψ in the operator norm whenever $\{\psi_n\}$ converges to ψ uniformly on $\overline{\Omega}$ it suffices to prove that H^Ω_ψ is compact whenever $\psi \in C^\infty(\overline{\Omega})$. Let us define

$$S_\psi(f)(z) = -\frac{1}{\pi}\int_\Omega \frac{\frac{\partial\psi}{\partial\overline{\xi}}(\xi)f(\xi)}{\xi - z}\,d\lambda(\xi)$$

for $f \in A^2(\Omega)$ and $z \in \Omega$. We will show that H^Ω_ψ is compact on $A^2(\Omega)$ by showing that S_ψ is a limit of compact operators (in the operator norm) and $S_\psi(f)$ solves $\overline{\partial}u = f\overline{\partial}\psi$ (because $H^\Omega_\psi = S_\psi - P^\Omega S_\psi$). To that end, for $\varepsilon > 0$ let χ_ε be a smooth cut-off function on $\mathbb{R}$ such that $\chi_\varepsilon \equiv 1$ on a neighborhood of the origin and $\chi_\varepsilon(t) = 0$ for $|t| \ge \epsilon$. Then $S_\psi = A^\varepsilon_\psi + B^\varepsilon_\psi$ where

$$A^\varepsilon_\psi(f)(z) = -\frac{1}{\pi}\int_\Omega \frac{\chi_\varepsilon(|\xi - z|)\frac{\partial\psi}{\partial\overline{\xi}}(\xi)f(\xi)}{\xi - z}\,d\lambda(\xi),$$
$$B^\varepsilon_\psi(f)(z) = -\frac{1}{\pi}\int_\Omega \frac{(1-\chi_\varepsilon(|\xi - z|))\frac{\partial\psi}{\partial\overline{\xi}}(\xi)f(\xi)}{\xi - z}\,d\lambda(\xi).$$

Then the operator B^{ε}_{ψ} is Hilbert–Schmidt and, in particular, compact because the kernel

$$-\frac{(1-\chi_{\varepsilon}(|\xi-z|))\frac{\partial\psi}{\partial\overline{\xi}}(\xi)}{\pi(\xi-z)}$$

is square integrable on $\Omega\times\Omega$.

Next, we will show that A^{ε}_{ψ} has a small norm. Let $\widehat{f}$ denote the trivial extension of f. That is, $\widehat{f}=f$ on Ω but $\widehat{f}=0$ otherwise. Since $\frac{\partial\psi}{\partial\overline{\xi}}$ is continuous on $\overline{\Omega}$ and Ω is bounded, using polar coordinates, we get

$$\left|A^{\varepsilon}_{\psi}(f)(z)\right|\lesssim\int_{\mathbb{C}}\frac{|\chi_{\varepsilon}(|\xi|)\widehat{f}(z+\xi)|}{|\xi|}\,d\lambda(\xi)\lesssim\int_0^{2\pi}\int_0^{\varepsilon}\left|\widehat{f}\big(z+re^{i\theta}\big)\right|dr\,d\theta.$$

Then the Cauchy–Schwarz inequality together with Fubini's theorem yield that

$$\left\|A^{\varepsilon}_{\psi}(f)\right\|^2\lesssim 2\pi\varepsilon\int_0^{2\pi}\int_0^{\varepsilon}\int_{\Omega}\left|\widehat{f}\big(z+re^{i\theta}\big)\right|^2d\lambda(z)\,dr\,d\theta\le 4\pi^2\varepsilon^2\|f\|^2.$$

Hence, $\|A^{\varepsilon}_{\psi}\|\lesssim\varepsilon$ and S_{ψ} is a limit (in the operator norm) of a sequence $\{B^{1/k}_{\psi}\}$ of compact operators.

Next, we want to show that $\overline{\partial}S_{\psi}(f)=f\overline{\partial}\psi$. Let $\{f_n\}$ be a sequence of functions that are smooth on $\overline{\Omega}$ and converging to f in $L^2(\Omega)$. Then the Cauchy integral with remainder formula (see [CS01, Theorem 2.1.2]) shows that $\overline{\partial}S_{\psi}(f_n)=f_n\overline{\partial}\psi$. On the other hand, $\{\overline{\partial}S_{\psi}(f_n)\}$ converges weakly to $\overline{\partial}S_{\psi}(f)$ and $\{f_n\overline{\partial}\psi\}$ converges to $f\overline{\partial}\psi$ in $L^2(\Omega)$. Therefore, $\overline{\partial}S_{\psi}(f)=f\overline{\partial}\psi$ for $f\in A^2(\Omega)$. $\square$

Acknowledgment. I am in debt to Daniel Jupiter for bringing the proof of [Jup03, Theorem VI.3] to my attention, to Mehmet Çelik and Trieu Le for fruitful conversations, to my advisor Emil Straube for the proof of Proposition 1, and to the referee for valuable comments.

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